Exercises in Superstring Theory

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Hand in: 11.12.2018

`http://www.th.physik.uni-bonn.de/klemm/strings1819/`

1 The Kac determinant and singular vectors

A representation of the Virasoro algebra is said to be unitary if it contains no negative-norm states. Unitarity as well as the presence of singular vectors can be studied using the **Kac determinant**. The matrix of inner products between all basis states

$$L_{-k_1}L_{-k_2}\dots L_{-k_n}|h\rangle\,, \tag{1.1}$$

of a given Verma module is called the **Gram matrix** M. Due to the orthogonality of descendant states at different levels the Gram matrix is of block diagonal form. We denote the block corresponding to states of level l by $M^{(l)}$. The determinant det $M^{(l)}$ is called the Kac determinant.

1. Calculate $M^{(l)}$ for $l = 0, 1, 2$ as a function of h and c. Argue that for a unitary representation $h > 0$.
2. Show that a unitary representation contains singular vectors at level two for
$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(1-c)(25-c)}\right)\,. \tag{1.2}$$
3. Given $|h\rangle = \phi(0)|0\rangle$, the descendant field associated with the state $L_{-n}|h\rangle$ is given by
$$\phi^{(-n)}(w) = \frac{1}{2\pi i}\oint_{\mathcal{C}_w} dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w)\,. \tag{1.3}$$
Show that for a string $X = \phi_1(w_1)\dots\phi_N(w_N)$ of primary fields with conformal dimensions h_i,
$$\langle \phi^{(-n)}(w) X\rangle = \mathcal{L}_{-n}\langle \phi(x) X\rangle\,, \tag{1.4}$$
with
$$\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i} \right\}\,. \tag{1.5}$$
4. Assume that the Verma module generated by $|h\rangle = \phi(0)|0\rangle$ is unitary and contains a singular vector at level two. Show that the correlators involving $\phi(z)$ satisfy a differential equation. Use the general form of the three-point function $\langle\phi(z)\phi_1(z_1)\phi_2(z_2)\rangle$ to show that it vanishes unless
$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1)\,. \tag{1.6}$$