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Advanced Condensed Matter Theory — SS10

Exercise 4

Please return your solutions during the lecture on June 2, 2010
to be discuss on June 3, 2010

<u>1.1. Screening in an electrongas I: Lindhard function</u> (15 points)

In 1^{st} order perturbation theory we obtained the self-energy

$$\Sigma_{k\sigma} \overset{T\to 0}{=} \frac{e_0^2}{2\pi} k_{\mathrm{F}} \left(2 + \frac{k_{\mathrm{F}}^2 - k^2}{k k_{\mathrm{F}}} \ln \left| \frac{k_{\mathrm{F}} + k}{k_{\mathrm{F}} - k} \right| \right).$$

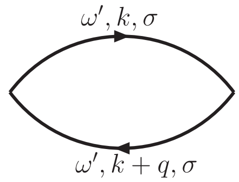
for a gas of free electrons interacting through Coulomb force.

a) The group velocity at the fermi surface is defined by

$$v_{\mathrm{F}} \equiv \left. \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \right|_{\mathbf{k}=k_{\mathrm{F}}}$$

where $\epsilon(\mathbf{k}) = k^2/2m + \Sigma_{k\sigma}$ is the energy-momentum relation. Show that group velocity diverges. Discuss the physical origin of this behavior. Why is it not correct?

b) The lowest order polarization contribution is given by the bubble

$\Pi(\mathbf{q}, \omega = 0) =$ 

,

which corresponds to the Fourier transform $\chi(\mathbf{q}, \omega = 0)$ of the respond function

$$\chi(\mathbf{r} - \mathbf{r}', t - t') = -i\theta(t - t') \left\langle [n(\mathbf{r}, t), n(\mathbf{r}', t')] \right\rangle.$$

Show that it yields

$$\begin{aligned} \Pi(\mathbf{q}, \omega = 0) &= 2 \sum_{\mathbf{k}} \frac{f(\epsilon(\mathbf{k} + \mathbf{q}) - \mu) - f(\epsilon(\mathbf{k}) - \mu)}{\epsilon(\mathbf{k} + \mathbf{q}) - \epsilon(\mathbf{k})} \\ &\overset{T\to 0}{=} 2 \int \frac{d^d k}{(2\pi)^d} \frac{\Theta(\mu - \epsilon(\mathbf{k} + \mathbf{q}/2)) - \Theta(\mu - \epsilon(\mathbf{k} - \mathbf{q}/2))}{\epsilon(\mathbf{k} + \mathbf{q}/2) - \epsilon(\mathbf{k} - \mathbf{q}/2)}. \end{aligned}$$

c) The main contribution to $\Pi(\mathbf{q}, \omega = 0)$ arises from small momentum $\mathbf{q}$. Assume $\epsilon(\mathbf{k}) = k^2/2m$ and neglect all terms of order $\mathcal{O}(q^2)$ in the denominator of the integrand and show

$$\Pi(\mathbf{q}, \omega = 0) \approx \frac{2m}{\pi q} \int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \int_{k_+}^{k_-} \frac{dk_\parallel}{k_\parallel} \qquad \text{with: } k_\pm = \sqrt{k_\mathrm{F}^2 - k_\perp^2} \pm \frac{q}{2}.$$

Hint: Use a coordinate system such that $\mathbf{k} = (\mathbf{k}_\perp, k_\parallel)$, where $k_\parallel$ denotes the component of $\mathbf{k}$ pointing in the direction of $\mathbf{q}$.

d) Finally, derive the *Lindhard function* in $d = 1, 3$ dimensions

$$\Pi(\mathbf{q}, \omega = 0) = \begin{cases} \frac{m}{\pi k_\mathrm{F}} \frac{1}{q/2k_\mathrm{F}} \ln \left| \frac{1-q/2k_\mathrm{F}}{1+q/2k_\mathrm{F}} \right| & , \quad d = 1 \\ -\frac{m\, k_\mathrm{F}}{2\pi^2} \left(1 + \frac{1-(q/2k_\mathrm{F})^2}{q/k_\mathrm{F}} \ln \left| \frac{1+q/2k_\mathrm{F}}{1-q/2k_\mathrm{F}} \right| \right) & , \quad d = 3 \end{cases},$$

which is plotted below.

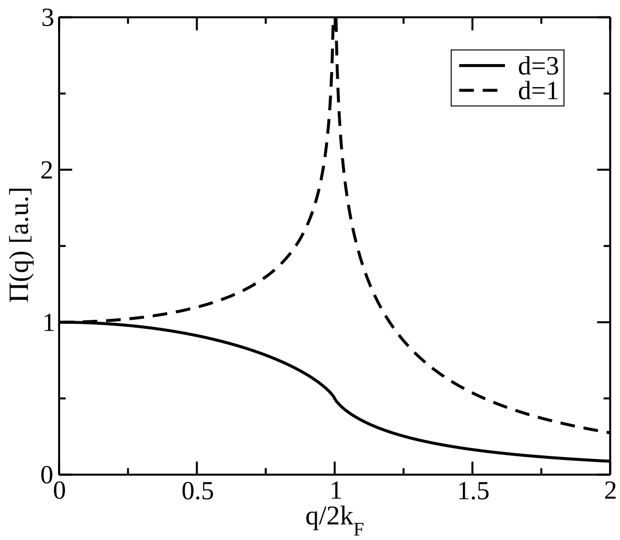


Figure 1: The Lindhard function in $d = 1, 3$ dimensions.

