Quantum field-theoretical methods in transport theory of metals*

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The authors review the Keldysh method of obtaining kinetic equations for normal and superconducting metals. The use of the method is illustrated by examples involving electron-impurity, electron-phonon, and electron-electron scattering, both within and beyond the quasiclassical approximation.

CONTENTS

*Based on part of a thesis submitted by J. Rammer for the degree of Lic. Scient. at the University of Copenhagen (November 1984).

I. INTRODUCTION

A. A brief history

The purpose of this tutorial review is to give an account of the use of real-time Green's-function methods in the transport theory of metals and to discuss the relevance of these methods for some problems of current interest. In so doing, we shall follow the formulation due to Keldysh (1964) because of its simplicity, although alternative formulations also exist in the literature. We shall deal with different transport phenomena by deriving the appropriate kinetic equations, starting from the Dyson equation in its general form. This method has the advantage of greater generality than linear-response theory by allowing nonlinear situations to be considered, and facilitates comparison with the conventional Boltzmann equation approach. In some cases, however, linear-response theory is more convenient to use, as we shall discuss below in specific cases.

The introduction of quantum field-theoretical methods in nonequilibrium statistical mechanics, in the form we adopt here, dates back to Martin and Schwinger (1959) and Schwinger (1961). Significant further developments were due to Kadanoff and Baym (1962). Parallel to this development in the USA was the work in the USSR by Konstantinov and Perel (1960), Dzyaloshinski (1962), Keldysh (1964), Abrikosov, Gorkov, and Dzyaloshinski (1965), Eliashberg (1971), and others.

Our main purpose in writing this review is to give a self-contained presentation of nonequilibrium theory which may be understood with the knowledge of standard

equilibrium Green's-function theory. For this purpose, the technique employed by Keldysh (1964) is particularly convenient, since the level of complication is reduced to that of standard quantum field theory for systems in equilibrium. To illustrate how the method works, we have chosen a few representative examples involving transport phenomena in normal and superconducting metals.

B. Survey of previous work

We summarize below the previous work in which techniques identical or similar to those introduced by Keldysh (1964) have been employed. For brevity, we refer to these techniques as the Keldysh method, although they are the results of related or independent developments involving the work of a number of authors, as already indicated.[1] The purpose of this survey of previous work is to give the reader a sense of the scope and potential of the Keldysh method. In our survey we thus deliberately focus on the formalism employed, which inherently puts a bias on our selection of references. Even so, we cannot hope to provide the reader with an exhaustive list, but only to include the important areas in which the Keldysh method has been utilized. We shall group the references by area of application.

1. Transport properties of normal metals

The transport equation for electrons interacting with phonons was discussed by Konstantinov and Perel (1960) by a diagrammatic technique. In his 1964 paper, Keldysh applied his technique to the derivation of the kinetic equation for electrons interacting with phonons in equilibrium. The early paper by Prange and Kadanoff (1964) has had a lasting influence on the transport theory of normal metals. Using the formulation due to Kadanoff and Baym (1962), these authors derived the transport equations beyond perturbation theory for the case of interacting electrons and phonons, a subject that we treat in detail in Sec. III. Fleurov and Kozlov (1978) employed the Keldysh formulation for the derivation of the transport equation for electrons in a metal. Derivations of the electron-phonon transport equation using the Kadanoff-Baym method have also been given by Hänsch and Mahan (1983a) and by Jauho (1983). An application of the Kadanoff-Baym formulation was used to treat spin resonance and diffusion in dilute magnetic alloys by Langreth and Wilkins (1972). A generalization of the Kadanoff-Baym formulation has been given by Langreth (1976), who also considered (1966) the derivation of the transport equation in a magnetic field. Interaction effects in weakly disordered metals were treated in a series of important papers by Altshuler and Aronov (1978a,1979a,1979b) with the Keldysh method, and also with the use of temperature Green's functions (Altshuler and Aronov, 1979c). We mention, in addition, work by Jauho and Wilkins (1983), who employed Langreth's formulation to investigate nonlinear effects of an electric field on the scattering of electrons from resonant impurities.

2. Nonequilibrium superconductivity

The Keldysh technique has been used extensively in recent years to derive kinetic equations for superconductors. A derivation of the quasiparticle kinetic equation by the Keldysh method has been given by Aronov and Gurevich (1974) and Aronov *et al.* (1981). The quasiparticle kinetic equation was also derived by Tremblay *et al.* (1980) using the Langreth formulation. Starting with the work of de Gennes (1964,1966), the use of quasiclassical methods was shown to be a powerful tool for studying equilibrium properties under conditions when the simple quasiparticle picture could not be applied. The use of the quasiclassical method to describe transport phenomena in superconductors was brought to full maturity through the work of Eilenberger (1968), Larkin and Ovchinnikov (1968,1975,1977), Eliashberg (1971), and Schmid and Schön (1975a). Some of these authors used temperature Green's functions as a starting point instead of real-time Green's functions. It turns out, however, that the use of the Keldysh technique, based on real-time Green's functions, allows the nonequilibrium theory to be formulated rather elegantly. This is demonstrated in the review article by Schmid (1981), who discussed the application of the kinetic equations obtained by the Keldysh technique to various nonequilibrium phenomena in dirty superconductors. Further details regarding the use of the method are given by Hu (1980). A different, but equivalent, set of kinetic equations was obtained by Shelankov (1980). The kinetic equations were solved under various conditions by Schmid and Schön (1975a) at temperatures close to the transition temperature. Approximate collision operators appropriate to lower temperatures were discussed by Eckern and Schön (1978). Solutions to the kinetic equation describing charge imbalance were obtained at arbitrary temperatures by Beyer Nielsen *et al.* (1982). The case of thermal conductivity was treated by Beyer Nielsen and Smith (1982,1985), who also considered strong coupling effects on both charge imbalance and thermal conductivity. The transition between the clean and dirty limits in the presence of supercurrents was discussed by Schön (1981) and by Beyer Nielsen *et al.* (1982,1986). Further applications of the kinetic equations include the study of collective modes (Schmid and Schön, 1975b), frequency-dependent conductivity (Ovchinnikov, Schmid, and Schön, 1981; Beyer Nielsen, Smith, and Yang, 1984), and gap relaxation (see the review by Schmid, 1981). An attempt to generalize the quasiclassical method by using single-particle Green's functions integrated over the

[1] In particular, we emphasize the significance of the paper by Schwinger (1961), in which the so-called "closed time path Green's function" was first introduced.

chemical potential instead of the magnitude of the relative momentum was made by Eckern and Schmid (1981).

3. Fermi liquids

The kinetic equation of a normal Fermi liquid has been derived with the use of Green's-function methods by Abrikosov, Gorkov, and Dzyaloshinski (1965), who used temperature Green's functions requiring analytic continuation from imaginary frequencies. The kinetic equation for a slightly nonideal Fermi gas was derived with the Keldysh technique by Lifshitz and Pitaevskii (1981). The review by Serene and Rainer (1983) on the quasiclassical approach to normal and superfluid ^{3}He contains a derivation by the Keldysh method of the kinetic equation appropriate to ^{3}He. The kinetic equation for the A phase of ^{3}He has also been discussed by Kopnin (1978) and by Eckern (1981,1982) on the basis of the Keldysh technique. When pair breaking may be neglected, the kinetic equation for superfluid ^{3}He reduces to the Landau-Boltzmann transport equation, which has been used extensively to discuss nonequilibrium phenomena in superfluid Fermi liquids (see, for example, the review by Pethick and Smith, 1977). When one considers low-frequency properties of superfluid ^{3}He, one does not need to use kinetic equations of more general validity than the Landau-Boltzmann equation, due to the smallness of the pair-breaking effects. However, when external frequencies are comparable to the magnitude of the gap, as may be the case for zero sound propagation, it is necessary to use more general quantum-kinetic equations. The work in this area has been reviewed by Wölfle (1978) and by Serene and Rainer (1983).

4. Other applications

The review by Langreth (1976) discusses applications of the Kadanoff-Baym method to photoemission and related processes, in which it is necessary to go beyond linear response in the treatment of the photon field (Chang and Langreth, 1972,1973; Yue and Doniach, 1973; McMullen and Bergersen, 1972; Caroli *et al.*, 1973). Blandin *et al.* (1976) have used the Keldysh formalism to treat problems in surface physics, when a nonperturbative treatment of time-dependent potentials is essential. The method was used by Caroli *et al.* (1971) to discuss the effect of inelasticity on the tunneling current between normal metals.

The Keldysh technique was used by Artemenko and Volkov (1981a,1981b) and by Eckern (1986) to derive kinetic equations for quasi-one-dimensional conductors with a charge-density wave resulting from the Peierls instability. The self-consistent method used by these authors is closely related to that used in superconductivity theory and requires for its validity that the fluctuations be negligible.

Altshuler and Aronov (1978b) employed the Keldysh method to study electron-electron collisions in a weakly ionized plasma. Similar methods have also been applied to relativistic plasmas (Bezzerides and DuBois, 1972), nuclear collisions (Danielewicz, 1984), and quark-gluon plasmas (Heinz, 1983). Other applications to nonequilibrium problems include derivation of Langevin equations by Schmid (1982) for a particle in a dissipative environment, and by Zhou *et al.* (1980) in the context of critical dynamics. A review of functional methods employing the Keldysh technique has recently been published by Chou *et al.* (1985). We also mention the work of Korenman (1966), who applied the method introduced by Schwinger (1961) to a model of a gas laser.

C. Summary of content

The outline of this review is as follows. In Sec. II we introduce the Green's-function theory of nonequilibrium statistical mechanics in a form due to Langreth (1976) and describe the Keldysh formulation. We derive the appropriate Feynman rules and show how the kinetic equation is obtained from the Dyson equations. Section III discusses the important quasiclassical approximation and the connection between the excitation and particle representations. As an example of the use of the quasiclassical approximation, we derive the electron-phonon renormalization of the ac optical conductivity in the limit of high frequencies, when collisions may be neglected. When particle-hole asymmetry is important, as in the case of thermoelectric effects, one must abandon the quasiclassical scheme. As an example, we consider the electron-phonon renormalization of the Nernst-Ettingshausen effect in high magnetic fields. In addition, we indicate how the effects of a periodic potential may be included in the kinetic equation. For the derivation we refer the reader to the thesis by Rammer (1984).

In Sec. IV we discuss some recent applications of the method to transport problems in normal metals. We consider localization and interaction effects arising from electron-impurity and electron-electron scattering, which may be treated within the framework of kinetic equations, although it is often simpler to use linear-response theory. A recent review of the effects of localization and interaction in disordered electronic systems has been given by Lee and Ramakrishnan (1985).

In Sec. V we obtain kinetic equations for superconductors within the quasiclassical limit and discuss the dirty limit when impurity scattering dominates. The technique is illustrated by studying the charge imbalance generated by thermal gradients. In Sec. VI we summarize our discussion of the Keldysh method in relation to linear-response theory and the conventional Boltzmann equation approach.

II. NONEQUILIBRIUM STATISTICAL MECHANICS

A. Green's functions and perturbation theory

Let us consider a physical system represented by the time-independent Hamiltonian

$$H = H_0 + H^i \,, \tag{2.1}$$

where H_0 represents free particles and H^i the interaction between the particles.

In thermodynamic equilibrium, the state of the system is described by the statistical operator ρ given by

$$\rho(H) = (\mathrm{Tr} e^{-\beta H})^{-1} e^{-\beta H}, \quad \beta = T^{-1} \tag{2.2}$$

(we use the grand canonical ensemble and measure particle energies from the chemical potential μ).[2]

We now employ the standard device for obtaining a nonequilibrium state: At time t_0, prior to which the system is assumed to be in thermodynamic equilibrium with a reservoir, the system is disconnected from the reservoir and exposed to a disturbance represented by the contribution $H'(t)$ to the Hamiltonian. The total Hamiltonian is thus given by

$$\mathscr{H}(t) = H + H'(t) \,, \tag{2.3}$$

where $H'(t) = 0$ for $t < t_0$.

Nonequilibrium statistical mechanics is concerned with calculating average values $\langle O_{\mathscr{H}}(t) \rangle$ of physical observables for times $t > t_0$,

$$\langle O_{\mathscr{H}}(t) \rangle = \mathrm{Tr}[\rho(H) O_{\mathscr{H}}(t)] \,, \tag{2.4}$$

where $O_{\mathscr{H}}(t)$ is the observable in the Heisenberg picture.

We also mention that the real-time method is not restricted to using the statistical equilibrium state at times prior to t_0 as the boundary condition. As shown by Korenman (1966), the real-time method is also suitable for describing a nonequilibrium situation maintained through contact with a reservoir. Schmid (1982) also applied the Keldysh technique to a discussion of the coupling of a system to a reservoir.

