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Exercise Sheet 7
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Exercises in Superstring Theory

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`http://www.th.physik.uni-bonn.de/klemm/strings1819/`

1 Operator Product Expansions

In a 2d conformal invariant theory, there are two non-vanishing components of the energy-momentum tensor: the chiral part $T(z)$ and the anti-chiral part $\bar{T}(\bar{z})$. They give rise to an infinite number of conserved currents from which the associated conserved charges generating infinitesimal conformal transformations $z \to z + \epsilon(z)$, $\epsilon(z) \ll 1$, are written as

$$Q_\epsilon = \oint_{C_0} \frac{dz}{2\pi i} \epsilon(z) T(z) \ . \tag{1.1}$$

Any field $\phi(z)$ on the complex plane has a mode expansion of the following form

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \qquad \text{with} \qquad \phi_n = \oint_{C_0} \frac{dz}{2\pi i} \phi(z) z^{n+h-1}, \tag{1.2}$$

where integration is counterclockwise around the origin,

1. For two generic operators $A(z)$ and $B(z)$ with

$$A = \oint_{C_0} \frac{dz}{2\pi i} A(z) \ , \qquad B = \oint_{C_0} \frac{dz}{2\pi i} B(z) \ , \tag{1.3}$$

show that radial ordering $\mathcal{R}$ implies that

$$\oint_{C_0} \frac{dz}{2\pi i} [A(z), B(w)] = \oint_{C_w} \frac{dz}{2\pi i} \mathcal{R}(A(z)B(w)) \ . \tag{1.4}$$

In particular, this leads to the important relation

$$[A, B] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} \mathcal{R}(A(z)B(w)) \ . \tag{1.5}$$

2. The variation (2.2) of exercise sheet 6 is encoded into the commutator $\delta_\epsilon \phi(w) = -[Q_\epsilon, \phi(w)]$. Use the result from item 1. and the Cauchy-Riemann formula given by

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w) \tag{1.6}$$

to show that the ($\mathcal{R}$-ordered) *operator product expansion* $T(z)\phi(w)$ is given by

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \text{reg.} \ , \tag{1.7}$$

where reg. stands for a holomorphic fct. of z regular at $z = w$, also called finite terms or non-singular terms.

Therefore, a primary field $\phi(z)$ of weight h is also defined as a field which has the operator product expansion (1.7) with the energy-momentum tensor.

Consider the Laurent expansion of the energy-momentum tensor $T(z)$ given by

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \ , \tag{1.8}$$

where the L_n's are the Virasoro generators given by

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \ . \tag{1.9}$$

3. Show that the (radial-ordered) OPE of the energy-momentum tensor with itself,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{finite terms} \ , \tag{1.10}$$

is equivalent to the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ . \tag{1.11}$$

2 The propagator of the free boson

The first simple system is that of a free boson. In two dimensions the free boson has the following Euclidean action

$$S = \frac{g}{2}\int d^2x (\partial_\alpha\varphi\partial^\alpha\varphi + m^2\varphi^2) \, , \tag{2.1}$$

where g is a normalization parameter. The two-point function called *propagator* is given by

$$K(\boldsymbol{x},\boldsymbol{y}) = \langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\rangle \, . \tag{2.2}$$

The propagator obeys

$$g(-\partial_x^2 + m^2)K(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \, . \tag{2.3}$$

1. Why is the propagator a function of only $r := |\boldsymbol{x}-\boldsymbol{y}|$, i.e. $K(\boldsymbol{x},\boldsymbol{y}) = K(r)$? Show that for $m=0$ (2.3) is solved by

$$K(r) = -\frac{1}{2\pi g}\log r + const \, . \tag{2.4}$$

   Hint: Proceed by integrating (2.3) over $\boldsymbol{x}$ within a disc D of radius r centered around $\boldsymbol{y}$

In terms of complex coordinates this reads

$$\langle\varphi(z,\bar z)\varphi(w,\bar w)\rangle = -\frac{1}{4\pi g}\Big(\log(z-w) + \log(\bar z - \bar w)\Big) + const \, . \tag{2.5}$$

This leads to the following OPE

$$\partial_z\varphi(z,\bar z)\partial_w\varphi(w,\bar w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \, . \tag{2.6}$$

2. Given the energy momentum tensor

$$T(z) = -2\pi g : \partial\varphi\partial\varphi : \tag{2.7}$$

   Show that the normal ordered operators $V_\alpha(z,\bar z) =: e^{i\alpha\varphi(z,\bar z)}$ are primary fields and determine their conformal weights h and $\bar h$.
   Hint: Determine the OPE with the energy momentum tensor $T(z)$.

3 The Free Fermion

The action for a free Majorana fermion reads

$$S = \frac{1}{2} g \int d^2 x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \,, \tag{3.1}$$

where the Dirac matrices γ^μ satisfy the so-called Dirac algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \,, \tag{3.2}$$

if $\eta^{\mu\nu} = \mathrm{diag}(1,1)$ a representation thereof is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

1. Write the action in S in terms of the two-component spinor $\Psi = (\psi, \bar{\psi})$ and calculate the equations of motion for ψ and $\bar{\psi}$. What do they imply?

2. Next we want to calculate the correlator $\langle \Psi_i(\boldsymbol{x}), \Psi_j(\boldsymbol{y}) \rangle$, where $i, j = 1, 2$ label the components of Ψ. To do so, express the kinetic terms in the derived action in 1. in terms of a matrix A_{ij} and write a differential equation for the Green's function.

3. We claim the the Green's function $G_{ij}(z, \bar{z})$ in complex coordinates for item 2. is given by

$$G = \frac{1}{2\pi g} \begin{pmatrix} \frac{1}{z-w} & 0 \\ 0 & \frac{1}{\bar{z}-\bar{w}} \end{pmatrix} \,. \tag{3.4}$$

   Prove this by using the techniques you used in the bosonic case.

4 The Ghost System

Another simple system is the so-called *ghost system* with the following action

$$S = \frac{g}{2} \int d^2 x b_{\mu\nu} \partial^\mu c^\nu \,, \tag{4.1}$$

where the field $b_{\mu\nu}$ is a traceless symmetric tensor, and both $b_{\mu\nu}$ and c^μ are anticommuting fields. The propagator of such system can be obtanied in a similar way as sketched in section 2. It is given by

$$\langle b(z) c(w) \rangle = \frac{1}{\pi g} \frac{1}{z-w} \,. \tag{4.2}$$

1. Determine the correlators $\langle b(z) \partial c(w) \rangle$, $\langle \partial b(z) c(w) \rangle$ and $\langle \partial b(z) \partial c(w) \rangle$.

2. The normal-ordered holomorphic energy-momentum tensor of the bc-system is given by

$$T(z) = \pi g : (2\partial c b + c \partial b) : \tag{4.3}$$

   Compute the OPEs $T(z)b(w)$ and $T(z)c(w)$ using Wick's Theorem. What are the conformal dimensions of the fields b and c?

3. Compute the OPE of the energy-momentum tensor with itself and bring it to the form

$$T(z)T(w) = \frac{\mathrm{c}/2}{(z-w)^4} + \frac{hT(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \mathrm{ref} \,. \tag{4.4}$$

   Read off the conformal weight h of the energy momentum tensor and the central charge c of the bc system.