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Advanced Condensed Matter Theory — SS10

Exercise 8

8.1. Temperature Dependence of the Spontaneous Magnetization

The spontaneous magnetization, in the *spin-wave approximation*, of a Heisenberg ferromagnet at low temperatures is given by:

$$\frac{M_0 - M_S(T)}{M_0} = \frac{1}{NS}\sum_{\mathbf{q}} \frac{1}{\exp[\beta\hbar\omega(\mathbf{q})] - 1} \tag{1}$$

where $M_0 = g_J \mu_B S \frac{N}{V}$ is the saturation magnetization, and $\hbar\omega(\mathbf{q}) = 2S\hbar^2(J_0 - J(\mathbf{q}))$ is the magnon energy. Prove Bloch's $T^{3/2}$ law:

$$\frac{M_0 - M_S(T)}{M_0} \sim T^{3/2} \tag{2}$$

Hints:

1. Transform the summation over $\mathbf{q}$ into an integral from 0 to infinity.
2. Use the approximation $J_0 - J(\mathbf{q}) \approx \frac{D}{2S\hbar^2} q^2$

8.2. The Antiferromagnetic Ground State

From the lecture we have seen that the *spin-wave approximation* has proved itself to be successful in the case of ferromagnets. The question arises as to whether the same can be applied to an antiferromagnetic ground state. To determine this we apply the spin-wave formalism, as a first approximate analysis, in this exercise to an antiferromagnet. Assuming the sub-lattice model $(\cdots ABABAB \cdots)$ we can write our Heisenberg Hamiltonian with an applied external field B_0 as

$$H = -\sum_{i,j}^{n.n.} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - g_J \frac{\mu}{\hbar}(B_0 + B_A)\sum_i^A S_i^z - g_J \frac{\mu}{\hbar}(B_0 - B_A)\sum_j^B S_j^z \tag{3}$$

where $n.n.$ denotes nearest neighbor summation, summation over A and B denotes summation over the respective sublattices, and B_A is an "anisotropy" field, which we here take simply as a constant.

a) Using the *Holstein-Primakoff* transformation (in our notation the a_i and $a_i^\dagger$ operators act on the A sublattice, and the b_i and $b_i^\dagger$ operators act on the B sublattice) in the *harmonic approximation*

$$\frac{1}{\hbar}S_i^+ = \sqrt{2S}a_i \quad ; \quad \frac{1}{\hbar}S_i^- = \sqrt{2S}a_i^\dagger \quad ; \quad \frac{1}{\hbar}S_i^z = S - a_i^\dagger a_i \tag{4}$$

$$\frac{1}{\hbar}S_j^+ = \sqrt{2S}b_j^\dagger \quad ; \quad \frac{1}{\hbar}S_j^- = \sqrt{2S}b_j \quad ; \quad \frac{1}{\hbar}S_j^z = -S + b_j^\dagger b_j \tag{5}$$

write (3) in the form

$$H = E_a + b_A \sum_i^A a_i^\dagger a_i + b_B \sum_j^B b_j^\dagger b_j - S\hbar^2 \sum_i^A \sum_j^B J_{ij}(a_i b_j + a_i^\dagger b_j^\dagger) \tag{6}$$

where here S is the total spin in the ground state of the respective lattices. What are the constants b_A, b_B and E_a? We note here that E_a is the energy of the fully ordered Neel (antiferromagnetic) state. (*Hint*: Remember the first sum in (3) is over the nearest neighbours only. Denote the number of nearest neigbours z, and the coupling strength between nearest neighbour spins as J_1. Give plausible arguments why

$$\sum_{i,j}^{n.n.} J_{ij} a_i^\dagger a_i = zJ_1 \sum_i a_i^\dagger a_i$$

$$\sum_{i,j}^{n.n.} J_{ij} = NzJ_1$$

Note: The transformations (4) and (5) are the Holstein-Primakoff transformations in the *harmonic approximation*. This means that it is derived by simply taking the "usual" definition of the transformations, as given below:

$$\frac{1}{\hbar}S_i^z = S - \hat{n}_i \tag{7}$$

$$\frac{1}{\hbar}S_i^+ = \sqrt{2S}\phi(\hat{n}_i)a_i \tag{8}$$

$$\frac{1}{\hbar}S_i^- = \sqrt{2S}a_i^\dagger \phi(\hat{n}_i) \tag{9}$$

where $\phi(\hat{n}_i) = \sqrt{1 - \frac{\hat{n}_i}{2S}}$, and taking the zeroth order term in the expansion of the square root. This yields us then simple approximations for the *exact* expressions (7) - (9).