1. Green's functions

Green's or correlation, functions play a fundamental role in statistical physics since they constitute the connection between experimentally relevant quantities and conveniently calculable quantities.

We start by introducing the following correlation functions:[3]

$$\begin{aligned} G^<(1,1') &= \mp i \langle \psi^\dagger_{\mathscr{H}}(1') \psi_{\mathscr{H}}(1) \rangle \,, \\ G^>(1,1') &= -i \langle \psi_{\mathscr{H}}(1) \psi^\dagger_{\mathscr{H}}(1') \rangle \,, \end{aligned} \tag{2.5}$$

where ψ is a Bose (upper sign) or Fermi (lower sign) field.

A quantity possessing a simple perturbation expansion is the so-called contour-ordered Green's function[4] defined by

$$G(1,1') = -i \langle T_c(\psi_{\mathscr{H}}(1) \psi^\dagger_{\mathscr{H}}(1')) \rangle \,, \tag{2.6}$$

where c is the contour along the real-time axis that starts and ends at t_0 and passes through t_1 and $t_{1'}$ once. The contour-ordering operator T_c orders the operators according to the position on the contour of their time arguments

$$T_c(\psi_{\mathscr{H}}(1) \psi^\dagger_{\mathscr{H}}(1')) \equiv \begin{cases} \psi_{\mathscr{H}}(1) \psi^\dagger_{\mathscr{H}}(1') & t_1 >_c t_{1'} \,, \\ \pm \psi^\dagger_{\mathscr{H}}(1') \psi_{\mathscr{H}}(1) & t_1 <_c t_{1'} \,. \end{cases} \tag{2.7}$$

The contour-ordering relation $t_1 >_c t_{1'}$ means that t_1 is further along the contour than $t_{1'}$. We observe that[5]

$$G(1,1') = \begin{cases} G^>(1,1') & t_1 >_c t_{1'} \,, \\ G^<(1,1') & t_1 <_c t_{1'} \,. \end{cases} \tag{2.8}$$

The contour c is depicted in Fig. 1 for the situation $t_1 >_c t_{1'}$.

2. Perturbation theory

To obtain a diagrammatic perturbation expansion of the contour-ordered Green's function, we notice that the standard way of expressing the relation between an observable $O_{\mathscr{H}}$ in the Heisenberg picture and the corresponding observable O_H in the interaction picture with respect to H is

$$O_{\mathscr{H}}(t) = u^\dagger(t,t_0) O_H(t) u(t,t_0) \,, \tag{2.9}$$

where

$$u(t,t_0) = T \exp\left[-i \int_{t_0}^{t} dt' H'_H(t') \right] . \tag{2.10}$$

Here T is the usual time-ordering operator and $H'_H(t)$ represents the operator $H'(t)$ in the interaction picture with respect to the Hamiltonian H.

The transformation (2.9) can be expressed as

$$O_{\mathscr{H}}(t) = T_{c_t}\left[\exp\left[-i \int_{c_t} d\tau H'_H(\tau) \right] O_H(t) \right] , \tag{2.11}$$

where c_t is the contour depicted in Fig. 2.

Using Eq. (2.11) we can express the contour-ordered Green's function in the interaction picture with respect to the Hamiltonian H

$$G(1,1') = -i \langle T_c(S_c^H \psi_H(1) \psi^\dagger_H(1')) \rangle \,, \tag{2.12}$$

where

$$S_c^H = \exp\left[-i \int_c d\tau H'_H(\tau) \right] \tag{2.13}$$

and c is the contour depicted in Fig. 1.

To obtain a perturbation expansion for the contour-ordered Green's function, we could employ the standard functional derivative method due to Schwinger (1951), as explained in detail by Kadanoff and Baym (1962) and employed for real times by Korenman (1966; see also

[2]Units are chosen to put $\hbar = k_B = c = 1$.

[3]The abbreviation $1 \equiv (t_1, \mathbf{x}_1)$ is used. Here $\mathbf{x}$ denotes the spatial variable, which may be generalized in a straightforward manner to include spin and other degrees of freedom.

[4]This function was originally named the "closed time path Green's function" (Schwinger, 1961).

[5]This will be our general prescription for relating contour-ordered quantities to greater and lesser quantities.

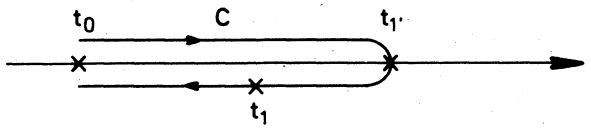


FIG. 1. The "closed time path" contour c.

Schwinger, 1961). In this review, we use instead the equivalent procedure based on the statistical Wick's theorem.

Since we wish to employ the statistical Wick's theorem, we must transform to the interaction picture with respect to the free-particle Hamiltonian H_0, which is quadratic in the field operators. Utilizing the relation

$$e^{-\beta H}=e^{-\beta H_0}v(t_0-i\beta,t_0)\,, \tag{2.14}$$

where

$$v(t,t_0)=T\exp\left[-i\int_{t_0}^{t}dt' H^i_{H_0}(t')\right]\,, \tag{2.15}$$

with $H^i_{H_0}(t)$ being the operator H^i in the interaction picture with respect to the Hamiltonian H_0, we obtain

$$\begin{aligned} G(1,1')&=-i\frac{\langle T_c(S_{c^i}S_c\psi_{H_0}(1)\psi^\dagger_{H_0}(1'))\rangle_0}{\langle T_c(S_{c^i}S_c)\rangle_0}\\ &\equiv -i\frac{\mathrm{Tr}[e^{-\beta H_0}T_c(S_{c^i}S_c\psi_{H_0}(1)\psi^\dagger_{H_0}(1'))]}{\mathrm{Tr}[e^{-\beta H_0}T_c(S_{c^i}S_c)]}\,, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} S_{c^i}&=\exp\left[-i\int_{c^i}d\tau H^i_{H_0}(\tau)\right]\,,\\ S_c&=\exp\left[-i\int_c d\tau H^i_{H_0}(\tau)\right]\,, \end{aligned} \tag{2.17}$$

and the contour c^i is depicted in Fig. 3 for the same situation as in Fig. 1, $t_1>_c t_{1'}$.

We can now use Wick's theorem, just as in equilibrium theory, to get a perturbation expansion for G. The only formal difference from equilibrium theory is the appearance of integration over a contour instead of integration over the inverse temperature interval for the case of finite temperature or the real axis for the case of zero temperature. Consequently, the contour-ordered Green's function is mapped onto its Feynman diagrams precisely as in equilibrium theory.

The different forms of nonequilibrium theory employ different methods for converting the internal contour integration into the usual time integration. In the formulation by Langreth (1976), which we have followed here, these forms appear as different choices of the contour, thus demonstrating their equivalence.

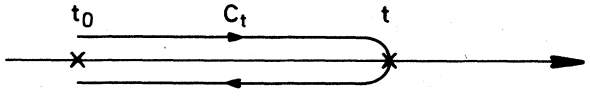


FIG. 2. The "transformation" contour c_t.

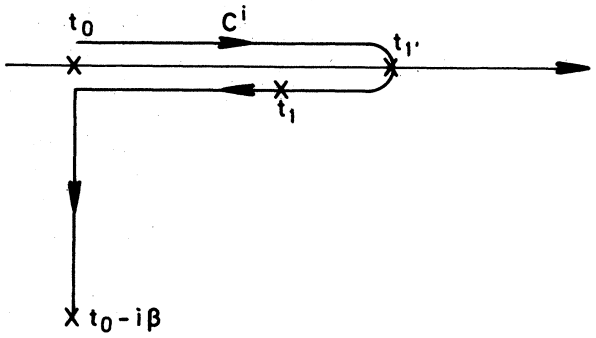


FIG. 3. The "interaction" contour c^i.

We conclude that equilibrium and nonequilibrium statistical mechanics are formally and structurally equivalent and that this equivalence is demonstrated by introducing a contour ordering to play the role of the usual time ordering.

B. The Keldysh formulation

In this section, we seek to reduce the equation of motion for the Green's function to a tractable kinetic equation. We do not consider initial correlations and therefore let t_0 approach minus infinity. Since we assume that the Green's function falls off sufficiently rapidly as a function of the separation of its time arguments, we can neglect the part of the contour c^i extending from t_0 to $t_0-i\beta$ (for a discussion of this point, see Mills, 1969). It has been shown by taking explicitly into account the initial correlations (Fujita, 1965,1971; Hall, 1975; Kukharenko and Tikhodeev, 1982) that the neglect of this part of the contour corresponds to the neglect of initial correlations.

In the functional derivative method, the contour from t_0 to $t_0-i\beta$ will not appear, since Eq. (2.12) forms the starting point, and the transformation (2.14) is not employed. The boundary condition, that the system is assumed to be in equilibrium before the external perturbation is turned on, can then be imposed directly on the Dyson equation in integral form. This provides an independent demonstration that, for cases where initial correlations can be neglected (corresponding to the assumed boundary condition), we can discard the contribution of the contour from t_0 to $t_0-i\beta$, which arose in the transformation from Eq. (2.12) to Eq. (2.16).

The contours c^i and c are now identical, as they both start and end at $-\infty$. They can be extended beyond the largest time by use of the unitarity of the time-development operator, and we then obtain the contour c_K introduced by Keldysh (1964). The contour c_K depicted in Fig. 4 consists of two parts: c_1 extending from $-\infty$ to $+\infty$ and c_2 extending from $+\infty$ to $-\infty$. The contour-

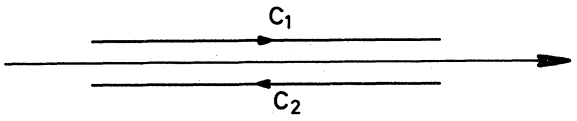


FIG. 4. The Keldysh contour c_K.

ordered Green's function G_{c_K} specified by the Keldysh contour can then be mapped onto the Keldysh space,

$$G_{c_K}(1,1') \mapsto \hat{G} \equiv \begin{Bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{Bmatrix}, \tag{2.18}$$

by the prescription that the ij component of $\hat{G}$ be defined as $G_{c_K}(1,1')$ for t_1 and $t_{1'}$ residing on c_i and c_j, respectively. Writing out the components, we observe[6]

$$\begin{aligned} \hat{G}_{11}(1,1') &= -i\langle T(\psi_{\mathscr{H}}(1)\psi^{\dagger}_{\mathscr{H}}(1'))\rangle , \\ \hat{G}_{12}(1,1') &= G^{<}(1,1') , \\ \hat{G}_{21}(1,1') &= G^{>}(1,1') , \\ \hat{G}_{22}(1,1') &= -i\langle \tilde{T}(\psi_{\mathscr{H}}(1)\psi^{\dagger}_{\mathscr{H}}(1'))\rangle . \end{aligned} \tag{2.19}$$

The components of $\hat{G}$ are not linearly independent, and by performing a rotation in Keldysh space it is possible to remove part of the redundancy. In the original article, Keldysh (1964) used the linear transformation

$$\hat{G} \mapsto \begin{Bmatrix} 0 & G^A \\ G^R & G^K \end{Bmatrix}, \tag{2.20}$$

where, besides the usual retarded and advanced Green's functions,[7]

$$\begin{aligned} G^R(1,1') &= -i\theta(t_1 - t_{1'})\langle [\psi_{\mathscr{H}}(1), \psi^{\dagger}_{\mathscr{H}}(1')]_{\mp}\rangle \\ &= \theta(t_1 - t_{1'})[G^{>}(1,1') - G^{<}(1,1')] , \\ G^A(1,1') &= +i\theta(t_{1'} - t_1)\langle [\psi_{\mathscr{H}}(1), \psi^{\dagger}_{\mathscr{H}}(1')]_{\mp}\rangle \\ &= -\theta(t_{1'} - t_1)[G^{>}(1,1') - G^{<}(1,1')] , \end{aligned} \tag{2.21}$$

we encounter the function

$$\begin{aligned} G^K(1,1') &= G^{>}(1,1') + G^{<}(1,1') \\ &= -i\langle [\psi_{\mathscr{H}}(1), \psi^{\dagger}_{\mathscr{H}}(1')]_{\pm}\rangle , \end{aligned} \tag{2.22}$$

central to the nonequilibrium formulation of Keldysh.

