Physikalisches Institut
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Exercise 3
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/klemm/strings2_19/

PRESENCE EXERCISE

Modular invariance and the GSO projection

Let us first recall the Hamiltonians in the NS and R sector from the light-cone quantization

$$H_{\rm NS} = \sum_{i=2}^{9} \sum_{r\in\mathbb{Z}+\frac{1}{2}>0} r b^i_{-r} b^i_r - \frac{1}{6}, \tag{1}$$

$$H_{\rm R} = \sum_{i=2}^{9} \sum_{r\in\mathbb{Z}} r b^i_{-r} b^i_r + \frac{1}{3}. \tag{2}$$

The 1-loop partition function is simply the trace over the states $|\psi\rangle$ in the Hilbert space.

$$Z(\tau) = \sum_{\psi} \langle\psi| e^{iH\tau} |\psi\rangle. \tag{3}$$

We will be interested in the case where $Z(\tau)$ is non-diagonally modular invariant. Thus, the partition function can be factorized into a product of the right and left moving part.

$$Z(\tau) = ({\rm Tr}_{\rm L} q^{H})({\rm Tr}_{\rm R} q^{\bar{H}}) \tag{4}$$

where $q = e^{2\pi i\tau}$. As you have already seen from the previous exercise that, on a torus, there are four different spin structures. Therefore, we have

$$\text{NS} \quad \begin{cases} Z_{-+}(\tau) = \eta_{-+} {\rm Tr}_{\rm NS}\left(q^{H_{\rm NS}}(-1)^F\right) \\ Z_{--}(\tau) = \eta_{--} {\rm Tr}_{\rm NS} q^{H_{\rm NS}} \end{cases} \tag{5}$$

$$\text{R} \quad \begin{cases} Z_{+-}(\tau) = \eta_{+-} {\rm Tr}_{\rm R} q^{H_{\rm R}} \\ Z_{++}(\tau) = \eta_{++} {\rm Tr}_{\rm R}\left(q^{H_{\rm R}}(-1)^F\right) \end{cases} \tag{6}$$

The factor of $(-1)^F$ ensures the periodicity in the second direction. The following might be useful. The generalize Jacobi theta-function is given by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \eta(\tau) e^{2\pi i\alpha\beta} q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta}\right)\left(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta}\right) \tag{7}$$

$$= \sum_{n=-\infty}^{\infty} e^{i\pi(n+\alpha)^2\tau+2\pi i(n+\alpha)\beta} \tag{8}$$

The theta functions satisfy the Riemann identity

$$\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}^4 - \vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}^4 = 0. \tag{9}$$

1. Show explicitly that

$$Z_{--}(\tau) = \eta_{--}\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4}{\eta^4(\tau)} \quad \text{and} \quad Z_{+-}(\tau) = \eta_{+-}\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^4(\tau)}. \tag{10}$$

Homework: Following the same computations, one can obtain the other two partition functions

$$Z_{++}(\tau) = \eta_{++}\frac{\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4}{\eta^4(\tau)} \quad \text{and} \quad Z_{-+}(\tau) = \eta_{-+}\frac{\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4}{\eta^4(\tau)} \tag{11}$$

2. **Homework**: Show that under the T transformation

$$Z_{--}(\tau+1) = -\frac{\eta_{-+}}{\eta_{--}}Z_{-+}(\tau) \tag{12}$$

$$Z_{-+}(\tau+1) = -\frac{\eta_{--}}{\eta_{-+}}Z_{--}(\tau) \tag{13}$$

$$Z_{+-}(\tau+1) = Z_{+-}(\tau), \tag{14}$$

and under the S transformation

$$Z_{--}(-1/\tau) = Z_{--}(\tau) \tag{15}$$

$$Z_{-+}(-1/\tau) = \frac{\eta_{+-}}{\eta_{-+}}Z_{+-}(\tau) \tag{16}$$

$$Z_{+-}(-1/\tau) = \frac{\eta_{+-}}{\eta_{-+}}Z_{-+}(\tau). \tag{17}$$

Also, show that Z_{++} transforms irreducibly under the modular transformations.

3. Fix $\eta_{\pm\pm}$ by requiring that the total partition function is modular invariant. Arrive at the following result from which you should see that this is precisely that same as applying the GSO projection to the states in the partition function

$$Z(\tau) = \frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4}{\eta^4(\tau)} - \frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^4(\tau)} - \frac{\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4}{\eta^4(\tau)} \pm \frac{\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4}{\eta^4(\tau)}. \tag{18}$$

4. It is easy to see that

$$\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix} = 0. \tag{19}$$

With the above and the Riemann identity, one finds that the 1-loop partition function obtained from the previous question vanishes. Give a physical explanation about the result.