Physikalisches Institut
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Exercise 3
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/klemm/strings2_19/
PRESENCE EXERCISE

Spin structure & spin manifolds

Consider a string worldsheet with the topology $\mathbb{P}^1 \simeq S^2$ on which fermionic states live. The rotation group of the manifold is $SO(2) \simeq U(1)$. The representations of $SO(2)$ are given by specifying the eigenvalues of the only generator W of $SO(2)$.

1. Show that a vector field V^μ decomposes into components V^+ and V^- of $W = 1$ and $W = -1$, respectively. A two-component spinor field ψ^A has components ψ^+ and ψ^- with $W = \frac{1}{2}$ and $W = -\frac{1}{2}$, respectively. How many degrees of freedom does ψ^A have if it is Majorana.

In the following, we consider V^+ parallel transported around a closed path γ on the sphere. Then, V^+ will pick up a phase $e^{i\alpha}$.

2. What happens to ψ^+ when parallel transported around γ? Explain why one could naively assume there would be a sign ambiguity for the parallel transported ψ^+.

3. Now shrink the path γ to a point and show that the sign must be $+1$.

4. Now consider a closed path γ along an enclosed handle on the worldsheet Σ_g, where Σ_g is an oriented Riemann surface with genus $g \in \mathbb{Z}_{>0}$. What would be the sign choice due to the phase obtained by parallel transport?

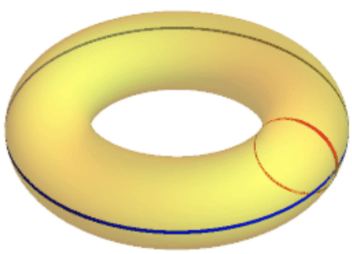

Figure 1: Closed paths along the A and B cycle of the torus.

5. Determine all Spin structures on a two dimensional torus T^2. Parametrise the torus by the worldsheet coordinates τ and σ and identify the boundary conditions for the Neveu-Schwarz and Ramond sector.

A Spin bundle

The **spin group** $Spin(n)$ is the double cover of the special orthogonal group $SO(n)$ such that there exists a short exact sequence of Lie groups

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\phi} Spin(n) \xrightarrow{\psi} SO(n) \longrightarrow 1$$

Let $\pi : TM \to M$ be a tangent bundle with $\dim(M) = n$. The bundle TM is assumed to have a fibre metric and structure group $G = SO(n)$, when M is orientable. Let LM be the frame bundle associated with TM. Let $\Lambda_{\alpha\beta}$ be the transition function of LM which satisfies

$$\Lambda_{\alpha\beta}\Lambda_{\beta\gamma}\Lambda_{\gamma\alpha} = 1\,, \quad \Lambda_{\alpha\alpha} = 1\,. \tag{1}$$

A spin structure on M is defined by the transition function $\tilde{\Lambda}_{\alpha\beta} \in Spin(n)$ such that

$$\psi(\tilde{\Lambda}_{\alpha\beta}) = \Lambda_{\alpha\beta}\,, \quad \tilde{\Lambda}_{\alpha\beta}\tilde{\Lambda}_{\beta\gamma}\tilde{\Lambda}_{\gamma\alpha} = 1\,, \quad \tilde{\Lambda}_{\alpha\alpha} = 1\,. \tag{2}$$

The set of $\tilde{\Lambda}_{\alpha\beta}$ defines a **spin bundle** $PS(M)$ over M and M is said to admit a **spin structure**[1]. M may admit many spin structures depending on the choice of $\tilde{\Lambda}_{\alpha\beta}$. Not all manifolds admit spin structures.

[1] If M admits a spin structure, it might also be called a **spin manifold**.