For many purposes there exists a more convenient representation of G_{c_K} introduced by Larkin and Ovchinnikov (1975). It has the advantage that the matrix structure in Keldysh space of a one-body coupling is the simplest possible. To obtain this representation we first perform a transformation in Keldysh space[8]

$$\check{G} \equiv \tau^3 \hat{G} , \tag{2.23}$$

followed by a rotation

$$\underline{G} \equiv L\check{G}L^{\dagger} , \tag{2.24}$$

where

$$L = \frac{1}{\sqrt{2}}(\tau^0 - i\tau^2) . \tag{2.25}$$

With the help of the following identities,

$$\begin{aligned} G^R(1,1') &= \hat{G}_{11}(1,1') - \hat{G}_{12}(1,1') \\ &= \hat{G}_{21}(1,1') - \hat{G}_{22}(1,1') , \\ G^A(1,1') &= \hat{G}_{11}(1,1') - \hat{G}_{21}(1,1') \\ &= \hat{G}_{12}(1,1') - \hat{G}_{22}(1,1') , \\ G^K(1,1') &= \hat{G}_{21}(1,1') + \hat{G}_{12}(1,1') \\ &= \hat{G}_{11}(1,1') + \hat{G}_{22}(1,1') , \end{aligned} \tag{2.26}$$

which are easily proved, we obtain

$$\underline{G} = \begin{Bmatrix} G^R & G^K \\ 0 & G^A \end{Bmatrix} . \tag{2.27}$$

The representation (2.27) with one off-diagonal element equal to zero will be used below.

C. Feynman rules

In order to establish the Feynman rules in Keldysh space it is sufficient to consider simple diagrams. As pointed out in Sec. II.A.2 we need only add to the standard Feynman rules the additional features brought about by the contour. In addition we note that the cancellation of the unlinked diagrams is obtained automatically in the Keldysh technique. Let us start with the simplest example, coupling to an external potential U. In standard notation we have the diagrammatic expansion of G_{c_K} shown in Fig. 5 with the crosses denoting the external potential.

The first-order term in U is given by the second diagram in the infinite series shown in Fig. 5,

[6]The ordering along the contour c_1 is given by the usual time-ordering operator T, while the ordering along c_2 is given by the anti-time-ordering operator $\tilde{T}$ according to

$$\tilde{T}(\psi(1)\psi^{\dagger}(1')) = \begin{cases} \psi(1)\psi^{\dagger}(1') & t_1 < t_{1'} \\ \pm\psi^{\dagger}(1')\psi(1) & t_1 > t_{1'} . \end{cases}$$

[7]We note that the original article by Keldysh (1964) contains the misprint of interchanging the definition of retarded and advanced.

[8]We use τ^i ($i = 0,1,2,3$) instead of σ^i to designate the Pauli matrices, to stress that they have nothing to do with spin degrees of freedom (τ^0 is the unit matrix).

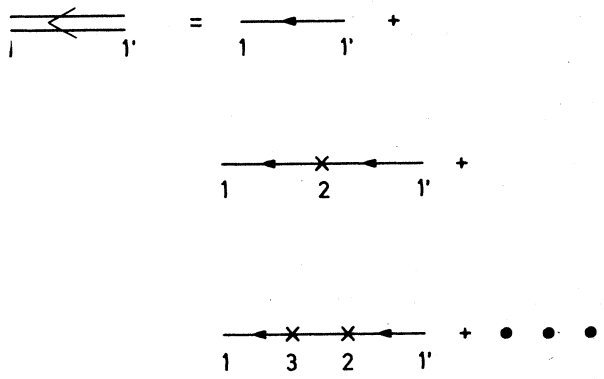


FIG. 5. Perturbation expansion of the Green's function.

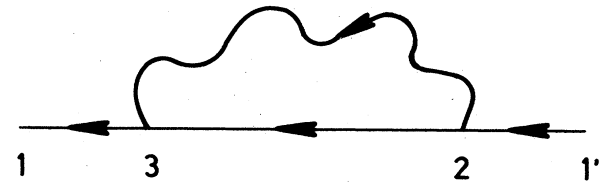


FIG. 6. Electron-phonon diagram.

$$G_{c_K}^{(1)}(1,1')=\int d\mathbf{x}_2\int_{c_K} d\tau_2 G_{c_K}^{(0)}(1,2)U(2)\times G_{c_K}^{(0)}(2,1')\ . \tag{2.28}$$

Interchanging the limits of integration on the lower branch of the contour c_K,

$$\int_{c_K} d\tau \to \int_{-\infty}^{\infty} dt - \int_{-\infty}^{\infty} dt\ , \tag{2.29}$$

we split $G_{c_K}^{(1)}$ into two terms whose components in Keldysh space are uniquely represented as

$$\hat{G}_{ij}^{(1)}(1,1')=\int d\mathbf{x}_2\int_{-\infty}^{\infty} dt_2 \hat{G}_{ik}^{(0)}(1,2)\hat{U}_{kk'}(2)\times\hat{G}_{k'j}^{(0)}(2,1')\ , \tag{2.30}$$

where

$$\hat{U}_{ij}(2)=U(2)\tau_{ij}^3\ , \tag{2.31}$$

and the Einstein summation rule for repeated Keldysh indices is assumed.

In matrix form, and using the condensed notation

$$(A\otimes B)(1,1')=\int d\mathbf{x}_2\int_{-\infty}^{\infty} dt_2 A(1,2)B(2,1')\ , \tag{2.32}$$

we can write Eq. (2.30) as

$$\hat{G}^{(1)}=\hat{G}^{(0)}\otimes\hat{U}\hat{G}^{(0)}\ . \tag{2.33}$$

We have thus established in the representation given by Eq. (2.18) that an external potential in Keldysh space couples to particles through a τ^3 matrix.

For the electron-phonon interaction described by the Hamiltonian

$$H_{e\text{-ph}}=g\int d\mathbf{x}_1\psi^{\dagger}(1)\psi(1)\varphi(1)\ , \tag{2.34}$$

where g is the coupling constant and φ the phonon field, we have for the simplest diagram shown in Fig. 6

$$G_{c_K}^{(1)}=ig^2\int_{c_K} d\tau_3\int_{c_K} d\tau_2\int d\mathbf{x}_3\int d\mathbf{x}_2 G_{c_K}^{(0)}(1,3)G_{c_K}^{(0)}(3,2)D_{c_K}^{(0)}(3,2)G_{c_K}^{(0)}(2,1')\ , \tag{2.35}$$

where $D_{c_K}^{(0)}$ is the Keldysh contour-ordered Green's function for free phonons,

$$D_{c_K}^{(0)}(1,1')=-i\langle T_{c_K}(\varphi_{H_0}(1)\varphi_{H_0}(1'))\rangle_0\equiv -i(\mathrm{Tr}(e^{-\beta H_0}))^{-1}\mathrm{Tr}[e^{-\beta H_0}T_{c_K}(\varphi_{H_0}(1)\varphi_{H_0}(1'))]\ . \tag{2.36}$$

The ij component in Keldysh space of Eq. (2.35) can now be expressed as

$$\hat{G}_{ij}^{(1)}=ig^2\hat{G}_{ii'}^{(0)}\otimes\hat{\gamma}_{i'l'}^{k}\hat{G}_{l'l}^{(0)}\hat{D}_{kk'}^{(0)}\hat{\gamma}_{lj'}^{k'}\otimes\hat{G}_{j'j}^{(0)}\ , \tag{2.37}$$

where the bare electron-phonon vertex $\hat{\gamma}$ is the third rank tensor,

$$\hat{\gamma}_{ij}^{k}=\delta_{ij}\tau_{jk}^3 \tag{2.38}$$

(no summation over j). The reason for the form of Eq. (2.38) is as follows. The vertex is a unit tensor in the lower indices (the electronic Keldysh indices) reflecting that the electron, at the vertex, enters and leaves the same (space-) time point. The change of the upper index (phonon Keldysh index) can change the sign of the tensor, reflecting that the contour point at the other end of the phonon propagator can reside on either the upper or the lower branch of the Keldysh contour.[9] As a consequence, higher-order diagrams are mapped onto the Keldysh space in accordance with the same rule, Eq. (2.38).

It is readily shown that the Feynman rules in Keldysh space for a two-body interaction can be formulated identically to the electron-phonon rules.

It is now straightforward, by transforming according to Eqs. (2.23) and (2.24), to obtain the Feynman rules in Keldysh space in the representation given by Eq. (2.27). For later reference, we state them below.

[9]Note that the argument for the form of the vertex (2.38), although given explicitly for the lowest-order diagram in Fig. 6, carries through for any diagram of higher order.

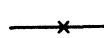
Electron propagator:

$$= \underline{G} \ . \tag{2.39}$$

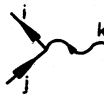
Phonon or Coulomb propagator:


$$= \underline{D}, \underline{V} \ . \tag{2.40}$$

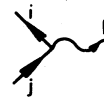
Scalar potential:


$$= U\tau^0 \ . \tag{2.41}$$

Electron-phonon or Coulomb vertices:



$$= \gamma^k_{ij} \ \text{(absorption)} \ , \tag{2.42}$$



$$= \tilde{\gamma}^{\,k}_{ij} \ \text{(emission)} \ . \tag{2.43}$$

The convenience of the representation given by Eq. (2.27) has been paid for by the loss of identity in the description of emission and absorption, since

$$\gamma^1_{ij} = \tilde{\gamma}^{\,2}_{ij} = \frac{1}{\sqrt{2}}\delta_{ij} \ , \tag{2.44}$$

$$\gamma^2_{ij} = \tilde{\gamma}^{\,1}_{ij} = \frac{1}{\sqrt{2}}\tau^1_{ij} \ . \tag{2.45}$$

We emphasize that the diagrammatic formulation of nonequilibrium theory presented above is identical in form to standard diagrammatic theory, except that both propagators and vertices are tensors. The tensor structure results in the need to carry out internal sums over Keldysh indices in addition to the usual integration over internal variables when evaluating diagrams.

D. Dyson equations

The Dyson equations are obtained, just as in equilibrium theory, by considering the time evolution of the field operators. Transforming to the representation given by Eq. (2.27), we obtain for the right-hand Dyson equation[10]

$$(\underline{G}_0^{-1} - \underline{\Sigma}) \otimes \underline{G} = \delta(1-1') \ , \tag{2.46}$$

where the inverse matrix Green's function is given by

$$\underline{G}_0^{-1}(1,1') = [i\partial_{t_1} - \varepsilon(1)]\delta(1-1') \ , \tag{2.47}$$

and

$$\underline{\Sigma} = \begin{Bmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{Bmatrix} \tag{2.48}$$

consists of the usual retarded and advanced self-energies,

$$\Sigma^R(1,1') = \theta(t_1 - t_{1'})[\Sigma^>(1,1') - \Sigma^<(1,1')] \ , \tag{2.49}$$

$$\Sigma^A(1,1') = -\theta(t_{1'} - t_1)[\Sigma^>(1,1') - \Sigma^<(1,1')] \ , \tag{2.50}$$

and a Keldysh component

$$\Sigma^K = \Sigma^> + \Sigma^< \ , \tag{2.51}$$

which is nontrivial for nonequilibrium states.

For the case of electrons coupled to an electromagnetic field described by the potentials $\mathbf{A}$ and φ, the single-particle Hamiltonian is given by

$$\varepsilon(1) = \frac{1}{2m}\left[\frac{1}{i}\nabla_{\mathbf{x}_1} - e\mathbf{A}(1)\right]^2 + e\varphi(1) - \mu \ , \tag{2.52}$$

where e is the charge of the electron, $e = -|e|$, and m its mass. Likewise, we get for the left-hand or conjugate Dyson equation

$$\underline{G} \otimes (\underline{G}_0^{-1} - \underline{\Sigma}) = \delta(1-1') \ . \tag{2.53}$$

E. Kinetic equations

The Dyson equations provide an exact description of the system. Generally speaking, the diagonal components of $\underline{G}$ characterize the states, and the off-diagonal component (Keldysh component) contains information on the occupation of these states, although the precise validity of this interpretation depends on further approximations such as the gradient expansion and the quasiclassical approximation. The equation of motion for G^K therefore generally constitutes the quantum-kinetic equation. Approximations to this equation will lead us to the ordinary Boltzmann equation and its generalizations.

1. The gradient expansion

We now introduce the ingredients of the gradient approximation scheme and use, as an illustration, the very simplest transport problem, where the electrons are driven out of equilibrium by a scalar potential U and relax by impurity scattering.

First, we introduce the mixed or Wigner representation. When the equilibrium system possesses translational invariance, this is done by shifting the frame of reference to the center-of-mass system, defined by variables $\mathbf{R}$ and T given by

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_{1'}), \quad T = \tfrac{1}{2}(t_1 + t_{1'}) \ ,$$

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_{1'}, \quad t = t_1 - t_{1'} \ , \tag{2.54}$$

and Fourier transforming with respect to relative coordinates[11] $\mathbf{r}$ and t

[10] Unit matrices in Keldysh space are suppressed throughout.

