Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/klemm/strings2_19/
SPECIAL HOMEWORK #2

1 Type I Strings & O-planes

In this section we consider the open-closed duality for the case of unoriented superstrings, i.e. Type I strings. We consider Type IIB string theory and the worldsheet-parity operator acting on the worldsheet coordinates as follows

$$\Omega : (\sigma, \tau) \mapsto (\ell - \sigma, \tau) \,. \tag{1}$$

As a starting point we compute the loop-vacuum amplitude of closed strings. Recall that for Type II string theories we must insert the GSO projection operator P_{GSO} when performing the trace sum over the Hilbert space $\mathcal{H}$ of concern. In addition to that, for Type I strings, we have to add the projection onto even states under Ω. Hence, the closed loop-vacuum partition function for Type I strings reads

$$Z_I = \mathrm{Tr}_{\mathcal{H}_{cl}} \left(\frac{1+\Omega}{2} P_{GSO} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) . \tag{2}$$

- Show that Z_I separates in two parts as follows $Z_I = \frac{1}{2} Z_{T^2}(\tau, \bar{\tau}) + \frac{1}{2} Z_{\mathcal{K}}(t)$. Here Z_{T^2} is the one-loop partition function of the torus you already know from previous course. Using the level matching condition take $Z_{\mathcal{K}}$ into the trace factor appearing in (3), with $t := \mathrm{Im}\tau$.

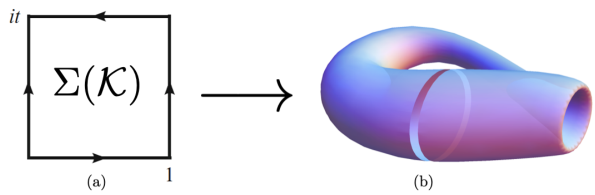

Figure 1: (a) Topology associated to $Z_{\mathcal{K}}$. (b) A closed string following a loop, such that it sweeps a Klein bottle, reverses its orientation along the σ direction.

Note that Ω flips the orientation over the length direction of the worldsheet for the outgoing states on the trace. We interpret this action as a change of topology for the worldsheet associated to the one-loop partition function as depicted in Figure 1-(a). We denote such a worldsheet as $\Sigma(\mathcal{K})$, which in fact has topology of a Klein bottle. Taking the measure of the Klein bottle, the corresponding vacuum diagram reads

$$\mathcal{K} = \int_0^\infty \frac{dt}{2t} Tr_{\mathcal{H}_{cl}} \left(\Omega P_{GSO} e^{-4\pi t (L_0 - \frac{c}{24})} \right) . \tag{3}$$

- Compute the action of Ω on the ground states of closed strings in Type IIB.
- Using previous item and level matching condition due to the first item, show that

$$\mathcal{K} = \frac{V_{10}}{(4\pi\alpha')^5}\int_0^\infty \frac{dt}{2t}\frac{1}{t^5}\left(\frac{\theta_3^4-\theta_4^4-\theta_2^4}{2\eta^{12}}\right)(2it)\,. \tag{4}$$

Here $V_D = \langle p|p\rangle$. In the following we take a look into the analogous of loop-channel $\leftrightarrow$ tree-chanel duality for the cylinder diagram corresponding to $\Sigma(\mathcal{K})$.

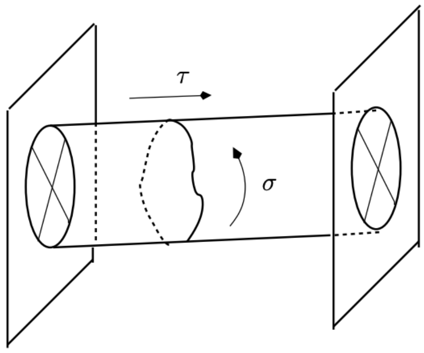

Figure 2: Exchange of a closed strings between two O-planes. Note the crosscaps located at the ends of the cylinder. A crosscap is a circle with diagonally opposite points identified.

- Take the necessary cuts, flips and/or gluings in order to take $\Sigma(\mathcal{K})$ into a tree-channel diagram form. You might get some aid for this task from Figure 6.16 in *Basic Concepts of String Theory* by Blumenhagen, Lüst and Theisen. Your diagram should correspond to the tube depicted in Figure 2. Moreover, parametrize the time direction of such a tree diagram with l and relate the latter with the parameter t.

