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Advanced Condensed Matter Theory — SS10

Exercise 2

1.1. Spectral Properties of Matsubara Green's Functions (6 points)

In this exercise we study the spectral properties of the Matsubara's Green's Functions (MGF).

a) From the definition of the spectral function $A(\omega)$ from the lecture prove the following sum rule

$$\int \frac{d\omega'}{2\pi} A(\omega') = 1 \tag{1}$$

b) Using the result (1), show that

$$G(\omega) \to \frac{1}{\omega} \tag{2}$$

in the high-frequency limit $\omega \to \infty$

1.2. The Spectral Weight: A Physical Picture, Part I (15 points)

In the lecture Advanced Theoretical Condensed Matter Physics part 1 you learned that the Fermi liquid theory relies on the assumption that the excitation created by adding a particle to the system can be described by a free particle, a quasiparticle, with a long lifetime. The function that measures precisely the density of states for adding particles is the retarded Green's function G^R. In other words, if the retarded Green's function of the interacting system turns out to be similar to that of free particles, the quasiparticle picture will have a real physical meaning. We will explore this in more detail in this exercise and one on the next problem sheet.

A useful way of deriving Green's functions is to compute the *equations of motion*. We note that in what follows we will work with zero temperatures; however the steps in the derivation follows completely analogously for the $T \neq 0$ regime.

a) Derive the equation of motion for the retarded Green's function for an arbitrary Hamiltonian H.

b) From your results above write down the equation of motion for a system of free (noninteracting) electrons with the Hamiltonian H_0

$$H_0 = \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{3}$$

Subsequently solve for $G^{R0}_{\mathbf{k}\sigma}(\omega)$ (the superscript 0 denotes the noninteracting Green's function).

c) Now consider a system of mutually interacting electrons with the Hamiltonian

$$H = H_0 + V \tag{4}$$

where the interaction term V is given by

$$V = \frac{1}{2} \sum_{\substack{\mathbf{kpq} \\ \sigma\sigma'}} v_{\mathbf{kp}}(\mathbf{q}) c^{\dagger}_{\mathbf{k+q}\sigma} c^{\dagger}_{\mathbf{p-q}\sigma'} c_{\mathbf{p}\sigma'} c_{\mathbf{k}\sigma} \tag{5}$$

What do you obtain for the equation of motion? Is it possible to solve for the *interacting* Green's function as in the free case? Show that the equation of motion depends in this case on the higher-order Green's function

$$\Gamma^{\sigma\sigma'}_{\mathbf{pkq}} = -i\theta(\tau - \tau') \left\langle \left[\left(c^{\dagger}_{\mathbf{p+q}\sigma'} c^{\dagger}_{\mathbf{k+q}\sigma} c_{\mathbf{p}\sigma'} \right)(\tau), c^{\dagger}_{\mathbf{k}\sigma}(\tau') \right] \right\rangle \tag{6}$$

d) In order to be able to complete the derivation of the interacting Green's function, we need to close our equation of motion. One *ad hoc* way to do this is to simply replace the higher order Green's function (6) with the following quantity

$$\Gamma^{\sigma\sigma'}_{\mathbf{pkq}}(\omega) \to \Sigma_{\sigma}(\mathbf{k}, \omega) G_{\mathbf{k}\sigma}(\omega) \tag{7}$$

Substituting (7) into your equation of motion, solve for $G^{R}_{\mathbf{k}\sigma}(\omega)$. Show that you obtain a Green's function of the form

$$G^{R}_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\varepsilon_{\mathbf{k}} - \mu) + \Sigma_{\sigma}(\mathbf{k}, \omega)} \tag{8}$$

$$= \frac{1}{\omega - \xi_{\mathbf{k}} + \Sigma_{\sigma}(\mathbf{k}, \omega)} \tag{9}$$

The quantity $\Sigma_{\sigma}(\mathbf{k}, \omega)$ is called the *self-energy.* It is an important quantity that characterizes the interaction between the particles in a particular system. It can be more rigorously derived via Dyson's equation, but here it suffices to make this simple substitution. We note also that the self-energy is a complex function.

e) At low energies only the processes around the Fermi edge ($\omega = 0$, $k = k_F$) are physically relevant. Assume that the non-interacting states are free particles

$$\varepsilon_k = \frac{k^2}{2m} \tag{10}$$

and furthermore assume that one can expand the denominator in the vicinity of ($\omega = 0$, $k = k_F$). Perform the expansion of (8), and show that one can write the Green's function as

$$G^{R}(\omega) \approx \frac{Z}{\omega - \tilde{\xi}_{\mathbf{k}} + \frac{i}{2\tilde{\tau}_{\mathbf{k}}(\omega)}} \tag{11}$$

Express the quantities Z, $\tilde{\xi}_k$ and $\tilde{\tau}_k(\omega)$ in terms of the real and imaginary of the self-energy and it derivatives thereof. (Hint: keep the imaginary part of the self-energy undifferentiated, and assume that $\xi_{k_F} + \mathrm{Re}\Sigma(k_F, 0) = 0$, i.e., the real part of the energy vanishes.)

f) Assume that the imaginary part of the self-energy is small. Write down the spectral function using the definition of the spectral function $A(k, \omega) = -2\mathrm{Im}G^{R}(k, \omega)$ and the result (11) above.