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1st Special HW
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/klemm/strings2_19/
SPECIAL HOMEWORK #1

1 Lorentz invariance and the LCQ

Here, you will compute the critical dimension of an open superstring theory in the light-cone quantization (LCQ) by requiring a quantum anomaly in the Lorentz algebra to vanish. **In this section, we will be working the light-cone coordinates and the Greek indices always denote $-, +, i$ where $i = 1, \ldots, d-2$.** The light-cone coordinates are defined by

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}) \quad \text{with} \quad \eta_{-+} = \eta_{+-} = -1 \quad \text{and} \quad \eta_{ij} = \delta_{ij}. \tag{1}$$

Let us start with the super-Virasoro constraint.

$$\psi_\mu \partial_+ X^\mu = 0 \quad \text{and} \quad \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi^\mu \partial_+ \psi_\mu = 0, \tag{2}$$

where $\partial_\pm = \frac{\partial}{\partial \sigma^\pm}$ with $\sigma^\pm = \tau \pm \sigma$. Recall that, for open strings,

$$\partial_+ X^\mu = \frac{1}{2}\sum_n \alpha_n^\mu e^{-in\sigma^+}, \tag{3}$$

$$\partial_+ \psi^\mu = \frac{1}{\sqrt{2}}\sum_r b_r^\mu e^{-ir\sigma^+}. \tag{4}$$

Choose the gauge

$$X^+ = x^+ + p^+\tau \quad \text{and} \quad \psi^+ = 0. \tag{5}$$

- Solve α_n^- and b_r^- from (2). You should find that

$$\begin{aligned} \alpha_n^- &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\left(\sum_m :\alpha_{n-m}^i \alpha_m^i: + \sum_r (r - \frac{n}{2}) :b_{n-r}^i b_r^i:\right) - \frac{a}{2p^+}\delta_n, \\ b_r^- &= \frac{1}{p^+}\sum_{i=1}^{d-2}\sum_s \alpha_{r-s}^i b_s^i, \end{aligned} \tag{6}$$

  where a is the normal ordering constant for the zero modes.

Let us the revisit the Lorentz algebra,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho}). \tag{7}$$

- Show that, in the light-cone coordinates, (7) gives

$$[J^{i-}, J^{j-}] = 0. \tag{8}$$

The generator $J^{\mu\nu}$ can be expressed through the modes α^μ and b^μ via

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu}, \tag{9}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{10}$$

$$E^{\mu\nu} = \frac{1}{i}\sum_{n>0}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), \tag{11}$$

$$K^{\mu\nu} = \frac{1}{i}\sum_{r>0}\frac{1}{r}(b^\mu_{-r}b^\nu_r - b^\nu_{-r}b^\mu_r). \tag{12}$$

- Write down the following relations: $[p^+a^-_m, a^i_n]$, $[p^+a^-_m, x^-]$, $[p^+a^-_m, b^i_n]$, $[p^+b^-_m, a^i_n]$, $[p^+b^-_m, x^-]$, and $[p^+b^-_m, b^i_n]$.
- Compute $[p^+a^-_m, p^+b^-_r]$, $[p^+a^-_m, p^+a^-_n]$, and $\{p^+b^-_r, p^+b^-_s\}$ **only for** $m+n \neq 0$ **and** $r+s \neq 0$.
  Hint: $[x^-, 1/p^+] = i/(p^+)^2$.

You should obtain the following

$$[p^+a^-_m, p^+a^-_n] = (m-n)p^+a^-_{m+n} + A(m)\delta_{m+n}, \tag{13}$$

$$\{p^+b^-_r, p^+b^-_s\} = 2p^+a^-_{r+s} + B(r)\delta_{r+s}, \tag{14}$$

with

$$A(m) = \frac{d-2}{8}(m^3 - m) + 2am, \tag{15}$$

$$B(r) = \frac{d-2}{2}(r^2 - \frac{1}{4}) + 2a \tag{16}$$

and

$$[p^+a^-_m, p^+b^-_r] = (\frac{m}{2} - r)p^+b^-_{m+r} \tag{17}$$

with

- Using the previous results to compute $[l^{\mu\nu}, b^\rho_p]$, $[l^{\mu\nu}, \alpha^\rho_p]$, $[E^{\mu\nu}, b^\rho_p]$, $[E^{\mu\nu}, \alpha^\rho_p]$, $[K^{\mu\nu}, b^\rho_p]$, and $[K^{\mu\nu}, \alpha^\rho_p]$.
- Compute all the commutators of $l^{\mu\nu}$, $E^{\mu\nu}$, and $K^{\mu\nu}$.
- Show that J^{ij} and J^{kl} satisfy the Lorentz algebra.
- Arrive at the following

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2}\sum_{n=1}^{\infty}\left(n\frac{d-10}{8} + \frac{1}{n}(2a - \frac{d-2}{8})\right)(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n). \tag{18}$$

  Hence, one immediately sees that $d = 10$ and $a = \frac{1}{2}$ leads to a vanishing anomaly in the Lorentz algebra.

2 The RNS formalism

In the lecture the LCQ formulation has been used for the description of Type II string theories. Another complementary description of the latter is the so-called RNS formulation, which is heavily discussed in *String Theory Volume II* by Polchinski. Such a formulation will be useful in the future when discussing Vertex operators and string scattering amplitudes.

Let $H(z)$ be the holomorphic part of a scalar field with the following Operator Product Expansion (OPE)

$$H(z)H(w) \sim \log(z-w)\,, \tag{19}$$

Consider the Majorana-Weyl fermions

$$\Psi^{\pm a}(z) = \frac{1}{\sqrt{2}}\Big(\psi^{2a+1}(z) \pm i\psi^{2a}(z)\Big)\,. \tag{20}$$

Here $a = 0 \ldots \frac{d-2}{2}$, ψ^μ are the holomorphic anticommuting worldsheet Majorana spinor fields. Due to previous section, we fix $d = 10$.