[11] We introduce the abbreviated notation (no confusion of the "center-of-mass" time and the temperature should arise) $X = (T, \mathbf{R})$, $x = (t, \mathbf{r})$, $p = (E, \mathbf{p})$, $px = -Et + \mathbf{p}\cdot\mathbf{r}$, $\partial^A_X = (-\partial_T, \nabla_{\mathbf{R}})$, $\partial^A_p = (-\partial_E, \nabla_{\mathbf{p}})$. Here the upper index refers to the function operated on. Note that $\partial^A_X \partial^B_p = -\partial^A_T \partial^B_E + \nabla^A_{\mathbf{R}} \cdot \nabla^B_{\mathbf{p}}$ and $x\partial_X = t\partial_T + \mathbf{r}\cdot\nabla_{\mathbf{R}}$.

$$\underline{G}(X,p)=\int dx\, e^{-ipx}\underline{G}(X+x/2,X-x/2)\ . \tag{2.55}$$

Performing a Taylor expansion, we observe that the convolution $A\otimes B$ in this representation is given by

$$(A\otimes B)(X,p)=e^{i(\partial_X^A\partial_p^B-\partial_p^A\partial_X^B)/2}A(X,p)B(X,p)\ . \tag{2.56}$$

Physical quantities such as densities and particle currents may be expressed in terms of the equal-time, one-particle density matrix, which is essentially our Green's function $G^<$ in the mixed representation integrated over E. However, the Dyson equation cannot be reduced to one involving equal-time Green's functions. As a first step towards obtaining an equation that may be integrated over E, we subtract the Dyson equation and its conjugate, whereupon we obtain

$$[\underline{G}_0^{-1}-\underline{\Sigma}\underset{,}{\otimes}\underline{G}]_-=0\ , \tag{2.57}$$

where, in the mixed representation,

$$\underline{G}_0^{-1}=E-\xi_{\mathbf{p}}-U(\mathbf{R},T)\ , \tag{2.58}$$

and, since we consider a free electron model below,

$$\xi_{\mathbf{p}}=\frac{\mathbf{p}^2}{2m}-\mu\ . \tag{2.59}$$

To obtain a kinetic equation, we consider the Keldysh component of Eq. (2.57),

$$[G_0^{-1}-\mathrm{Re}\Sigma\underset{,}{\otimes}G^K]_--[\Sigma^K\underset{,}{\otimes}\mathrm{Re}G]_-$$
$$=\frac{i}{2}[\Sigma^K\underset{,}{\otimes}A]_+-\frac{i}{2}[\Gamma\underset{,}{\otimes}G^K]_+\ , \tag{2.60}$$

where we have introduced the spectral weight function

$$A=i(G^R-G^A) \tag{2.61}$$

together with the other abbreviations

$$\Gamma=i(\Sigma^R-\Sigma^A)\ , \tag{2.62}$$

$$\mathrm{Re}\Sigma=\tfrac{1}{2}(\Sigma^R+\Sigma^A)\ , \tag{2.63}$$

$$\mathrm{Re}G=\tfrac{1}{2}(G^R+G^A)\ , \tag{2.64}$$

specifying the real and imaginary parts of the retarded and advanced Green's functions and self-energies.

The self-energies appear in two ways in Eq. (2.60). They describe scattering between states as accounted for by the right-hand side, and they also appear on the left-hand side, resulting in renormalization effects, which reflect the fact that the electrons between collisions do not behave as noninteracting particles. Notice that in equilibrium the exact quantum-kinetic equation (2.60) is an empty statement as a consequence of the equation

$$G_0^<(p)=-e^{-\beta E}G_0^>(p)\ , \tag{2.65}$$

which is valid in equilibrium.[12] Translating to our choice of Green's functions, we have[13]

$$G_0^K(p)=h_0(E)[G_0^R(p)-G_0^A(p)]\ , \tag{2.66}$$

with the consequence

$$\Sigma_0^K(p)=h_0(E)[\Sigma_0^R(p)-\Sigma_0^A(p)]\ , \tag{2.67}$$

where

$$h_0(E)=\tanh\tfrac{1}{2}\beta E\ . \tag{2.68}$$

Thus the two terms on the right-hand side of Eq. (2.60) cancel in equilibrium, while the left-hand side is trivially zero in equilibrium. The gradient approximation is obtained from (2.56) by keeping the first two terms in the expansion of the exponential function in (2.56). We thus have

$$[A\underset{,}{\otimes}B]_+=2AB\ ,$$
$$-i[A\underset{,}{\otimes}B]_-=[A,B]_p\ , \tag{2.69}$$

where the generalized Poisson brackets is defined as

$$[A,B]_p=\partial_X^A A\partial_p^B B-\partial_p^A A\partial_X^B B$$
$$=(\partial_E^A\partial_T^B-\partial_T^A\partial_E^B-\nabla_{\mathbf{p}}^A\cdot\nabla_{\mathbf{R}}^B+\nabla_{\mathbf{R}}^A\cdot\nabla_{\mathbf{p}}^B)AB\ . \tag{2.70}$$

2. The Boltzmann equation for impurity scattering

We now turn to an illustration of how the classical Boltzmann equation for the impurity problem can be obtained. Treating the impurity scattering problem within the self-consistent Born approximation, the impurity-averaged self-energy is given by the diagram shown in Fig. 7,

$$\underline{\Sigma}(E,\mathbf{p},\mathbf{R},T)=n_i\int\frac{d\mathbf{p}'}{(2\pi)^3}|v(\mathbf{p}-\mathbf{p}')|^2\underline{G}(E,\mathbf{p}',\mathbf{R},T)\ , \tag{2.71}$$

where $v(\mathbf{p})$ is the Fourier transform $v(\mathbf{p})=\int d\mathbf{r}\,e^{-i\mathbf{p}\cdot\mathbf{r}}v(\mathbf{r})$ of the impurity potential $v(\mathbf{r})$ and n_i is the concentration of impurities. Considering the dilute limit (low impurity concentration), and assuming a slowly varying external perturbation, we can treat the self-energies appearing on the left-hand side of Eq.(2.60) as constants and obtain in the gradient approximation

$$[G_0^{-1},G^K]_p=\Sigma^K A-\Gamma G^K\ . \tag{2.72}$$

To proceed further, we examine the spectral weight function. Subtracting the diagonal components of Eq. (2.57) gives

$$[G_0^{-1}-\mathrm{Re}\Sigma\underset{,}{\otimes}A]_--[\Gamma\underset{,}{\otimes}\mathrm{Re}G]_-=0\ . \tag{2.73}$$

The detailed solution of this equation within the gra-

[12]The subscript refers to equilibrium. Equation (2.65) is a rephrasing of the Kubo-Martin-Schwinger boundary condition, $\langle\psi^\dagger(t_1)\psi(t_{1'})\rangle=\langle\psi(t_{1'})\psi^\dagger(t_1+i\beta)\rangle$, following from the cyclic invariance of the trace.

[13]We assume here and in the following that the particles are spin-$\frac{1}{2}$ fermions.

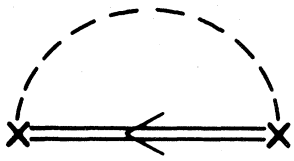


FIG. 7. Impurity self-energy diagram.

dient approximation is given by Kadanoff and Baym (1962). One finds that the solution has the same form as in equilibrium,

$$A=\frac{\Gamma}{(E-\xi_{\mathbf{p}}-\mathrm{Re}\Sigma-U)^2+(\Gamma/2)^2}\ . \tag{2.74}$$

The spectral weight function thus approaches a delta function in the dilute limit

$$A=2\pi\delta(E-\xi_{\mathbf{p}}-U)\ . \tag{2.75}$$

This is also seen by observing that the solution to Eq. (2.73), in the case where the self-energies are neglected, is the function given by Eq. (2.75); in other words, G_0^{-1} and A given by Eq. (2.75) commute.

Exploiting the delta-function character of the spectral weight and the similarly peaked character of G^K, we can easily integrate Eq. (2.72) with respect to E from minus to plus infinity and obtain[14]

$$\partial_T h+\nabla_{\mathbf{p}}\xi\cdot\nabla_{\mathbf{R}}h-\nabla_{\mathbf{R}}U\cdot\nabla_{\mathbf{p}}h =-2\pi n_i\int\frac{d\mathbf{p}'}{(2\pi)^3}\,|v(\mathbf{p}-\mathbf{p}')|^2 \times\delta(\xi_{\mathbf{p}}-\xi_{\mathbf{p}'})(h_{\mathbf{p}}-h_{\mathbf{p}'})\ , \tag{2.76}$$

where (suppressing space and time arguments) we have introduced the distribution function

$$h_{\mathbf{p}}=-\int\frac{dE}{2\pi i}G^K\ , \tag{2.77}$$

and a free-particle form for the spectral weight, corresponding to

$$G^K=-2\pi i\delta(E-\xi_{\mathbf{p}}-U)h_{\mathbf{p}}\ , \tag{2.78}$$

has been inserted in the self-energies.

Introducing the more familiar distribution function

$$f_{\mathbf{p}}=\tfrac{1}{2}(1-h_{\mathbf{p}})\ , \tag{2.79}$$

which reduces to the Fermi function in equilibrium, we obtain the Boltzmann equation for impurity scattering

$$(\partial_T+\mathbf{v}\cdot\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}}U\cdot\nabla_{\mathbf{p}})f_{\mathbf{p}} =-2\pi n_i\int\frac{d\mathbf{p}'}{(2\pi)^3}\,|v(\mathbf{p}-\mathbf{p}')|^2\,\delta(\xi_{\mathbf{p}}-\xi_{\mathbf{p}'})(f_{\mathbf{p}}-f_{\mathbf{p}'})\ , \tag{2.80}$$

[14]Notice that the first term on the left-hand side of Eq. (2.76) is exact and so is the second, insofar as the dispersion is quadratic.

where $\mathbf{v}=\nabla_{\mathbf{p}}\xi$ is the (group) velocity.

The dilute-limit approximation that went into the derivation of Eq. (2.80) is equivalent to the usual criterion for the validity of the Boltzmann equation for a degenerate Fermi gas,

$$\frac{1}{\tau}\ll\mu\ , \tag{2.81}$$

where $1/\tau$ is the impurity scattering rate at the Fermi surface ($p=p'=k_F$),

$$\frac{1}{\tau}=2\pi n_i N_0\int\frac{d\hat{p}'}{4\pi}\,|v(\mathbf{p}-\mathbf{p}')|^2\ . \tag{2.82}$$

Here $N_0=mk_F/2\pi^2$ is the density of states per spin at the Fermi energy and k_F the Fermi momentum. Furthermore, the gradient approximation is valid when the external perturbation is slowly varying in space and time, so that its characteristic frequency ω and wave vector q satisfy the conditions

$$\omega\ll\mu,\quad q\ll k_F\ . \tag{2.83}$$

The condition (2.83) for a degenerate Fermi gas is sufficient to ensure the validity of the gradient expansion. It is not necessary, however, to assume that the external frequency is small in the sense of $\omega\ll 1/\tau$ or $\omega\ll T$, as the question of the magnitude of the frequency with respect to the collision rate or the temperature never enters the approximations leading to Eq. (2.72). Slow variations in time are therefore, within the context of the impurity problem, to be understood in the sense of the conditions (2.83). In subsequent sections, the condition $q\ll k_F$ can be used throughout, while the restriction on ω may depend on the particular system under consideration. Thus in discussing the electron-phonon problem (Sec. III.B), we assume $\omega\ll T$, while in the context of superconductivity (Sec. V), the inclusion of an external frequency within the gradient expansion would require $\omega\ll\Delta$. The validity of the gradient expansion, as far as the external frequency is concerned, will therefore depend on the physical system under consideration. It should be noted, however, that it is never necessary to assume the frequency to be small in comparison to the collision rate, as one would also expect from the simple physical arguments that lead to the semiclassical Boltzmann equation.

Since we, in our approximations, have freed the Green's functions from their quantum-mechanical origin, it is no surprise that we have obtained a classical equation (2.80). However, the absence of any feature due to quantum statistics is a well-known property of impurity scattering. If we had treated electron-phonon scattering by the same lowest-order perturbation treatment, we would have encountered the characteristics of Fermi-Dirac and Bose-Einstein statistics. We note that, except for this statistical feature, integration of the quantum-kinetic equation with respect to E leads to the classical Boltzmann equation. The basic approximation was the use of the free-particle form of the spectral weight in Eq. (2.78).

3. Currents and densities

The physical quantities we shall encounter are expressible in terms of the Green's function $G^<$: the fermion charge density ρ,

$$\rho(1) = -2iG^<(1,1) , \tag{2.84}$$

and the electric current density $\mathbf{j}$,

$$\mathbf{j}(1) = -\frac{e}{m}\{\nabla_1 - \nabla_{1'} - ie[\mathbf{A}(1)+\mathbf{A}(1')]\}G^<(1,1')\,|_{1'=1} . \tag{2.85}$$

We have assumed spin independence, which accounts for factors of 2. To change to our choice of basic Green's functions we observe that

$$G^< = \tfrac{1}{2}G^K + \frac{i}{2}A . \tag{2.86}$$

The spectral weight A in Eq. (2.86) represents a contribution to Eqs. (2.84) and (2.85) that does not depend on the state of the system and shall henceforth be dropped when nonequilibrium contributions are considered.