b) Fourier transform (6) using the *Ansatz*

$$a_i = \sqrt{\frac{2}{N}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}_i} a_{\mathbf{q}}$$

$$b_j = \sqrt{\frac{2}{N}} \sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}_j} b_{\mathbf{q}}$$

$$J_{ij} \equiv \frac{2}{N} \sum_{\mathbf{q}} J(\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{R}_i - \mathbf{R}_j)}$$

$$\delta_{\mathbf{q},\mathbf{q}'} = \frac{2}{N} \sum_i^A e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{R}_i} = \frac{2}{N} \sum_j^B e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{R}_j}$$

$$J(\mathbf{k}) = J(-\mathbf{k})$$

Show that (6) in wavevector space reads

$$H = E_a + b_A \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + b_B \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} + \sum_{\mathbf{q}} c(\mathbf{q})(a_{\mathbf{q}} b_{\mathbf{q}} + a_{\mathbf{q}}^\dagger b_{\mathbf{q}}^\dagger) \tag{10}$$

where $c(\mathbf{q}) = -J(\mathbf{q})\hbar^2 S$.

c) Define the following operator transformations:

$$\begin{aligned} a_{\mathbf{q}} &= c_1 \alpha_{\mathbf{q}} + c_2 \beta^\dagger_{\mathbf{q}} \\ b_{\mathbf{q}} &= d_1 \alpha^\dagger_{\mathbf{q}} + d_2 \beta_{\mathbf{q}} \end{aligned} \tag{11}$$

with $c_1 = d_2 = \cosh \eta_{\mathbf{q}}$ and $c_2 = d_1 = \sinh \eta_{\mathbf{q}}$, and $\eta_{\mathbf{q}}$ defined via $\tanh 2\eta_{\mathbf{q}} = -\frac{2c(\mathbf{q})}{b_A + b_B}$. Show that, when you substitute the transformation (11) into the result you obtained in exercise (b), you obtain a transformed Hamiltonian in terms of the $\alpha_{\mathbf{q}}$ and $\beta_{\mathbf{q}}$ operators:

$$H = \tilde{E}_a + \sum_{\mathbf{q}} \hbar\omega_\alpha(\mathbf{q}) \alpha^\dagger_{\mathbf{q}} \alpha_{\mathbf{q}} + \sum_{\mathbf{q}} \hbar\omega_\beta(\mathbf{q}) \beta^\dagger_{\mathbf{q}} \beta_{\mathbf{q}} \tag{12}$$

with a "new" energy:

$$\tilde{E}_a = E_a - \frac{N}{4}(b_A + b_B) + \frac{1}{2} \sum_{\mathbf{q}} \sqrt{(b_A + b_B)^2 - 4c^2(\mathbf{q})}$$

Hint: You may find the following identities useful:

$$\begin{aligned} \sinh \eta_{\mathbf{q}} \cdot \cosh \eta_{\mathbf{q}} &= \frac{1}{2} \sinh 2\eta_{\mathbf{q}} \\ \cosh^2 \eta_{\mathbf{q}} &= \frac{1}{2}(\cosh 2\eta_{\mathbf{q}} + 1) \\ \sinh^2 \eta_{\mathbf{q}} &= \frac{1}{2}(\cosh 2\eta_{\mathbf{q}} - 1) \end{aligned}$$

This is an extremely interesting result which shows that the ground state energy $\tilde{E}_a$ is smaller than the energy E_a of the fully ordered Neel state, in which the sublattice magnetizations are oriented exactly antiparallel to each other. This means that the Neel state is not the ground state of the antiferromagnet, but the ground state should rather include some spin disorder.