Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/klemm/strings2_19/
SPECIAL HOMEWORK #3

Appetizer: Toroidal compactification of bosonic strings

Consider the simplest situation in which we compactify the 25th direction of closed bosonic string theory (in 26 dimensions) on a circle with radius R.

$$X^{25}(\sigma + 2\pi, \tau) \sim X^{25}(\sigma, \tau) + 2\pi RL, \tag{1}$$

where L is the winding number. Therefore, the string mass satisfy the relation $m^i m_i = m_R^2 + m_L^2$ with

$$\alpha' m_L^2 = \frac{\alpha'}{2}\Big(\frac{M}{R} + \frac{LR}{\alpha'}\Big)^2 + 2(N_L - 1) \quad \text{and} \quad \alpha' m_R^2 = \frac{\alpha'}{2}\Big(\frac{M}{R} - \frac{LR}{\alpha'}\Big)^2 + 2(N_L - 1), \tag{2}$$

where M is the mass of X^{25}.

- Show that $N_L - N_R = ML$
- Show that the mass $m^i m_i$ is invariant under under $R \to R/\alpha'$ and $L \leftrightarrow M$ (T-duality).
- Write the down the massless spectrum of this theory. How many gauge bosons do we have? This implies which gauge symmetry?
- Set $R = \sqrt{\alpha'}$ and repeat the previous question (The gauge group is now $SU(2)_L \times SU(2)_R$).
- We consider the vertex operators of the $SU(2)_L \times SU(2)_R$ given by

$$\begin{aligned} &\partial X^{25}(z),\ \exp\Big(\frac{\pm 2iX^{25}(z)}{\sqrt{\alpha'}}\Big) \\ &\bar\partial X^{25}(\bar z),\ \exp\Big(\frac{\pm 2iX^{25}(\bar z)}{\sqrt{\alpha'}}\Big). \end{aligned} \tag{3}$$

  Compute the OPE of the above operators, using

$$\langle X^{25}(z) X^{25}(w)\rangle = -\frac{\alpha'}{2}\log(z-w). \tag{4}$$

- Show the current algebra from the current modes of the operators in (3) is the Kac-Moody algebra of $SU(2) \times SU(2)$ at level 1.
  Using $J(z) = \sum J_n z^{-n-1}$.

Hors d'oeuvres: Heterotic strings, modular invariance & T-duality

Toroidal compactifications of string theory and the heterotic string contain a subsector that can be described in terms of chiral and antichiral bosons with non-trivial zero-mode sector. In order to understand such CFTs better, we now consider a CFT of N chiral bosons and M antichiral bosons whose zero modes (p_L, p_R) take values in a lattice $\Gamma_{N,M} \subset R^{N+M}$. We can endow this lattice with a Lorentzian metric $||(p_L,p_R)||^2 = p_L^2 - p_R^2$. Moreover, we assume that this lattice is integral, i.e. all scalar products between lattice points take integer values only. We call $N+M$ the rank of the lattice and $N-M$ its signature. If $||x||^2$ is even for all $x \in \Gamma_{N,M}$, we call the lattice even (type II) otherwise odd (type I). A lattice is called unimodular or self-dual if its unit cell has volume 1. In this problem we assume that $\Gamma_{N,M}$ is even and uni-modular.

- Calculate the partition function of this CFT $Z(\tau,\bar\tau)$.
- Prove that $Z(\tau,\bar\tau)$ is modular invariant.
  Hint: Use Poisson resummation.
- The orthogonal group $O(N,M;\mathbb{R})$ acts on lattices of rank $N+M$ and signature $N-M$. Check that being even and unimodular is invariant under such a transformation and argue that the partition function $Z(\tau,\bar\tau)$ does not change if we replace the lattice $\Gamma_{N,M}$ with $\Lambda\cdot\Gamma_{N,M}$, where $\Lambda \in O(N,M;\mathbb{Z}) = GL(N+M,\mathbb{Z}) \cap O(N,M;\mathbb{R})$ is a Lorentz transformation that is a lattice automorphism at the same time (T-duality).
- The Hasse-Minkowski classification of unimodular, indefinite lattices states that up to isomorphism there is exactly one lattice for a given rank, signature and type (see for example *J.-P Serre: A course in arithmetics*, theorem V.2.2.5. and theorem V.2. Use this result to argue, why the moduli space $\mathcal{N},\mathcal{M}$ of physically inequivalent lattices is given by $(N, M > 0)$

$$\begin{aligned} \mathcal{M}_{N,M} &= \mathcal{M}^0_{N,M}/O(N,M;\mathbb{Z})\,, \\ \mathcal{M}^0_{N,M} &= O(N,M;\mathbb{R})/\Big(O(N;\mathbb{R})\times O(M;\mathbb{R})\Big)\,. \end{aligned} \tag{5}$$

  What is its dimension? Describe the moduli spaces $\mathcal{M}^0_{1,1}$ and $\mathcal{M}_{1,1}$.