III. THE QUASICLASSICAL LIMIT AND BEYOND

There exists a consistent and self-contained approximation scheme for a degenerate Fermi system, valid for describing a wide range of phenomena, that does not employ a restrictive quasiparticle approximation such as Eq. (2.78). It is called the quasiclassical approximation[15] and is of interest since it allows one to obtain simple transport equations even in cases where the quasiparticle approximation fails. As an example of this approximation scheme, we derive in this section the electronic kinetic equation for the case of electron-phonon interaction. In a later section (V) we treat the related case of superconductors with large pair breaking.

A. The quasiclassical approximation

The breakdown of the quasiparticle approximation is associated with the deviation from a delta function of the spectral weight function

$$A = \frac{\Gamma}{(E-\xi-\mathrm{Re}\Sigma)^2+(\Gamma/2)^2} . \tag{3.1}$$

If A differs significantly from a delta function, a kinetic equation involving only equal-time quantities cannot be obtained by integration with respect to E. It is well known that the electron-phonon interaction leads to an important structure in the functions $\mathrm{Re}\Sigma$ and Γ as functions of the variable E. In contrast, as noted by Migdal (1958), the momentum dependence is very weak as a consequence of the phonon energy's being small compared to the Fermi energy. The spectral function thus becomes a peaked function in the variable ξ. We shall use this peaked character to restrict all other quantities in conjunction with the spectral weight function to the Fermi surface.[16] Just as the subtraction of the Dyson equations in Eq. (2.57) eliminated the strong E dependence, it also eliminated the strong ξ dependence. Utilizing the peaked character of the spectral weight function and neglecting the momentum dependence of the self-energy, it is therefore possible to integrate the left-right subtracted Dyson equation with respect to ξ and obtain

$$[\underline{g}_0^{-1} + i\underline{\varrho}\circ\underline{g}]_- = 0 , \tag{3.2}$$

where

$$\underline{g}_0^{-1} = g_0^{-1}\tau^0 , \tag{3.3}$$

with

$$g_0^{-1} = \left\{\partial_{t_1} + \mathbf{v}_F\cdot[\nabla_{\mathbf{R}} - ie\,\mathbf{A}(\mathbf{R},t_1)] + ie\varphi(\mathbf{R},t_1) - \frac{e^2}{2m}[\mathbf{A}(\mathbf{R},t_1)]^2\right\}\delta(t_1-t_{1'}) , \tag{3.4}$$

where $\mathbf{v}_F$ denotes the Fermi velocity, while $\underline{g}$ is the quasiclassical Green's function

$$\underline{g}(\mathbf{R},\hat{p},t_1,t_{1'}) = \frac{i}{\pi}\int d\xi\,\underline{G}(\mathbf{R},\mathbf{p},t_1,t_{1'}) . \tag{3.5}$$

One often refers to $\underline{g}$ as the ξ-integrated Green's function. The integrand in Eq. (3.5) is not well behaved for large values of ξ, since it falls off only as $1/\xi$. A high-energy cutoff must therefore be introduced, as discussed by, for example, Serene and Rainer (1983). Equivalently, we can use the decomposition employed by Eilenberger (1968) and illustrated diagrammatically in Fig. 8. The integration over ξ in Fig. 8 from $-\infty$ to $+\infty$ is replaced by two closed semicircles [Fig. 8(a)] yielding the low-energy contribution of physical interest, provided the cutoff energy is suitably chosen. The remaining high-energy contribution [Fig. 8(b)] contains terms that do not depend on the nonequilibrium state and therefore can be dropped in the equation of motion (3.2). The quasiclassical equations may also be derived without explicit use of ξ integration, as shown by Shelankov (1985).

In obtaining an equation which only involves the center-of-mass spatial coordinates, it has been assumed

[15]This scheme was first applied by Prange and Kadanoff (1964) in their treatment of transport phenomena in the electron-phonon system, although they did not use the name quasiclassical. It was later extended to describe transport phenomena in superfluid systems by Eilenberger (1968).

[16]In doing so, we are, for instance, unable to account for the Lorentz force effect of the magnetic field on the electronic motion.

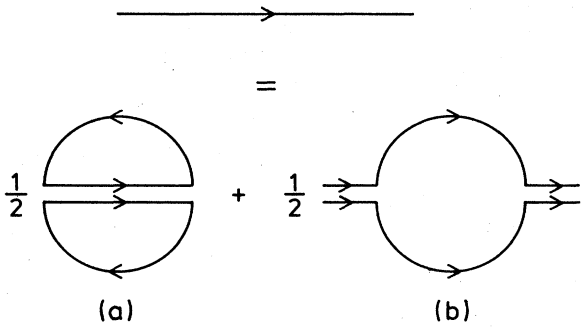


FIG. 8. High- and low-energy integration decomposition.

that the wavelength of the applied perturbation is large in comparison with the Fermi wavelength. We stress that the quasiclassical approximation does not involve the temporal coordinates. This is reflected in the fact that the symbol $\circ$ appearing in Eq. (3.2) is short for internal time integration. In order to have a closed set of equations and a complete quasiclassical theory we must make sure that the self-energy can be expressed solely in terms of the quasiclassical Green's function and thus account for the transformation $\underline{\Sigma}[\underline{G}]\to\underline{\sigma}[\underline{g}]$, $\underline{\Sigma}$ being a functional of $\underline{G}$ and $\underline{\sigma}$ being a functional of $\underline{g}$.

According to the Feynman rules of Sec. II.C, the electron-phonon self-energy in the Migdal approximation is

$$\Sigma_{ij}^{e\text{-ph}}(1,1')=ig^2\gamma_{ii'}^{k}G_{i'j'}(1,1')D_{kk'}(1,1')\tilde{\gamma}_{j'j}^{k'} . \tag{3.6}$$

We now transform to the mixed representation with respect to the spatial coordinates. Using the fact that the sound velocity is small compared to the Fermi velocity, and the substitution

$$\int \frac{d\mathbf{p}}{(2\pi)^3}\mapsto N_0\int d\xi\int\frac{d\hat{p}}{4\pi} , \tag{3.7}$$

which is valid in the case where particle-hole symmetry applies, we obtain

$$\begin{aligned}\sigma_{e\text{-ph}}^{R(A)}(\mathbf{R},\hat{p},t_1,t_{1'})=&\frac{\lambda}{8}\int d\hat{p}\,'[g^K(\mathbf{R},\hat{p}\,',t_1,t_{1'})D^{R(A)}(\mathbf{R},p_F(\hat{p}-\hat{p}\,'),t_1,t_{1'})\\&+g^{R(A)}(\mathbf{R},\hat{p}\,',t_1,t_{1'})D^K(\mathbf{R},p_F(\hat{p}-\hat{p}\,'),t_1,t_{1'})] ,\\ \sigma_{e\text{-ph}}^K=&\frac{\lambda}{8}\int d\hat{p}\,'[(g^R-g^A)(D^R-D^A)+g^KD^K] ,\end{aligned} \tag{3.8}$$

where $\lambda=g^2N_0$ is the dimensionless electron-phonon coupling constant. For brevity, the arguments in the expression for $\sigma_{e\text{-ph}}^K$ have been left out. Furthermore, to improve convergence, the terms of the form $D^{R(A)}G^{R(A)}$ have been rewritten using identities of the type $D^R(t_1,t_{1'})G^A(t_1,t_{1'})=0$.

To obtain the equation obeyed by the quasiclassical Green's function g^K we take the Keldysh component of Eq. (3.2), yielding

$$[g_0^{-1}+i\,\mathrm{Re}\sigma\circ g^K]_-=2i\sigma^K-\frac{i}{2}[\sigma^R-\sigma^A\circ g^K]_+ , \tag{3.9}$$

since

$$[\sigma^K\circ\mathrm{Re}g]_-=0 . \tag{3.10}$$

This follows from introducing the spectral representation for $\mathrm{Re}G$ and interchanging the integration with respect to ξ and E,

$$\begin{aligned}\mathrm{Re}g=&\frac{i}{\pi}\int d\xi P\int\frac{dE'}{2\pi}\frac{A(\mathbf{p},E')}{E-E'}\\=&\frac{i}{\pi}P\int\frac{dE'}{2\pi}\frac{1}{E-E'} .\end{aligned} \tag{3.11}$$

The resulting $\mathrm{Re}g$ is thus seen to be a state-independent constant, and all derivatives arising in Eq. (3.10) will therefore give zero. Note that the interchange of the order of the ξ and the E integration may affect the value of the state-independent constant, but does not involve any nonequilibrium contribution [cf. the discussion below Eq. (3.5)].

A simplification that arises in normal systems and that should be contrasted with the much more complicated situation in superconductors is the lack of structure in the retarded and advanced quasiclassical Green's functions

$$g^{R(A)}=\pm 1 . \tag{3.12}$$

This fact leaves Eq. (3.9), together with the self-energy expression Eq. (3.8), as a closed set of equations for g^K.

We again remark that in obtaining the quasiclassical equation of motion we have used only the degeneracy of the Fermi system in restricting the characteristic wave vector and the frequency of the applied perturbation to satisfy the conditions

$$q\ll k_F,\quad \omega\ll\mu . \tag{3.13}$$

B. The particle representation

In contrast to the convolution in space, there is in general no simple approximation for the convolution in time. Two different approximation schemes are immediately available. One consists in a linearization with respect to a perturbation such as an electric field, allowing frequencies ω much less than μ to be treated. The other assumes perturbations to be sufficiently slowly varying in time that a gradient expansion is valid and requires ω to be much less than the temperature T. For definiteness, we shall here

employ the second scheme. In order to reduce Eq. (3.9) to a simpler looking transport equation, we introduce the mixed representation with respect to the temporal coordinates and perform a gradient expansion as in Sec. II.E. This approximation restricts us to frequencies smaller than the characteristic frequency of the system, in this case the temperature. After transformation to a gauge where the vector potential is absent, Eq. (3.9) is reduced to

$$[(1-\partial_E \mathrm{Re}\sigma)\partial_T+\partial_T \mathrm{Re}\sigma\partial_E+\mathbf{v}_F\cdot\nabla_{\mathbf{R}}+e\partial_T\varphi\partial_E]g^K=I\ , \tag{3.14}$$

where the collision integral is

$$I=2i\sigma^K-\gamma g^K\ , \tag{3.15}$$

with

$$\gamma=i(\sigma^R-\sigma^A)\ . \tag{3.16}$$

The two terms in the collision integral constitute the scattering-in and -out terms, respectively. According to Eq. (3.8) they are determined by (space and time coordinates suppressed)

$$i\sigma^K_{e\text{-ph}}(E,\hat{p})=-\pi\int\frac{d\hat{p}'}{4\pi}\int dE'\mu('p_F(\hat{p}-\hat{p}'),E-E')\times\left[\coth\frac{E'-E}{2T}h(E',\hat{p}')-1\right], \tag{3.17}$$

$$\gamma(E,\hat{p})=-\pi\int\frac{d\hat{p}'}{4\pi}\int dE'\mu(p_F(\hat{p}-\hat{p}'),E-E')\times\left[\coth\frac{E'-E}{2T}-h(E',\hat{p}')\right], \tag{3.18}$$

where[17]

$$\mu(\mathbf{q},E)=\frac{iN_0|g_{\mathbf{q}}|^2}{2\pi}[D_0^R(\mathbf{q},E)-D_0^A(\mathbf{q},E)]\ , \tag{3.19}$$

and we have introduced the distribution function

$$h=\tfrac{1}{2}g^K\ . \tag{3.20}$$

We have further assumed the phonons to be in thermal equilibrium (although this is not necessary) and utilized the equilibrium property

$$D_0^K=(D_0^R-D_0^A)\coth\frac{E}{2T}\ . \tag{3.21}$$

The Fermi-surface average of the function μ is the Eliashberg function α^2F well known from superconductivity.

[17]At this point we allow for a more general electron-phonon coupling than before. The coupling is denoted by $g_{\mathbf{q}}$, corresponding to a momentum transfer $\mathbf{q}$. In the model described by (2.34) we have $|g_q|=g(\omega_q/2)^{1/2}$, where ω_q is the phonon frequency.

