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Physikalisches Institut
Prof. Dr. H. Kroha, Z. Y. Lai, M. Trujillo Martínez
http://www.th.physik.uni-bonn.de/th/Groups/kroha

Advanced Condensed Matter Theory — SS10

Exercise 1

1.1. Bosonic coherent states (14 points)

The coherent states $|\phi\rangle$ are defined as the eigenstates of the bosonic annihilation operator a.

$$a_i|\phi\rangle = \phi_i|\phi\rangle \qquad , \phi_i \in \mathbb{C}$$

a) Since $|\phi\rangle$ is a state of the Fock space, it can be expanded

$$|\phi\rangle = \sum_{n_1,n_2,\ldots} C_{n_1,n_2,\ldots}|n_1,n_2,\ldots\rangle \tag{1}$$

where $|n_1,n_2,\ldots\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}}\ldots|0\rangle$. Show that $|\phi\rangle$ can be written as

$$|\phi\rangle = \exp\left[\sum_i \phi_i a_i^\dagger\right]|0\rangle$$

Hint: By applying the annihilation operator on (1) one can obtain a recursive relation for the coefficients $C_{n_1,n_2,\ldots}$. Assume $C_{0,0,\ldots} = 1$

b) Show that the action of the creation operator on $|\phi\rangle$ is given by

$$a_i^\dagger|\phi\rangle = \frac{\partial}{\partial\phi_i}|\phi\rangle$$

and

$$\langle\phi|a_i = \frac{\partial}{\partial\bar{\phi}_i}\langle\phi|$$

c) Consider the following operator

$$\hat{E} \equiv \int\prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i\phi_i}|\phi\rangle\langle\phi| \tag{2}$$

where $d\bar{\phi}_i d\phi_i = d(\text{Re }\phi_i)d(\text{Im }\phi_i)$ and show that it commutes a_i

d) Use Schur's lemma[1] to argue that $\hat{E} = \lambda\cdot\mathbb{1}$, $\lambda\in\mathbb{C}$.
Calculate the constant λ from the vacuum expectation value of $\hat{E}$.

[1] If the matrices $D(R)$ are an irreducible representation of a group G ($R \in G$), and if

$$D(R)A = AD(R) \quad \forall R \in G,$$

then $A = \text{constant}\cdot\mathbb{1}$. In other words, if a matrix commutes with all the matrices of an irreducible representation, the matrix must be a multiple of the unit matrix.

From these we can deduce that coherent states form a complete set of states in the Fock space, i. e.

$$\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| = \mathbb{1} \qquad (3)$$

e) Since we are interested in computing physical quantities, we have to calculate the traces of operators corresponding to the physical observables.
Show that by inserting the completeness relation (3) the trace of an operator A is converted to an integral over the coherent state eigenvalues.

$$\text{tr}[A] = \sum_n \langle n|A|n\rangle = \int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} \langle\phi|A|\phi\rangle$$

1.2. Fermionic coherent states - the Grassmann algebra (8 points)

Analogously to the bosonic case, one can define the fermionic coherent states in the same way than for the bosonic case

$$a_i|\eta\rangle = \eta_i|\eta\rangle$$

But the anticommutativity of the fermionic operators $[a_i, a_j^\dagger]_+ = \delta_{ij}$ implies that

$$\eta_i\eta_j = -\eta_j\eta_i \qquad \text{for} \quad i \neq j \qquad \Rightarrow \eta_i^2 \;=\; 0$$

Clearly these objects can not be ordinary numbers. They are so called *Grassmann variables.* The Grassmann algebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ together with an anticommutative and associative product. In addition a conjugation operation is defined by $\overline{(\eta)} = \bar{\eta}$ and $\bar{\bar{\eta}} = \eta$, such that $(\eta a)^\dagger = a^\dagger \bar{\eta}$. Functions of Grassmann variables exhibit an amazingly simple property: Since any power of η higher than one vanishes, a function of, e.g., two Grassmann variables can be written as (Taylor series):

$$f(\eta_i, \eta_j) = c_0 + c_1\eta_i + c_1\eta_j + c_2\eta_i\eta_j \qquad (4)$$

where the constants $c_i \in \mathbb{C}$.

