Exercises Advanced Topics in String Theory
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1 Warm up - String Theory on the circle

The mode expansion for the closed string reads

$$X^\mu = \frac{x_0^\mu}{2} + \frac{\tilde{x}_0^\mu}{2} - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n^\mu e^{-in\sigma^-} + \tilde{\alpha}_n^\mu e^{-in\sigma^+}\Big) \quad (1.1)$$

The momentum is expressed as

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu). \quad (1.2)$$

Finally we consider the 25th direction to be a circle of radius R, i.e. we impose the condition

$$X^{25} \sim X^{25} + 2\pi w R, \quad w \in \mathbb{Z}. \quad (1.3)$$

1. Show that

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\big(\frac{n}{R} + \frac{wR}{\alpha'}\big) = \sqrt{\frac{\alpha'}{2}}P_L, \quad \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\big(\frac{n}{R} - \frac{wR}{\alpha'}\big) = \sqrt{\frac{\alpha'}{2}}P_R \quad (1.4)$$

2. Starting form the mass shell condition

$$M^2 = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) = \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\bar{N}-1), \quad (1.5)$$

   where N and $\bar{N}$ denote the number of left respectively right-running modes, derive

$$M^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\big(N + \tilde{N} - 2\big), \quad nw + N - \bar{N} = 0 \quad (1.6)$$

3. Determine the massless spectrum and explain how it is obtained from the uncompactified 26-dimensional theory by Kaluza-Klein reduction.

2 String Theory on the orbifold $S^1/\mathbb{Z}_2$

Now we truncate our theory by dividing out the $\mathbb{Z}_2$ symmetry that acts as

$$R_{25} : X^{25} \longrightarrow -X^{25}. \quad (2.1)$$

2.1 Untwisted states

1. Draw a picture to show which points are identified under this symmetry and how the resulting geometry looks like. How many fixed points are there?

2. How does R_{25} act on the modes of the expansion (1.1)?

3. Which massless states do survive the projection? Does the tachyon survive?

2.2 Twisted states

The orbifold projection allows for a second type of boundary condition which gives rise to so-called twisted states.

$$X^{25}(\sigma + 2\pi, \tau) = -X^{25}(\sigma, \tau) + 2\pi w R \tag{2.2}$$

1. Draw a picture that shows such a string configuration.

2. Solve the Laplace equation

$$\Big(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\Big) X^{25}(\tau, \sigma) = 0 \tag{2.3}$$

   respecting the boundary condition (2.2).

2.3 The partition function

The partition function of the circle is given by

$$Z_{S^1}(q, R) = (\eta\bar{\eta})^{-1} \sum_{n,w} q^{\frac{\alpha'}{4}P_L^2} \bar{q}^{\frac{\alpha'}{4}P_R^2}, \quad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) \tag{2.4}$$

The orbifold partition function is given by

$$Z_{orb} = \mathrm{Tr}_{\text{untwisted}}\Big(\frac{1 + R_{25}}{2} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}}\Big) + \mathrm{Tr}_{\text{twisted}}\Big(\frac{1 + R_{25}}{2} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}}\Big) \tag{2.5}$$

1. Evaluate the partition function in the untwisted sector. Is it modular invariant? What do you conclude?

2. How is the trace over the twisted sector evaluated in principle? (You are not asked to do this explicitly.)

3. The whole partition function ist given by

$$Z_{orb}(R, q) = \frac{1}{2}\Big(Z_{S^1}(R, q) + 2\Big|\frac{\eta(q)}{\theta_{10}(0, q)}\Big|^2 + 2\Big|\frac{\eta(q)}{\theta_{01}(0, q)}\Big|^2 + 2\Big|\frac{\eta(q)}{\theta_{00}(0, q)}\Big|^2\Big) \tag{2.6}$$

4. Check that this is modular invariant. What is the interpretation of the factor 2?