If we introduce the distribution function f, which reduces to the Fermi distribution in equilibrium,

$$f=\tfrac{1}{2}(1-h)\ , \tag{3.22}$$

and the Bose distribution N, given by

$$N(E)=-\frac{1}{2}\left[1-\coth\frac{E}{2T}\right], \tag{3.23}$$

the kinetic equation takes the form

$$[(1-\partial_E \mathrm{Re}\sigma)\partial_T+\partial_T \mathrm{Re}\sigma\partial_E+\mathbf{v}_F\cdot\nabla_{\mathbf{R}}+e\partial_T\varphi\partial_E]f=I_{e\text{-ph}}[f]\ , \tag{3.24}$$

with the collision integral

$$I_{e\text{-ph}}[f]=-2\pi\int\frac{d\hat{p}'}{4\pi}\int dE'\mu(\hat{p}-\hat{p}',E-E')R^{E,\hat{p}}_{E',\hat{p}'}\ , \tag{3.25}$$

where

$$R^{E,\hat{p}}_{E',\hat{p}'}=[1+N(E-E')]f(E,\hat{p})[1-f(E',\hat{p}')]-N(E-E')[1-f(E,\hat{p})]f(E',\hat{p}')\ . \tag{3.26}$$

Finally, to introduce a distribution function that is invariant with respect to gauge transformations, we make the replacement

$$f(E)=\tilde{f}(E-e\varphi) \tag{3.27}$$

(cf. Sec. III.E), whereby Eq. (3.24) becomes

$$[(1-\partial_E \mathrm{Re}\sigma)\partial_T+\partial_T \mathrm{Re}\sigma\partial_E+\mathbf{v}_F\cdot(\nabla_{\mathbf{R}}+e\mathbf{E}\partial_E)]f=I_{e\text{-ph}}[f]\ , \tag{3.28}$$

where $\mathbf{E}=-\nabla\varphi$. Here we have dropped the tilde on f for convenience.

As noted previously, we have split off the self-energy terms in the kinetic equation. The terms on the right-hand side describe scattering, whereas the self-energy terms on the left-hand side, which are absent in the usual electron-phonon Boltzmann equation, describe the renormalization effects due to the electron-phonon interaction. Another difference from the usual electron-phonon Boltzmann equation is the dependence of the distribution function on energy and position on the Fermi surface (besides space and time). This feature of the kinetic equation is characteristic of the quasiclassical theory and reflects the fact that we do not rely on a definite relation between energy and momentum, that is, the validity of the quasiparticle description.

From the kinetic equation (3.24), Prange and Kadanoff (1964) drew the conclusion that many-body effects can be seen only in time-dependent transport properties and that static transport coefficients, such as dc conductivity and thermal conductivity, are correctly given by the usual Boltzmann result. However, there is a restriction to the generality of this statement. In deriving the quasiclassical equation of motion we assumed particle-hole symmetry.

Within the quasiclassical theory, all thermoelectric coefficients therefore vanish, and no conclusions can be drawn about many-body effects on the thermoelectric properties. We shall see in Sec. III.F, by going beyond the quasiclassical limit, how the thermoelectric properties do get renormalized by the electron-phonon interaction.

Before proceeding to consider physical problems, we express the electric charge and current densities in terms of the quasiclassical Green's function.

Referring to Sec. II.E.3 we get the following expression for the charge density in the mixed representation:

$$\rho(\mathbf{R},t)=-2eN_0\left[\frac{1}{4}\int\frac{d\hat{p}}{4\pi}\int dE\, g^K(E,\hat{p},\mathbf{R},T)+e\varphi(\mathbf{R},T)\right]. \tag{3.29}$$

The second term in Eq. (3.29) stems from energies far from the Fermi surface and is thus not accounted for by the quasiclassical Green's function. This is established directly (see Eliashberg, 1971) or by invoking the appropriate symmetry, which here is gauge invariance.

For the electric current density we have in the mixed representation

$$\mathbf{j}(\mathbf{R},T)=-\tfrac{1}{2}eN_0\int\frac{d\hat{p}}{4\pi}\int dE\,\mathbf{v}_F g^K . \tag{3.30}$$

Here the high-energy contribution is cancelled by what is sometimes called the diamagnetic current (as we shall verify in a specific case in Sec. IV.A).

The origin of the form of Eqs. (3.29) and (3.30) is the interchange of the order of integration over E and ξ, brought about by the use of the quasiclassical Green's function. This reverses the sequence of integration from the correct one, in which the E integration is done first. Later, in the context of superconductivity, we shall use Eq. (3.29) and interpret ρ as the difference between the charge density of a nonequilibrium superconducting state and that of the normal state in equilibrium. Such a subtraction procedure ensures the validity of the interchange of the order of integration.

C. An example: Electron-phonon renormalization of the ac conductivity

As an example of electron-phonon renormalization of time-dependent transport coefficients we shall consider the ac conductivity in the frequency range given by

$$\gamma \ll \omega , \tag{3.31}$$

where γ denotes the electron-phonon scattering rate at thermal excitation energies. For definiteness, we also consider the temperature to be low compared to the Debye temperature θ_D.

The inequality (3.31) enables us to neglect the collision integral. The linearized kinetic equation then takes the simple form

$$(1-\partial_E \mathrm{Re}\sigma)\partial_T h-\partial_T \mathrm{Re}\sigma\,\partial_E h_0+e\mathbf{v}_F\cdot\mathbf{E}\partial_E h_0=0 . \tag{3.32}$$

Except for a real constant (renormalizing the chemical potential), we have by using the Feynman rules of Sec. II.C

$$\mathrm{Re}\sigma=\tfrac{1}{2}N_0\int\frac{d\hat{p}'}{4\pi}\int dE'\,|g_{\mathbf{p}_F-\mathbf{p}'_F}|^2 h(\hat{p}',E',\mathbf{R},T)\times\mathrm{Re}D(\mathbf{p}_F-\mathbf{p}'_F,E-E') , \tag{3.33}$$

where

$$\mathrm{Re}D=\tfrac{1}{2}(D^R+D^A) . \tag{3.34}$$

Letting the applied electric field $\mathbf{E}$ be

$$\mathbf{E}(t)=\mathbf{E}_0 e^{-i\omega t} \tag{3.35}$$

we seek a solution to Eq. (3.32) of the form $h=h_0+h_1$ with

$$h_1=ae\mathbf{E}\cdot\mathbf{v}_F\partial_E h_0 , \tag{3.36}$$

where the constant a remains to be determined. Inserting Eq. (3.36) into Eq. (3.32) we obtain

$$a=\frac{1}{-i\omega(1+\lambda^*)} , \tag{3.37}$$

where

$$\lambda^*=2N_0\int\frac{d\hat{p}'}{4\pi}\frac{|g_{\mathbf{p}_F-\mathbf{p}'_F}|^2}{\omega_{\mathbf{p}_F-\mathbf{p}'_F}}(1-\hat{p}\cdot\hat{p}') . \tag{3.38}$$

We can now evaluate the current and obtain for the conductivity

$$\sigma(\omega)=\frac{ne^2}{-i\omega m_{\mathrm{opt}}} , \tag{3.39}$$

where n is the number density of electrons and the optical mass m_{opt} is renormalized according to

$$m_{\mathrm{opt}}=m(1+\lambda^*) . \tag{3.40}$$

This result was originally obtained by Holstein (1964). We note that it is the nonequilibrium electron contribution to the real part of the self-energy that makes the optical mass renormalization different from the specific-heat mass renormalization.

D. The excitation representation

The quasiclassical theory leads to equations that are more general than the usual Boltzmann equation. We have shown that the basic variables, besides space and time, are energy and position on the Fermi surface. Although the electron-phonon interaction does not permit the quasiparticle approximation *a priori*, we shall here recapitulate the derivation by Prange and Kadanoff (1964), showing that it is still possible to cast the electron-phonon problem in the Landau-Boltzmann form.

1. The kinetic equation

We start by defining a quasiparticle energy $E_{\mathbf{p}}$, which is given implicitly[18] by

$$E_{\mathbf{p}}=\xi_{\mathbf{p}}+\mathrm{Re}\sigma(E_{\mathbf{p}}+e\varphi,\hat{p},\mathbf{R},T)\ . \tag{3.41}$$

This satisfies the equations

$$\nabla_{\mathbf{p}}E_{\mathbf{p}}=Z_{\mathbf{p}}\nabla_{\mathbf{p}}\xi_{\mathbf{p}}\ , \tag{3.42}$$

$$\nabla_{\mathbf{R}}E_{\mathbf{p}}=Z_{\mathbf{p}}(e\nabla_{\mathbf{R}}\varphi\partial_E\mathrm{Re}\sigma+\nabla_{\mathbf{R}}\mathrm{Re}\sigma)\,|_{E=E_{\mathbf{p}}+e\varphi}\ , \tag{3.43}$$

$$\partial_T E_{\mathbf{p}}=Z_{\mathbf{p}}(e\partial_T\varphi\partial_E\mathrm{Re}\sigma+\partial_T\mathrm{Re}\sigma)\,|_{E=E_{\mathbf{p}}+e\varphi}\ , \tag{3.44}$$

where in (3.42) we have neglected any angular dependence of the real part of the self-energy and introduced the wave-function renormalization constant

$$Z_{\mathbf{p}}=(1-\partial_E\mathrm{Re}\sigma)^{-1}\,|_{E=E_{\mathbf{p}}+e\varphi}\ . \tag{3.45}$$

We now set E equal to $E_{\mathbf{p}}+e\varphi$ in the kinetic equation (3.24) and introduce the distribution function

$$n_{\mathbf{p}}=f(E,\hat{p},\mathbf{R},T)\,|_{E=E_{\mathbf{p}}+e\varphi}\ . \tag{3.46}$$

By using the relations[19]

$$\nabla_{\mathbf{p}}n=\nabla_{\mathbf{p}}E_{\mathbf{p}}(\partial_E f)\,|_{E=E_{\mathbf{p}}+e\varphi}\ , \tag{3.47}$$

$$\nabla_{\mathbf{R}}n=[\nabla_{\mathbf{R}}f+\nabla_{\mathbf{R}}(E_{\mathbf{p}}+e\varphi)\partial_E f]\,|_{E=E_{\mathbf{p}}+e\varphi}\ , \tag{3.48}$$

$$\partial_T n=[\partial_T f+\partial_T(E_{\mathbf{p}}+e\varphi)\partial_E f]\,|_{E=E_{\mathbf{p}}+e\varphi}\ , \tag{3.49}$$

and Eqs. (3.42)—(3.44), we obtain

$$Z_{\mathbf{p}}^{-1}[\partial_T+\nabla_{\mathbf{p}}E_{\mathbf{p}}\cdot\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}}(E_{\mathbf{p}}+e\varphi)\cdot\nabla_{\mathbf{p}}]n_{\mathbf{p}}=\tilde{I}_{e\text{-ph}}[n]\ , \tag{3.50}$$

where

$$\tilde{I}_{e\text{-ph}}[n]=-\frac{2\pi}{N_0}\int\frac{d\mathbf{p}'}{(2\pi)^3}Z_{\mathbf{p}'}\tilde{\mu}(\mathbf{p}-\mathbf{p}')R^{\mathbf{p}}_{\mathbf{p}'}\ , \tag{3.51}$$

and

$$R^{\mathbf{p}}_{\mathbf{p}'}=[1+N(E_{\mathbf{p}}-E_{\mathbf{p}'})]n_{\mathbf{p}}(1-n_{\mathbf{p}'})-N(E_{\mathbf{p}}-E_{\mathbf{p}'})(1-n_{\mathbf{p}})n_{\mathbf{p}'}\ , \tag{3.52}$$

$$\tilde{\mu}(\mathbf{p}-\mathbf{p}')=\frac{iN_0\,|\,g_{\mathbf{p}-\mathbf{p}'}\,|^2}{2\pi}[D^R(\mathbf{p}-\mathbf{p}',E_{\mathbf{p}}-E_{\mathbf{p}'})-D^A(\mathbf{p}-\mathbf{p}',E_{\mathbf{p}}-E_{\mathbf{p}'})]\ . \tag{3.53}$$

[18] We shall suppress the space-time variables and use the shorthand notation $E_{\mathbf{p}}=E(\mathbf{p},\mathbf{R},T)$.

[19] In using (3.47) to transform the gradient term in (3.24) we have ignored the explicit dependence of f on $\hat{p}$. This is necessary to preserve consistency in the quasiclassical scheme in the present form, where the momentum is fixed at the Fermi surface, cf. (3.5).