Annalogous to the annulus diagram for two parallel D-branes discussed in **Exercise 5**, we interpret the diagram obtained in previous item, as a locus in target spacetime (two of them) where strings become unoriented as they hit it (see crosscaps definition). Such an object is called **Orientifold plane** or **O-plane** for shortness.

- Now, we invoke the loop-channel $\leftrightarrow$ tree-chanel duality. Denote $\mathcal{O}_9$ the tree-channel amplitude corresponding to the diagram in Figure 2. Using modular transformation and the parametrization assigned with l, obtain $\mathcal{O}_9$ as an integral expression over l.

Taking the appropriate Fourier expansions of the modular objects within the integral in $\mathcal{O}_9$, we find

$$\mathcal{O}_9 = \frac{V_{10}}{(4\pi\alpha')^5}\int_0^\infty dl 2^4(1_{NS}-1_R)\Big(16+\mathcal{O}(e^{-2\pi l})\Big)\,. \tag{5}$$

Note the volume factor appearing in (5) suggests the objects considered here are a spacetime filling ones, i.e. **O9-planes**. So far, everything is good-looking here. Similar as in the discussion of D-branes closed strings exchange in **Exercise 5**, we could naively think the O9-plane is charged under an RR-field. This means, an O9-plane couples electromagnetically, with charge μ_{O9}, to a C_{10} form via the following action

$$S_\times = \mu_{O9}\int_{M_{10}} C_{10}\,. \tag{6}$$

Here M_{10} is the 10-dimensional target space.

- Explain what goes wrong when introducing the action $S_\times$. What kind of divergence/inconsistency would you obtain when considering charged/uncharged O9 planes?

We must include another source different than (6) to remove possible inconsistencies when regarding O9 planes. For that purpose we introduce a stack of N D9-branes. In the following we consider the loop-channel annulus amplitude for open strings stretched between the stack of D9-branes. Such expression reads

$$\int_0^\infty \frac{dt}{2t} \mathrm{Tr}_{\mathcal{H}_{op}} \left(\frac{1+\Omega}{2} P_{GSO} e^{-2\pi t (L_0 - \frac{c}{24})} \right) = \frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{M} \,. \tag{7}$$

Here we separated the open string vacuum diagram in two parts:

(a) Without Ω insertion: $\mathcal{A}$

(b) With Ω insertion: $\mathcal{M}$

- Recall the Chan-Paton factors introduced in **Exercise 6**. Argue that the open string states with Chan-Paton labels $|n, k; i, j\rangle$ add a factor N^2 to $\mathcal{A}$ (previously computed in **Exercise 5**) when taking the trace over $\mathcal{H}_{op}$.

We are interested in the exchange of closed strings in the tree-channel. Therefore, we perform the S transformation, by exchange of parameters using loop-channel $\leftrightarrow$ tree-chanel duality, we find

$$\mathcal{A}\Big|_{closed} = N^2 \frac{V_{10}}{(4\pi\alpha')^5} \int_0^\infty dl 2^{-6} (1_{NS} - 1_R) \Big(16 + \mathcal{O}(e^{-2\pi l}) \Big) \,. \tag{8}$$

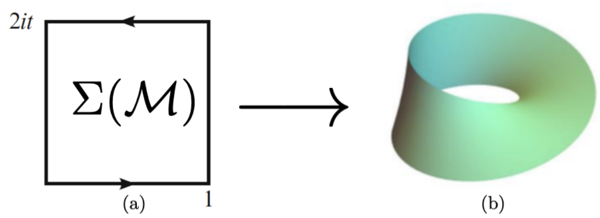

Figure 3: (a) Topology associated to $\mathcal{M}$. (b) An open string following a loop, such that it sweeps a Möbius strip, reverses its orientation along the σ direction

Lastly, we conisder the amplitude due to the the Ω insertion case, i.e.

$$\mathcal{M} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}_{\mathcal{H}_{op}} \Big(\Omega P_{GSO} e^{-2\pi t (L_0 - \frac{c}{24})} \Big) \,. \tag{9}$$

Similarly as the Klein bottle for the closed strings, the Ω operator flips the orientation for the open string outgoing states. This effectively turns the annulus diagram into a Möbius strip, which we denote by $\Sigma(\mathcal{M})$.