- Show the following OPEs

$$\begin{aligned} \Psi^{+a}(z)\Psi^{-a}(w) &\sim \frac{1}{z-w}\,, \\ \Psi^{+a}(z)\Psi^{+a}(w) &\sim 0\,, \\ \Psi^{-a}(z)\Psi^{-a}(w) &\sim 0\,. \end{aligned} \tag{21}$$

Notice that we obtain the same OPEs given above by considering the following vertex operators $e^{\pm iH(z)}$. Their OPEs read

$$\begin{aligned} e^{iH(z)}e^{-iH(w)} &\sim \frac{1}{z-w}\,, \\ e^{iH(z)}e^{iH(w)} &\sim 0\,, \\ e^{-iH(z)}e^{-iH(w)} &\sim 0\,. \end{aligned} \tag{22}$$

Hence we find the equivalence of arbitrary local operators

$$\Psi^{+a}(z) \simeq e^{iH_a(z)}\,, \quad \Psi^{-a}(z) \simeq e^{-iH_a(z)}\,. \tag{23}$$

In order for these theories to describe the same CFTs, their energy-momentum tensor must be equivalent.

- Show that the following OPEs hold

$$\begin{aligned} e^{iH_a(z)}e^{-iH_a(-z)} &= \frac{1}{2z} + i\partial H(0) + 2zT^H(0) + \mathcal{O}(z^2)\,, \\ \Psi^{+a}(z)\Psi^{-a}(-z) &= \frac{1}{2z} + \Psi^{+a}\Psi^{-a}(0) + 2zT^\Psi(0) + \mathcal{O}(z^2)\,. \end{aligned} \tag{24}$$

Hence we make the following identification

$$:\Psi^{+a}\Psi^{-a}(z): \simeq i\partial H_a(z)\,, \quad T^\Psi \simeq T^H\,. \tag{25}$$

The equivalence we find in (23) and (25) is known as **bosonization**. Recall the R ground state $|0\rangle_R$ in the LCQ formulation.

- Using the RNS formulation, argue that the vertex operator $\Theta_{\boldsymbol{s}} = \exp\left(i \sum_a s_a H_a\right)$ can be identified with the highest weight states (those you encountered in LCQ) given by

$$|\boldsymbol{s}\rangle = |s_0, s_1, s_2, s_3, s_4\rangle \ , \quad \text{with} \quad s_a = \pm\frac{1}{2} \, . \tag{26}$$

In the covariant quantization approach of the bosonic string theory, the ghosts system bc has to be added into the Polyakov action after fixing the conformal gauge. Similarly for the superstring we have to add the superconformal ghost system $bc + \beta\gamma$. A bosonization for the latter system can be made as well, which effectively adds a factor on the states depending on another bosonic field $\phi(z)$ with OPE $\phi(z)\phi(w) \sim -\log(z-w)$. Adding the contribution of the ghosts, the R ground state vertex operators read

$$\mathcal{V}_{\boldsymbol{s}} = e^{-\frac{\phi}{2}} \Theta_{\boldsymbol{s}} \, . \tag{27}$$

On the other hand, the fermionic parts of the tachyon and massless NS vertex operators are respectively given by

$$\mathcal{V}_{\times} = e^{-\phi} \, , \quad \mathcal{V}_{-1} = e^{-\phi} e^{\pm i H_a} \, . \tag{28}$$

- Recall the physical state condition, which states that physical states requires conformal weight $h = 1$. As a consistency check, compute the conformal weight of the vertex operators $\mathcal{V}_{\boldsymbol{s}}, \mathcal{V}_{\times}$, and $\mathcal{V}_{-1}$.
Hint: You might need the following energy momentum tensor for the bosonic field ϕ,

$$T_\phi(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : -\partial^2\phi(z) \, . \tag{29}$$

The NS sector works out much as in the bosonic string. The lowest state is $|0; k\rangle_{NS}$, labeled by the matter state and momentum coming from the asymptotic of the vertex operator $V_k = e^{ik\cdot X}$. In the R sector the lowest states are $|u; k\rangle_R = u_{\boldsymbol{s}} \, |\boldsymbol{s}; k\rangle_R$. where $u_{\boldsymbol{s}}$ is the polarization, and the sum on $\boldsymbol{s}$ is implicit.

- Explain why the Ramond vacuum $|u; k\rangle_R$ is a spacetime spinor, while the Neveu-Schwarz vacuum $|0; k\rangle_{NS}$ is a spacetime scalar.

- Consider the Dirac equation as follows

$$k_\mu \Gamma^\mu \, |u; k\rangle_R = (k_0 \Gamma^0 + k_1 \Gamma^1) \, |u; k\rangle_R = 0 \, , \tag{30}$$

such that $k^2 = 0$, and show that it can be written as

$$\left(S_0 - \frac{1}{2}\right) |u; k\rangle_R = 0 \, , \quad S_0 = \Gamma^{0,+}\Gamma^{0,-} - \frac{1}{2} \, , \tag{31}$$

where $\Gamma^{0,\pm} = \pm\Gamma^0 + \Gamma^1$.

- Consider the decomposition of the original 2^5-component Majorana spinor subject to the Dirac equation into spinor representations of $SO(1,1) \times SO(8)$ as appearing in the LCQ approach. Based on this decomposition, how many spinor component does $|u; k\rangle_R$ have? Are they real or complex? Is $|u; k\rangle_R$ chiral, antichiral or neither of them?