In transforming the collision integral we have utilized the substitution

$$N_0\int\frac{d\hat{p}'}{4\pi}\int dE'\mapsto N_0\int\frac{d\hat{p}'}{4\pi}\int d\xi_{\mathbf{p}'}\frac{dE_{\mathbf{p}'}}{d\xi_{\mathbf{p}'}}\mapsto\int\frac{d\mathbf{p}'}{(2\pi)^3}Z_{\mathbf{p}'}\ . \tag{3.54}$$

Since the sound velocity is much less than the Fermi velocity, the phonon damping is negligible, and the phonon spectral weight function therefore has a delta-function character

$$\tilde{\mu}(\mathbf{p}-\mathbf{p}')=N_0\,|\,g_{\mathbf{p}-\mathbf{p}'}\,|^2[\delta(E_{\mathbf{p}}-E_{\mathbf{p}'}-\omega_{\mathbf{p}-\mathbf{p}'})-\delta(E_{\mathbf{p}}-E_{\mathbf{p}'}+\omega_{\mathbf{p}-\mathbf{p}'})]\ , \tag{3.55}$$

where $\omega_{\mathbf{q}}$ denotes the phonon frequency. We can then write the kinetic equation in the final form

$$[\partial_T+\nabla_{\mathbf{p}}E_{\mathbf{p}}\cdot\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}}(E_{\mathbf{p}}+e\varphi)\cdot\nabla_{\mathbf{p}}]n_{\mathbf{p}}=I_{e\text{-ph}}[n]\ , \tag{3.56}$$

with

$$I_{e\text{-ph}}[n]=-2\pi\int\frac{d\mathbf{p}'}{(2\pi)^3}Z_{\mathbf{p}}Z_{\mathbf{p}'}\,|\,g_{\mathbf{p}-\mathbf{p}'}\,|^2R^{\mathbf{p}}_{\mathbf{p}'}\times[\delta(E_{\mathbf{p}}-E_{\mathbf{p}'}-\omega_{\mathbf{p}-\mathbf{p}'})-\delta(E_{\mathbf{p}}-E_{\mathbf{p}'}+\omega_{\mathbf{p}-\mathbf{p}'})]\ . \tag{3.57}$$

This has the form of the familiar Boltzmann equation, except for the fact that the matrix elements are renormalized.

We stress that only the quasiclassical approximation was used to derive the above kinetic equation. In particular, we have not assumed any relation between the scattering rate γ and the temperature. This would have been necessary for invoking a quasiparticle description in order to justify the existence of long-lived electronic momentum states. It has thus been established from microscopic theory that the validity of the Boltzmann description of the electron-phonon system is determined not by the Peierls criterion ($\gamma\ll T$), but by the Landau criterion,

$$\gamma\ll\mu\ . \tag{3.58}$$

2. Densities and currents—the conservation law

The Landau-Boltzmann expressions for the density and current density have the well-known form

$$n(\mathbf{R},T)=2\int\frac{d\mathbf{p}}{(2\pi)^3}n_{\mathbf{p}} \tag{3.59}$$

and

$$\mathbf{j}(\mathbf{R},T)=2\int\frac{d\mathbf{p}}{(2\pi)^3}\nabla_{\mathbf{p}}E_{\mathbf{p}}n_{\mathbf{p}}\ . \tag{3.60}$$

The collision integral [Eq. (3.57)] has the invariant

$$\int d\mathbf{p}\,I_{e\text{-ph}}[n]=0\ , \tag{3.61}$$

which expresses the conservation of the number of particles.

Integrating the kinetic equation (3.56) with respect to the momentum shows that the density and current-density expressions (3.59) and (3.60) satisfy the continuity equation

$$\partial_T n + \nabla_{\mathbf{R}} \cdot \mathbf{j} = 0 \ . \tag{3.62}$$

In order to establish that Eqs. (3.59) and (3.60) are indeed the correctly identified densities (in the excitation representation), we should connect one of them with the microscopic expression. According to Sec. III.B we have for the density in the particle representation (we suppress space-time variables)

$$n = 2N_0 \left[\int \frac{d\hat{p}}{4\pi} \int dE \, f(E,\hat{p}) - e\varphi \right]$$

$$= 2N_0 \int \frac{d\hat{p}}{4\pi} \int dE \, f(E + e\varphi, \hat{p}) \ . \tag{3.63}$$

The last equality seems to require that $e\varphi \ll T$. However, this is an artifact of our choice of representation. We could equally well have represented the electric field in a gauge where the scalar potential is absent.

In order to compare the particle and the excitation representations, we transform Eq. (3.59) to the particle representation by using Eqs. (3.41) and (3.54) and obtain

$$2 \int \frac{d\mathbf{p}}{(2\pi)^3} n_{\mathbf{p}} = 2N_0 \int \frac{d\hat{p}}{4\pi} \int dE \, (1 - \partial_E \mathrm{Re}\sigma) f(E,\hat{p}) \ . \tag{3.64}$$

Since Eqs. (3.63) and (3.64) appear to be different, we also transform Eq. (3.60) to the particle representation

$$2 \int \frac{d\mathbf{p}}{(2\pi)^3} \nabla_{\mathbf{p}} E_{\mathbf{p}} n_{\mathbf{p}} = 2eN_0 \int \frac{d\hat{p}}{4\pi} \int dE \, \mathbf{v}_F f(E,\hat{p}) \ . \tag{3.65}$$

Comparing Eq. (3.65) with Eq. (3.30), we observe that this is identical to the quasiclassical current-density expression. The only possibility for the above-mentioned discrepancy not to lead to a violation of the continuity equation is the existence of the identity

$$\partial_T \int d\hat{p} \int dE \, f(E,\hat{p}) \partial_E \mathrm{Re}\sigma(E,\hat{p}) = 0 \ . \tag{3.66}$$

We shall now prove this identity. Inserting the expression from Eq. (3.33) into the left-hand side of Eq. (3.66) we are led to consider

$$\int d\hat{p} \int dE \int d\hat{p}\,' \int dE' \, |g_{\mathbf{p}_F - \mathbf{p}'_F}|^2 \mathrm{Re}D(\mathbf{p}_F - \mathbf{p}'_F, E - E')[\partial_T f(E,\hat{p}) \partial_{E'} f(E',\hat{p}\,') - \partial_E f(E,\hat{p}) \partial_T f(E',\hat{p}\,')] \ . \tag{3.67}$$

By interchanging the variables $E,\hat{p}$ and $E',\hat{p}\,'$ one sees that Eq. (3.67) vanishes and the identity (3.66) is thus established.

E. Beyond the quasiclassical limit

The quasiclassical description of transport phenomena has some limitations. It relies on the assumption of particle-hole symmetry and it is unable to account for the Lorentz force due to a magnetic field. In this section we shall show how these limitations can be avoided within the gradient approximation, using results due to Langreth (1966) and Altshuler (1978). In the next section we treat, as an example, thermoelectric effects in a magnetic field. We only mention here that the effects of a periodic potential may also be included within the present framework. As shown by Rammer (1984), the kq representation (Zak, 1972) may be used in conjunction with field-theoretical methods to derive transport equations for Bloch electrons. One finds, as expected, that the velocity $\mathbf{v}$ in the Lorentz force is replaced by the group velocity $\partial\varepsilon_n(\mathbf{k})/\partial\mathbf{k}$, where $\varepsilon_n(\mathbf{k})$ is the band energy, provided the distribution matrix is assumed to be diagonal in the band index n. In the general case, the transport equation obtained by this method contains interband terms and a more complicated form of the Lorentz force. For details the reader is referred to Rammer (1984).

We start from the general equation (2.57)

$$[\underline{G}_0^{-1} - \underline{\Sigma} \underset{,}{\otimes} \underline{G}]_- = 0 \tag{3.68}$$

with $\underline{G}_0^{-1}$ given by Eqs. (2.47) and (2.52). A distribution function is introduced by the ansatz[20]

$$G^K = G^R \otimes h - h \otimes G^A \ . \tag{3.69}$$

To obtain a kinetic equation we take the Keldysh component of Eq. (3.68). Utilizing the equation of motion for the diagonal (retarded and advanced) components of Eq. (3.68) and the property that the composition $\otimes$ is associative, we can write the equation in the form

$$G^R \otimes B - B \otimes G^A = 0 \ , \tag{3.70}$$

where

$$B[h] = -i[G_0^{-1} - \mathrm{Re}\Sigma \underset{,}{\otimes} h]_- + \tfrac{1}{2}[\Gamma \underset{,}{\otimes} h]_+ - i\Sigma^K \ . \tag{3.71}$$

In the gradient approximation we obtain

$$(G^R - G^A)B + [B, \mathrm{Re}G]_p = 0 \ . \tag{3.72}$$

Inserting the solution of the equation

$$(G^R - G^A)B = 0 \tag{3.73}$$

[20] In the desired approximation, such a representation is always possible. In the mixed representation we have $G^K = (G^R - G^A)h + i[\mathrm{Re}G, h]_p$.

into Eq. (3.72) we observe that the second term of Eq. (3.72) has the form of a double Poisson bracket and thus can be dropped in the gradient approximation. We thus seek solutions of the equation

$$B[h]=0 . \tag{3.74}$$

However, since the distribution function is not gauge invariant, we shall not succeed in obtaining an appropriate kinetic equation with the usual expression for the Lorentz force. Performing a gradient expansion of the term in (3.71) containing G_0^{-1}, we obtain in the mixed representation

$$-i[G_0^{-1}\underset{\circ}{\otimes}h]_-=[E-e\varphi-\xi_{\mathbf{p}-e\mathbf{A}},h]_p , \tag{3.75}$$

a result first derived by Langreth (1966).

Within the gradient approximation, we thus can introduce a gauge-invariant distribution function $\tilde{h}$, defined by

$$\tilde{h}(\Omega,\mathbf{P},\mathbf{R},T)=h(E,\mathbf{p},\mathbf{R},T) , \tag{3.76}$$

by the change of variables

$$\mathbf{P}=\mathbf{p}-e\mathbf{A}(\mathbf{R},T),\quad \Omega=E-e\varphi(\mathbf{R},T) . \tag{3.77}$$

As a result of the transformation, we observe that it is the kinematic (not the canonical) momentum that enters the kinetic equation, as one would expect. Using the identity[21] (Langreth, 1966)

$$[A,B]_{\mathbf{p},E}=[\tilde{A},\tilde{B}]_{\mathbf{P},\Omega}+e\mathbf{E}\cdot(\partial_\Omega\tilde{A}\nabla_{\mathbf{P}}\tilde{B}-\partial_\Omega\tilde{B}\nabla_{\mathbf{P}}\tilde{A}) +e\mathbf{H}\cdot(\nabla_{\mathbf{P}}\tilde{A}\times\nabla_{\mathbf{P}}\tilde{B}) , \tag{3.78}$$

where $\mathbf{H}=rot\,\mathbf{A}$ and $\mathbf{E}=-\nabla\varphi-\partial\mathbf{A}/\partial t$, and $\tilde{A}$ and $\tilde{B}$ are related to A and B by equations analogous to Eq. (3.76), we get the following driving terms in the gradient approximation:

$$-i[G_0^{-1}-\mathrm{Re}\Sigma\underset{\circ}{\otimes}h]_-=[\Omega-\xi_{\mathbf{P}}-\mathrm{Re}\tilde{\Sigma},\tilde{h}]_{\mathbf{P},\Omega} +e\mathbf{E}\cdot[(1-\partial_\Omega\mathrm{Re}\tilde{\Sigma})\nabla_{\mathbf{P}}\tilde{h}+\mathbf{v}^*\partial_\Omega\tilde{h}] +e\mathbf{v}^*\times\mathbf{H}\cdot\nabla_{\mathbf{P}}\tilde{h} , \tag{3.79}$$

where we have introduced

$$\mathbf{v}^*=\nabla_{\mathbf{P}}[\xi_{\mathbf{P}}+\mathrm{Re}\tilde{\Sigma}(\Omega,\mathbf{P},\mathbf{R},T)] . \tag{3.80}$$

We could equally well, following Altshuler (1978), have chosen to introduce the mixed representation

$$G(X,p)=\int dx\, e^{-ir\cdot[\mathbf{p}+e\mathbf{A}(X)]+it[E+e\varphi(X)]}G(X,x) , \tag{3.81}$$

whereupon, in accordance with Eq. (3.78), we can express the Poisson bracket as

$$[A,B]_p=\partial_E A\{\partial_T+\mathbf{u}\cdot\nabla_{\mathbf{R}}+[e\mathbf{E}\cdot\mathbf{u}-(\partial_E A)^{-1}\partial_T A]\partial_E +[e\mathbf{E}+e\mathbf{v}\times\mathbf{H}+(\partial_E A)^{-1}\nabla_{\mathbf{R}}A]\cdot\nabla_{\mathbf{p}}\}B , \tag{3.82}$$

with

$$\mathbf{u}=(\partial_E A)^{-1}\nabla_{\mathbf{p}}A . \tag{3.83}$$

The kinetic equation thus has the form

$$[(1-\partial_E\mathrm{Re}\Sigma)\partial_T+\partial_T\mathrm{Re}\Sigma\partial_E+\mathbf{v}^*\cdot(\nabla_{\mathbf{R}}+e\mathbf{E}\partial_E) +(e\mathbf{E}+e\mathbf{v}^*\times\mathbf{H})\cdot\nabla_{\mathbf{p}}]h=I[h] , \tag{3.84}$$

where the collision integral is given by

$$I[h]=+i\Sigma^K-\Gamma h . \tag{3.85}$$

F. An example: Electron-phonon renormalization of the high-field Nernst-Ettingshausen coefficient

As an example of electron-phonon renormalization of a static transport coefficient we shall consider the high-field Nernst-Ettingshausen coefficient, which relates the current density to the vector product of the temperature gradient and the magnetic field. For now, we shall neglect any momentum dependence of the self-energy. The system is driven out of equilibrium by a temperature gradient. The magnetic field is assumed to satisfy the condition[22]

$$\gamma\ll\omega_c , \tag{3.86}$$

where γ is the collision rate and ω_c is the cyclotron frequency $\omega_c=|e|H/m$, so that the collision integral can be neglected. The kinetic equation (3.84) then reduces to

$$[\mathbf{v}\cdot\nabla_{\mathbf{R}}+e(\mathbf{v}\times\mathbf{H})\cdot\nabla_{\mathbf{p}}]h=0 . \tag{3.87}$$