1.2 Screening in an electrongas II: Thomas-Fermi approximation and Friedel oscillations (10 points)

a) Consider a point charge located at the origin. This charge, with its polarization, produces an electric potential ϕ_{eff} in its surrounding. This change the electron density $n(x)$ and the fermi momentum $p_F(x)$ at the point x, such that

$$\epsilon_F = \frac{p_F^2}{2m} = e_0 \phi_{eff}(x) + \frac{p_F^2(x)}{2m}.$$

The background positive ion density is given by $n_0 = 2 \times \frac{4\pi}{3}\frac{p_F^3}{(2\pi)^3}$ and the net charge density is $e_0(n(x) - n_0)$. Show that

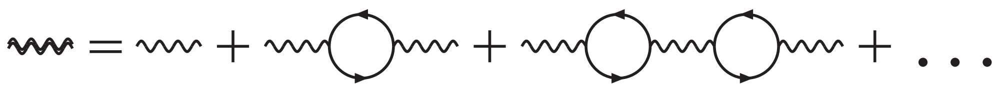
$$e_0(n(x) - n_0) \approx -\frac{m e_0^2 p_F}{\pi^2}\phi_{eff}$$

for small ϕ_{eff}. Insert this result in the Poisson equation for the system in consideration and argue that

$$\phi_{eff}(\mathbf{r}) = 4\pi e_0^2 \,\frac{\mathrm{e}^{-q_{\mathrm{TF}} r}}{4\pi r}.$$

Interpret the result. Which implications does it have for the group velocity at fermi surface?

b) In exercise 1.1 we computed the lowest order contribution to the polarization in a gas of free electron interaction through Coulomb force. Now, in order to obtain a physically meaningful result we must take into account the screening of the Coulomb interaction by the electron gas. For that purpose, we will resum the leading contributions (*random phase approximation*) to get an effective interaction



where the thin wavy lines correspond to the bare Coulomb interaction

$$\phi_{el} = \frac{e_0^2}{r},$$

the bubbles to the polarization diagram $\Pi(\mathbf{q}, \omega = 0)$ computed above and the thick wavy line to a effective screened Coulomb potential ϕ_{eff}.

Now we restrict our analysis to the $d = 3$. Show that

$$\phi_{eff}(\mathbf{q}) = \frac{4\pi e_0^2}{\mathbf{q}^2 - 4\pi e_0^2 \Pi(\mathbf{q}, \omega = 0)}.$$

c) Obtain a rough estimate for the asymptotic behavior of ϕ_{eff} for large distances ($\mathbf{q} \to 0$), i. e. show that

$$\phi_{eff}(\mathbf{r}) \overset{r\to\infty}{\approx} 4\pi e_0^2 \,\frac{\mathrm{e}^{-q_{\mathrm{TF}} r}}{4\pi r}.$$

Compare this with the result obtained in a). Give q_{TF} in terms of e_0 and k_F.

d) Even though the result of a) and c) is neat, unfortunately, it is not completely correct. The result above works only in the average sense. The sharp cutoff near the fermi surface makes its fourier transform oscillate. This causes the induced change of the charge density

$$\Delta n(\mathbf{r}) = -e_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\mathbf{r}} \phi_{eff}(\mathbf{q}) \Pi(\mathbf{q}, \omega = 0)$$

to oscillate in space. This oscillations are called *Friedel oscillations.*

A careful evaluation of the Fourier Transform shows that the correct result is

$$\Delta n(\mathbf{r}, t) \overset{r\to\infty}{\approx} -\frac{4e_0}{\pi}\,\frac{q_{\mathrm{TF}}^2/k_{\mathrm{F}}^2}{(8 + q_{\mathrm{TF}}^2/k_{\mathrm{F}}^2)^2}\,\frac{\cos(2k_{\mathrm{F}} r)}{r^3}.$$

To obtain the oscillations one has to take into account the singularity of $\Pi(x)$, $\Pi'(x) \approx -\delta(x-2)$. Using the asymptotics of $\Pi(x)$ we approximate

$$\Delta n(\mathbf{r},t) \;=\; -\int \frac{d^3q}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{r}}\, \frac{4\pi e_0^2 \Pi(\mathbf{q},\omega=0)}{\mathbf{q}^2 - 4\pi e_0^2 \Pi(\mathbf{q},\omega=0)} \simeq -\int \frac{d^3q}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{r}}\, \frac{4\pi e_0^2 \Pi(\mathbf{q},\omega=0)}{\mathbf{q}^2 + q_{TF}^2}$$

Use integration by parts and approximate

$$F(x) \;=\; \int_0^x dy\, \frac{y}{y^2 + r^2 q_{\mathrm{TF}}^2} \sin(y) \quad \Rightarrow \quad F(2k_{\mathrm{F}}r) \;\simeq\; -\frac{\cos(2k_{\mathrm{F}}r)}{r}$$

to obtain the Friedel oscillations of the density modulation.