The differentiation of Grassmann variables is defined in the following way.

$$\frac{\partial}{\partial\eta_i}\eta_j = \delta_{ij}$$

and the integration over Grassmann variables is defined by

$$\int d\eta_i = 0 \qquad \int d\eta_i\eta_i = 1.$$

Note that one cannot ignore the order of Grassmann variables:

$$\frac{\partial}{\partial\eta_i}\eta_j\eta_i = -\frac{\partial}{\partial\eta_i}\eta_i\eta_j = -\eta_j$$

$$\int d\eta_i\eta_j\eta_i = -\int d\eta_i\eta_i\eta_j = -\eta_j$$

a) The last lines indicate, that differentiation and integration are effectively the same for Grassmann variables. Show that

$$\frac{\partial}{\partial \eta_i} f(\eta_i, \eta_j) = \int d\eta_i f(\eta_i, \eta_j) = c_1 + c_2 \eta_j$$

holds. Calculate the integral

$$\int d\eta f(\eta + \xi) = \int d\eta f(\eta) \qquad \eta, \xi \in \mathcal{A} \tag{5}$$

b) To be consistent with the anticommutation relations the Grassmann variables has to anticommute with the fermion operators $[\eta_i, a_j]_+ = 0$. Show that

$$|\eta\rangle = \exp\left[\sum_i -\eta_i a_i^\dagger\right] |0\rangle.$$

1.3. Gaussian Integrals (8+5 points)

When evaluating functional integrals one is often faced with calculating Gaussian integrals of either complex numbers or Grassmann variables. Therefore, some important basic relations shall be derived in the following. We use the shorthand notations $(x, y) = \sum_{i=1}^n \bar{x}_i y_i$ and $(x, Ay) = \sum_{i,j=1}^n \bar{x}_i A_{ij} y_j$, respectively.

a) Let $A \in \mathbb{R}^{n\times n}$ be a positive-definite diagonal matrix. Show

$$\int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{dx_i}{\sqrt{\pi}} \, e^{-(x,Ax)} = \frac{1}{\sqrt{\det A}}.$$

b) Let $A \in \mathbb{C}^{n\times n}$ be a positive-definite Hermitian matrix and $y \in \mathbb{C}^n$. Show

$$\int \prod_{i=1}^{n} \frac{d\bar{z}_i dz_i}{\pi} \, e^{-(z,Az)+(y,z)+(z,y)} = \frac{e^{(y,A^{-1}y)}}{\det A}.$$

Hint: Use an unitary transformation $z' = Uz$, $y' = U^\dagger y$ to diagonalize A. Rewrite the integral with respect to the real and imaginary part of z, complete the square and use the result of a).

c) For $a \in \mathbb{C}$ and $\eta \in \mathcal{A}$ calculate the Grassmann integral

$$\int d\bar{\eta} d\eta \, e^{-\bar{\eta} a \eta} = a.$$

Hint: Expand the exponential in a power series to show that only one single term contributes to the integral. Pay attention to the order of the integration variables.

d) Now consider the case of a diagonal matrix $A \in \mathbb{C}^{n\times n}$. Show

$$\int \prod_{i=1}^{n} d\bar{\eta}_i d\eta_i \, e^{-(\eta,A\eta)} = \det A.$$

Hint: Again only one term of the power series can contribute. Which one? Reorder the integration variables to obtain the result.

e) Finally, deduce for a general $A \in \mathbb{C}^{n\times n}$ and $\xi \in \mathcal{A}$

$$\int \prod_{i=1}^{n} d\bar{\eta}_i d\eta_i \, e^{-(\eta,A\eta)+(\xi,\eta)+(\eta,\xi)} = \det A \cdot e^{(\xi,A^{-1}\xi)}.$$

Hint: Complete the square and use the linear transformation from (5). For reordering the only contributing term the Leibniz formula for determinants may be useful: $\det A = \sum_{P\in\mathcal{S}_n} \mathrm{sgn}(P)\, A_{1P(1)} \cdot \ldots \cdot A_{nP(n)}$, where the sum runs over all permutations of $\{1, .., n\}$.