Physikalisches Institut
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Theoretische Physik

Exercise Sheet 8
27.11.2018
WS 18/19

Exercises in Superstring Theory

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Hand in: 03.12.2018

http://www.th.physik.uni-bonn.de/klemm/strings1819/

1 Vertex operator

Recall the Vertex operator defined by

$$V_\alpha(z,\bar{z}) =: e^{i\alpha\phi(z,\bar{z})} : , \tag{1.1}$$

where ϕ is a free bosonic field. Show that for two free fields one has

$$: e^{a\phi_1} :: e^{b\phi_2} :=: e^{a\phi_1 + b\phi_2} : e^{ab\langle\phi_1\phi_2\rangle} . \tag{1.2}$$

Use this result to compute the OPE of two Vertex operator. Moreover, with such an OPE conclude that the 2-point function of two vertex operators is given by

$$\langle V_\alpha(z,\bar{z}) V_\beta(w,\bar{w})\rangle = \begin{cases} |z-w|^{-\frac{\alpha^2}{2\pi g}} & \text{if } \alpha = -\beta \\ 0 & \text{else.} \end{cases} \tag{1.3}$$

2 Physical meaning of the central charge

1. Show that an equivalent way to write the OPE (1.10) of exercise sheet 7 is

$$\delta_\epsilon T(z) = -\epsilon(z)\partial_z T(z) - 2\partial_z\epsilon(z)T(z) - \frac{\mathrm{c}}{12}\partial_z^3\epsilon(z) . \tag{2.1}$$

The finite version of (2.1) under $z \to w(z)$ is given by

$$T(z) \to T(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{\mathrm{c}}{12}\{w,z\}\right] , \tag{2.2}$$

where $\{w,z\}$ is the *Schwarzian derivative* ($w' = \partial_z w$)

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 . \tag{2.3}$$

Compare (1.10) of exercise sheet 7, (2.1) and (2.2) with the corresponding expressions for a primary field $\phi(z)$ of weight h you encountered in exercise sheet 7

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + \text{finite terms} , \tag{2.4}$$

$$\delta_\epsilon\phi(z) = -[\epsilon(z)\partial_z + h\partial_z\epsilon(z)]\phi(z) , \tag{2.5}$$

$$\phi(z) \to \phi(w) = \left(\frac{dw}{dz}\right)^{-h}\phi(z) . \tag{2.6}$$

From these you can clearly see that the energy-momentum tensor is NOT a primary field.

2. Show that $\{w, z\} = 0$ for $w = \frac{az+b}{cz+d}$, $ad - bc = 1$[1] and conclude that the energy-momentum tensor is a quasi-primary field. What is its conformal dimension h in this case?

Recall the map of the worldsheet of a free closed string – the cylinder – parametrized by $\sigma \in [0, l)$ and $\tau \in (-\infty, +\infty)$ to the complex plane via (after Wick rotation $\tau \to -i\tau$)

$$\begin{aligned} z &= e^{\frac{2\pi}{l}w} = e^{\frac{2\pi}{l}(\tau - i\sigma)} \, , \\ \bar{z} &= e^{\frac{2\pi}{l}\bar{w}} = e^{\frac{2\pi}{l}(\tau + i\sigma)} \, . \end{aligned} \tag{2.7}$$

3. Show that the map from the cylinder to the plane gives

$$T_{cyl.}(w) = \left(\frac{2\pi}{l}\right)^2 \left(z^2 T_{plane}(z) - \frac{\mathrm{c}}{24}\right) \, . \tag{2.8}$$

For the following discussion consider $\langle T_{plane}(z) \rangle = 0$.

4. Compute the *Casimir energy* of the cylinder given by

$$E = \int_0^l d\sigma \langle T_{\tau\tau} \rangle = -\frac{1}{2\pi} \int_0^l d\sigma \Big(\langle T_{cyl}(w) \rangle + \langle \bar{T}_{cyl}(\bar{w}) \rangle \Big) \, . \tag{2.9}$$

Generally speaking, for a 2d CFT defined on a curved 2d manifold Σ equipped with metric g, the expectation value of the trace of the energy-momentum tensor, instead of vanishing, is proportional to both the curvature R and the central charge c

$$\langle T^\mu{}_\mu(\boldsymbol{x}) \rangle_{(\Sigma, g)} = \frac{\mathrm{c}}{24\pi} R(\boldsymbol{x}) \, . \tag{2.10}$$

This quantum breaking of scale invariance is called the *trace anomaly.*

[1] The group $PSL(2, \mathbb{C})$ represents the globally defined conformal transformations acting on the conformal plane coordinates.