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Condensed Matter Theory I — WS09/10

Exercise 4

(Please return your solutions before Fr., 4.12., 12:00h)

4.1 Thermodynamic properties of a Fermi liquid at low T (15 points)

In the lecture the concept of quasiparticles was introduced. The thermodynamic properties of a Fermi liquid are determined by the properties of the quasiparticle gas. For one quasiparticle above the groundstate the energy can be linearly approached by

$$\epsilon_{\mathbf{k}} = v_F(k - k_F)\,,$$

with $v_F = k_F/m^*$. For more than one excited quasiparticle, interactions come into play. For the energy correction resulting from these quasiparticle interactions we derived

$$\delta E_{\mathbf{k}\sigma} \approx \frac{1}{N_F}\left[F_0^s \delta n + \sigma F_0^a \delta n_s + \frac{1}{k_F^2} F_1^s \mathbf{k}\delta\mathbf{g}\right]\,, \tag{1}$$

with $F_l^{s,a}$ the dimensionless Landau parameters. $\delta n = \sum_{\mathbf{k}\sigma}(n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma})$ is the particle density correction and $\delta n_s = \sum_{\mathbf{k}\sigma} \sigma(n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma})$ the spin density correction. For the particle current density it holds $\mathbf{j} = (1/m)\mathbf{g} = (1/m)\sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}\sigma}$.

The internal energy of the system is

$$U = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}\sigma} n(E_{\mathbf{k}\sigma})\,,$$

with $n(E)$ the Fermi-Dirac distribution. By differentiation with respect to the temperature T we can calculate the specific heat:

$$c_V = \frac{\partial}{\partial T}\sum_{\mathbf{k}\sigma} E_{\mathbf{k}\sigma} n(E_{\mathbf{k}\sigma}) \tag{2}$$

(a) Derive

$$c_V = \frac{1}{T}\sum_{\mathbf{k}\sigma} E^2_{\mathbf{k}\sigma}\left(-\frac{\partial n}{\partial E_{\mathbf{k}\sigma}}\right) + \sum_{\mathbf{k}\sigma}\frac{\partial E_{\mathbf{k}\sigma}}{\partial T} n\,. \tag{3}$$

(b) Show that the specific heat in leading order of T is given by

$$c_V = \frac{\pi^2}{3} N_F T\,,$$

with $N_F = (k_F m^*)/\pi^2$ the density of states at the Fermi level.
Hint: Start with equation (2), transform the $\mathbf{k}$-sum into an integral over ϵ and use a Sommerfeld expansion for the integral of the type $\int d\epsilon\, F(\epsilon) n(\epsilon)$.

Therefore, the specific heat is proportional to the temperature for all metals. The second term in (3) would lead to a correction $\delta c_V \propto T^2 \ln(T/E_F)$. In the lecture the spin susceptibility χ was derived by considering the system under the influence of a magnetic field. It was shown that χ is related to the Landau parameter F_0^a as

$$\chi = \mu_M^2 \frac{N_F}{1+F_0^a}\,.$$

In the following we want to derive the relation between the compressibility κ and F_0^s analogously. Therefore we consider the system under pressure.

(c) The compressibility is given by $\kappa = -\frac{1}{V}\frac{dV}{dP}$. Show that κ can be expressed by

$$\kappa = \frac{1}{n^2}\frac{dn}{d\mu}$$

Hint: Use $dG = -SdT - Nd\mu + VdP = 0$ and the fact that the chemical potential only depends on N/V, i.e. $\mu(N/V)$.

Due to the change of the particle density the energy is shifted by μ and the particle density is therefore changed by

$$\delta n = \frac{1}{V}\sum_{\mathbf{k}\sigma}\left(n(E_{\mathbf{k}\sigma}(\mu)) - n(E_{\mathbf{k}\sigma}(0))\right).$$

Since the energy itself depends on the particle density, δn results in an additional correction $\delta E_{\mathbf{k}\sigma}$. Thus, the energy reads

$$E_{\mathbf{k}\sigma}(\mu) = E_{\mathbf{k}\sigma}(0) - \mu + \delta E_{\mathbf{k}\sigma}\,.$$

(d) Expand $n(E_{\mathbf{k}\sigma}(\mu))$ for small μ around $\mu = 0$ up to second order.

(e) $\delta E_{\mathbf{k}\sigma}$ is given by the first term in (1). Plug this into your expansion and finally derive

$$\kappa = \frac{1}{n^2}\frac{N_F}{1+F_0^s}\,.$$

4.2 Quasiparticle distribution in equilibrium (5 points)

The Boltzmann equation reads (see lecture)

$$\frac{\partial n_{\mathbf{k}\sigma}}{\partial t} + \mathbf{v}_{\mathbf{k}}\cdot\frac{\partial n_{\mathbf{k}\sigma}}{\partial \mathbf{r}} - \frac{\partial E_{\mathbf{k}\sigma}}{\partial \mathbf{r}}\frac{\partial n^0_{\mathbf{k}\sigma}}{\partial \mathbf{k}} = I\{n_{\mathbf{k}\sigma}\}$$

with the collision integral

$$I\{n_{\mathbf{k}\sigma}\} = -\sum_{\substack{\mathbf{k}_1,\mathbf{k}',\mathbf{k}'_1\\ \sigma_1,\sigma',\sigma'_1}} f_{\substack{\mathbf{k}'\mathbf{k}'_1,\mathbf{k}\mathbf{k}_1\\ \sigma'\sigma'_1,\sigma\sigma_1}}\left[n_{\mathbf{k}\sigma}n_{\mathbf{k}_1\sigma_1}(1-n_{\mathbf{k}'\sigma'})(1-n_{\mathbf{k}'_1\sigma'_1}) - (1-n_{\mathbf{k}\sigma})(1-n_{\mathbf{k}_1\sigma_1})n_{\mathbf{k}'\sigma'}n_{\mathbf{k}'_1\sigma'_1}\right]$$
$$\cdot\,\delta(E_{\mathbf{k}\sigma} + E_{\mathbf{k}_1\sigma_1} - (E_{\mathbf{k}'\sigma'} + E_{\mathbf{k}'_1\sigma'_1}))$$

Show that for the distribution functions

$$n^0_{\mathbf{k}\sigma} = \frac{1}{e^{\frac{E_{\mathbf{k}\sigma}}{k_B T}} \pm 1}$$

('+' for fermions, '−' for bosons) the collision integral vanishes: $I\{n^0_{\mathbf{k}\sigma} = 0\}$. Conclude that $n^0_{\mathbf{k}\sigma}$ is an $\mathbf{r}$- and t-independent solution of the Boltzmann equation and that, therefore, $n^0_{\mathbf{k}\sigma}$ is the equilibrium distribution. Are other equilibrium distributions mathematically possible?

4.3 Sound waves in a Fermi liquid (part 1) (10 points)

Collision-less regime (zero sound): $\hbar\omega_{s0} \gg 1/\tau$

The zero sound dispersion was found in the lecture in the undamped regime

$$s = \frac{u_{s0}}{v_F} > 1$$

Using an analogous ansatz for damped sound waves in the regime $s < 1$

$$n_{\mathbf{k}\sigma}(\mathbf{r}, t) = \delta(E_{\mathbf{k}\sigma} - \epsilon_F)\, a(\hat{\mathbf{k}})\, e^{\mathrm{i}(\mathbf{k}_{s0}\mathbf{r} - \omega_{s0} t)}$$

with a complex frequency ω_{s0} (damping), determine the zero sound velocity and damping rate. The damping in this collisionless regime is called Landau damping.