Dessert: Compactifications & CY geometries

Consider the manifold $S^1 \otimes \mathbb{R}^n$ endowed with the metric

$$ds^2 = R^2 d\theta^2 + (dx^i + \Omega^i{}_j x^j d\theta)^2\,, \tag{6}$$

where $\Omega^i{}_j$ is a constant anti-symmetric matrix, i.e. a generator of the rotation group $SO(n)$ and R is the radios of S^1.

- Show that this metric has vanishing curvature, but nevertheless a vector, when parallel transported around the circle, i.e. it is rotated by an element of $SO(n)$.

Let M be a $2N$-dimensional manifold with $SU(N)$ holonomy. Such a manifold admits a spinor field ϵ that is covariantly constant, i.e. $\nabla_M\epsilon = 0$. Here $\nabla_M = \partial_M - \frac{1}{4}\omega_M^{AB}\Gamma_{AB}$, where ω denotes the spin connection, $\Gamma_{AB} = \frac{1}{2}[\Gamma_A,\Gamma_B]$ and the Γ's matrices fulfil a Clifford algebra.

- Show that

$$[\nabla_M,\nabla_N]\epsilon = \frac{1}{4}R_{MNPQ}\Gamma^{PQ} = 0\,. \tag{7}$$

  Here R_{MNPQ} denotes the Riemann tensor of the background M.

- Using the following identity

$$\Gamma^N \Gamma^{PQ} = \Gamma^{NPQ} + G^{NP}\Gamma^Q - G^{NQ}\Gamma^P\,, \tag{8}$$

  where $\Gamma^{NPQ} = \frac{1}{3!}\left(\Gamma^N\Gamma^P\Gamma^Q \pm \cdots\right)$, together with the Bianchi identity, show that (7) implies

$$R_{MQ}\Gamma^Q \epsilon = 0\,. \tag{9}$$

  Note that (9) implies that $R_{MQ} = 0$. This means that a metric of $SU(N)$ holonomy is necessarily Ricci-flat.

From physics, one wants solutions to Einstein's equation $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor derived from the metric g. On a Calabi-Yau manifold with a complex structure, we have a unique solution given by the Ricci-flat metric in that complex structure. Let us look at the space of all possible complex structure, and we can deform a solution by changing the complex structure. To see these two type of solutions, let us look at the nearby metric $g \to g + h$, and linearize $R_{\mu\nu}$ in this new metric.

- Assuming $R_{\mu\nu} = 0$ and $\nabla^\mu h_{\mu\nu} = 0$, perform this linearization to find he following equation for h:

$$\Delta h_{\mu\nu} + 2R_{\mu\ \nu}^{\ \alpha\ \beta} h_{\alpha\beta} = 0\,. \tag{10}$$

  Here $\Delta = \nabla_\alpha \nabla^\alpha$.

Let $E_\tau = \mathbb{C}/\Lambda_\tau$ be an elliptic curve with complex structure τ, where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$.

- Show that E_τ is a complex 1-dimensional Calabi-Yau manifold. Moreover, express the complex structure τ in terms of the periods of E_τ.

- Let E_τ and $E_{\tau'}$ be two elliptic curves with complex structure parameters τ and τ' respectively. Show that E_τ and $E_{\tau'}$ are biholomorphic iff $\tau' = \gamma\tau$ for some $\gamma \in SL_2\mathbb{Z}$.

- Construct a variety $X \subset \mathbb{P}^2$ given by the locus $P(x, y, z) = 0$, where $[x, y, z]$ are the homogeneous coordinates of $\mathbb{P}^2$. By setting the condition $c_1(TX) \overset{!}{=} 0$ determine the form of P. Argue that X is also holomorphically isomorphic to an elliptic curve E_τ for some complex structure τ.