- Recall 2.2) in **Exercise 6**. Argue that the Chan-Paton factors will add a factor $\sum \langle i', j' | \, \Omega \, | i, j \rangle = \pm N$ into $\mathcal{M}$ when taking the trace in $\mathcal{H}_{op}$.

The complete computation of $\mathcal{M}$ is straightforward, but a bit more tedious. Once again, we are interested in the D9 closed strings exchange tree diagram. For this, we might use the loop-channel $\leftrightarrow$ tree-chanel duality once more. For the latter purpose we need an intermediate step for making a triangulation of $\Sigma(\mathcal{M})$ into a tree diagram. At the end of the day the duality works by performing a modular transformation under the appropriate identification of parameters. We will skip the details of the computation $\mathcal{M} \leftrightarrow \mathcal{M}\big|_{closed}$, as this resembles the same one we did for $\mathcal{K} \leftrightarrow \mathcal{O}_9$. We take the final result as

$$\mathcal{M}\Big|_{closed} = \pm N \frac{V_{10}}{(4\pi\alpha')^5} \int_0^\infty dl (1_{NS} - 1_R)\Big(16 + \mathcal{O}(e^{-2\pi l})\Big) . \tag{10}$$

- Consider the total sum $\mathcal{O}_9 + \mathcal{A}\big|_{closed} + \mathcal{M}\big|_{closed}$. How does such a sum factorises? If we isolate the RR contributions in the sum, how can we make such a divergence to vanish? Lastly, explain what would be the implications of this result when comparing with 2.3) of **Exercise 6**.

2 The Graviton Propagator (Extra Points)

n the following, we do a field theory calculation to work out the amplitude for the exchange of the graviton and dilaton between a pair of parallel Dp-branes. Consider the action for a D-brane as follow

$$S = S_{\text{bulk}} + S_p \,, \tag{11}$$

where

$$S_{\text{bulk}} = \frac{1}{2\kappa_0} \int_{M_D} \sqrt{-G} e^{-2\Phi} \Big(\mathcal{R} + 4\nabla_\mu \Phi \nabla^\mu \Phi\Big) \,, \tag{12}$$

and

$$S_p = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{-\Phi} \det\Big(- X^* G_{ab} - X^* B_{ab} - 2\pi\alpha' X^* F_{ab}\Big) . \tag{13}$$

Here M_D denotes the D-dimensional target background spacetime and is endowed with a metric $G_{\mu\nu}$ and $\mathcal{R}$ denotes its Ricci scalar. Φ denotes the dilaton field. Σ_{p+1} is the world-volume of the Dp-brane with local coordiantes ξ^a. The target space coordinates $X^\mu(\xi^a)$ describe the embedding of the brane into space-time, i.e.

$$X : \Sigma_{p+1} \hookrightarrow M_D \,. \tag{14}$$

The action (13) is called the Dirac-Born-Infeld action and is written in terms of the pull-backs $X^* G_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$, similarly for the $B_{\mu\nu}$ antisymmetric field and the field-strength field $F_{\mu\nu}$.

1. Take the actions S_{bulk} and S_p into their Einstein frame form via the following redefinition
$$\tilde{G}_{\mu\nu}(X) = e^{-\frac{4\tilde{\Phi}}{D-2}} G_{\mu\nu}(X) \,, \quad \tilde{\Phi} = \Phi - \Phi_0 \,. \tag{15}$$

2. Linearise S^E_{bulk} and S^E_p about a flat background by writing $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, and expanding up to second order in $h_{\mu\nu}$.

3. Work out the propagators in momentum space for the graviton $\Delta_G(k)$ and the dilation $\Delta_\Phi(k)$. The result should read

$$\begin{aligned} \Delta_G(k) &= -\frac{2i\kappa^2}{k^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right), \\ \Delta_\Phi(k) &= -\frac{i\kappa^2(D-2)}{4k^2}. \end{aligned} \tag{16}$$