It follows from Eq. (2.85) that the electric current density in the gradient approximation is given by

$$\mathbf{j}=-e\int\frac{d\mathbf{p}}{(2\pi)^3}\int\frac{dE}{2\pi}\mathbf{v}(Ah-[\mathrm{Re}G,h]_p) . \tag{3.88}$$

In accordance with Eqs. (3.86) and (3.87) the last term in Eq. (3.88) can be neglected, since

$$[\mathrm{Re}G,h]_p=-\nabla_{\mathbf{p}}\mathrm{Re}G\cdot\nabla_{\mathbf{R}}h+e\mathbf{H}\cdot(\nabla_{\mathbf{p}}\mathrm{Re}G\times\nabla_{\mathbf{p}}h) =-\frac{\partial\,\mathrm{Re}G}{\partial\xi}[\mathbf{v}\cdot\nabla_{\mathbf{R}}h+e(\mathbf{v}\times\mathbf{H})\cdot\nabla_{\mathbf{p}}h] =0 . \tag{3.89}$$

Inserting the solution of Eq. (3.87),

[21] We have changed the subscript on the Poisson bracket to indicate the variables with respect to which the derivatives are taken.

[22] Describing the kinetics in the momentum representation assumes we are not in the quantum limit, that is, $\omega_c\ll T$.

$$h = h_0 - \frac{|\nabla T|}{eHT} p_y E \frac{\partial h_0}{\partial E} , \tag{3.90}$$

into the current expression and performing a Sommerfeld expansion,[23] we obtain

$$\mathbf{j} = \frac{(1+\lambda)S^0}{H^2} \nabla T \times \mathbf{H} , \tag{3.91}$$

where

$$\lambda = - \left. \frac{\partial \mathrm{Re}\Sigma}{\partial E} \right|_{E=0, \mathbf{p}=\mathbf{p}_F} , \tag{3.92}$$

and S^0 is the free-electron entropy. In the model described by Eq. (2.34) one has $\lambda = g^2 N_0$. Thus the enhancement of the high-field thermoelectric current in Eq. (3.91) is seen to be identical to the enhancement of the equilibrium specific heat. This result was conjectured by Lyo (1977) and demonstrated microscopically by Hänsch and Mahan (1983b).

Up to this point, we have not considered the possible momentum dependence of the self-energy. Taking this into account, we find that Eqs. (3.84)—(3.88) lead to correction terms to the coefficient in Eq. (3.91) of the type $\nabla_{\mathbf{p}} \mathrm{Re}\Sigma \,|_{\mathbf{p}=\mathbf{p}_F}$. However, the nonequilibrium contribution to the spectral weight is difficult to extract. A calculation within the context of Fermi-liquid theory leads to the appearance of two $\nabla_{\mathbf{p}} \mathrm{Re}\Sigma$-dependent terms that exactly cancel each other, thus suggesting the resulting equation (3.91) to be generally valid (further details may be found in Rammer, 1984).

Thermopower measurements by Opsal, Thaler, and Bass (1976) and Fletcher (1976) agree with the calculated mass enhancement in Eq. (3.91) due to the electron-phonon interaction.

IV. KINETIC EQUATIONS FOR NORMAL METALS

As examples of the use of field-theoretical methods for obtaining kinetic equations, we now discuss some effects of impurity scattering that have attracted considerable interest over the last few years. Recent reviews of these effects on localization and interaction in disordered systems have been given by Altshuler *et al.* (1982), Bergmann (1984), Altshuler and Aronov (1985), and Lee and Ramakrishnan (1985). Most of the work on these problems have taken the point of view of linear-response theory. Our objective here is to show how such effects may be incorporated in the kinetic equation derived by the Keldysh technique. We emphasize, however, that the linear-response method is often more convenient than the kinetic equation for deriving the results discussed below, and we shall therefore show how linear-response theory is formulated using the Keldysh technique.

[23]The coefficient relating the current density $\mathbf{j}$ to $\nabla T \times \hat{H}$ is the Nernst-Ettingshausen coefficient.

A. Impurity scattering and weak localization

The subject of weak localization in metals dates back to the pioneer work of Abrahams, Anderson, Licciardello, and Ramakrishnan (1979), who proposed a scaling theory for conductivity in disordered metals based on earlier ideas of Thouless (1977). A large amount of subsequent theoretical and experimental work has given support to the scaling approach to localization. Here we first discuss weak localization within linear-response theory using Keldysh propagators. Subsequently, we demonstrate how the effects of weak localization may be included in the kinetic equation derived by use of the Keldysh technique. We limit ourselves to considering the effects of weak localization on the frequency-dependent conductivity, as first obtained by Gorkov, Larkin, and Khmel'nitskii (1979), who used diagrammatic methods within linear-response theory. The Keldysh technique was employed by Vasko (1983) to treat nonlinear effects of the electric field. The derivation given in Sec. IV.A.2 essentially follows his approach.

1. Linear-response theory

In this section, we employ the linear-response method using Keldysh propagators to determine the effects of weak localization on the frequency-dependent conductivity. In the following section (IV.A.2), we derive the same result by the kinetic equation method. The latter is the more general method, since it allows, in principle, all nonlinear effects to be considered.

As a model for the impurities, we take a random potential $v(\mathbf{r})$ with zero mean value and a correlator $\langle v(\mathbf{r}) v(\mathbf{r}') \rangle$ with white-noise character, corresponding to the relations

$$\langle v \rangle = 0, \quad \langle v(\mathbf{r}) v(\mathbf{r}') \rangle = u^2 \delta(\mathbf{r} - \mathbf{r}') , \tag{4.1}$$

where $\langle \ \rangle$ denotes the impurity averaging. The constant u is a convenient measure of the strength of the impurity scattering, u^2 being equal to the square of the matrix element for scattering from a single impurity times the number of impurities per volume.

Linear-response theory may be formulated quite simply in terms of Keldysh propagators, since the current, Eq. (2.85) via Eq. (2.86), is determined by calculating the change in the Keldysh component of the matrix propagator to linear order in the external field, as we shall now demonstrate.

The use of Keldysh propagators makes the formulation at finite temperature analogous to the one used at zero temperature (cf. Abrikosov *et al.*, 1965). It is convenient to express the electric field by a vector potential $\mathbf{A} = \mathbf{A}_\omega e^{-i\omega t}$, where $\mathbf{A}_\omega = \mathbf{E}/i\omega$. The vector potential contributes a term of the form $\mathbf{j} \cdot \mathbf{A}$ to the Hamiltonian. In order to determine the corresponding first-order contribution G_1^K to the Keldysh component of the Green's

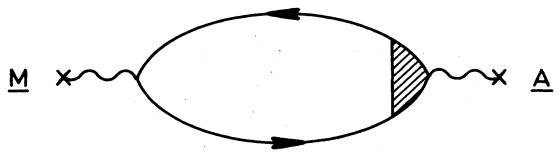


FIG. 9. The conductivity diagram.

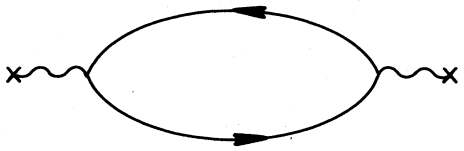


FIG. 10. The classical conductivity diagram.

function, we include for the moment the impurity potential in the free-particle Hamiltonian H_0, and since we do not consider interaction effects, we have $H^i=0$ and $S_{ci}=1$ in Eq. (2.16). Then we can repeat the steps given in Sec. II.C, Eqs. (2.28)—(2.30), with the vector potential replacing the scalar potential, as we expand the operator

$$S_c=\exp\left[-i\int_c d\tau\int d\mathbf{x}\,\mathbf{j}\cdot\mathbf{A}\right],$$

Eq. (2.17), to first order in $\mathbf{A}$ and express the current $\mathbf{j}$ in second-quantized form (within linear response, we need only consider the paramagnetic term in $\mathbf{j}$). As a result, we get

$$G_1^K(1,1')=\frac{ie}{2m}\mathrm{Tr}\left[\tau^1\int d2\,\mathbf{A}(2)[(\nabla_{\mathbf{x}_2}-\nabla_{\mathbf{x}_{2'}})\underline{G}(1,2')\underline{G}(2,1')]_{2'=2}\right]. \tag{4.2}$$

The trace in Keldysh space and the τ^1 matrix have been introduced to pick out the Keldysh component. To linear order in the electric field the current is thus

$$\mathbf{J}_1(1)=\frac{-e}{2m}(\nabla_1-\nabla_{1'})G_1^K(1,1')\,|_{1'=1}-\frac{ne^2}{m}\mathbf{A}, \tag{4.3}$$

where the last term is the usual diamagnetic current (cf. Abrikosov *et al.*, 1965). We now Fourier-transform Eq. (4.3) with the result

$$\mathbf{J}_1(\omega)=\frac{ie^2}{2\pi m^2}\mathrm{Tr}\left[\tau^1\int dE\sum_{\mathbf{pp}'}\mathbf{p}\underline{G}(\mathbf{p},\mathbf{p}',E+\omega)\mathbf{A}_\omega\cdot\mathbf{p}'\underline{G}(\mathbf{p}',\mathbf{p},E)\right]-\frac{ne^2}{m}\mathbf{A}_\omega. \tag{4.4}$$

Under isotropic conditions, the conductivity $\sigma(\omega)$ then becomes

$$\sigma(\omega)=\frac{e^2}{2\pi dm^2\omega}\mathrm{Tr}\left[\tau^1\int dE\sum_{\mathbf{pp}'}\mathbf{p}\cdot\mathbf{p}'\langle\underline{G}(\mathbf{p},\mathbf{p}',E+\omega)\underline{G}(\mathbf{p}',\mathbf{p},E)\rangle\right]-\frac{ne^2}{i\omega m}. \tag{4.5}$$

In Eq. (4.5) we have introduced the impurity averaging denoted by $\langle\ \rangle$.

We have thus demonstrated that, within the Keldysh formulation of linear response, the Feynman rules for interpreting the standard conductivity diagrams need only to be supplemented in accordance with the matrix structure in Keldysh space, where lines and vertices are interpreted as tensors, as explained in Sec. II. The structure of the first term in Eq. (4.5) is illustrated diagrammatically in Fig. 9. Note that in the representation given by Eq. (2.27) the Keldysh structure of the driving and measuring vertex in Fig. 9 is represented by

$$\underline{A}=\tau^0,\quad \underline{M}=\tau^1, \tag{4.6}$$

in accordance with Eq. (4.2). Finally, the closed loop implies that the trace in Keldysh space is to be taken.

The last term on the right-hand side of Eq. (4.5) diverges as $\omega\to 0$ and must cancel a similar term appearing in the first part of Eq. (4.5). To verify the cancellation of these terms, it is sufficient to consider the simplest diagram, depicted in Fig. 10, for which we obtain[24]

$$\sigma(\omega)+\frac{ne^2}{i\omega m}=\frac{e^2}{d\omega m^2}\int(dp)\mathbf{p}^2[(h^0_{E+\omega}-h^0_E)G^R_{p+q}G^A_p+G^A_{p+q}G^A_p(1-h^0_{E+\omega})+G^R_{p+q}G^R_p(h^0_E-1)], \tag{4.7}$$

[24]In this section we use the shorthand notation

$$\int(dp)=\int(d\mathbf{p}\,dE)=\int\frac{dE}{2\pi}\int(d\mathbf{p})=\int\frac{dE}{2\pi}\int\frac{d^d\mathbf{p}}{(2\pi)^d},$$

where d is the dimension.

where

$$q=(\mathbf{0},\omega)\ . \tag{4.8}$$

Exploiting the Ward identity for the vertex function, which in our approximation simply reads

$$\mathbf{p}=-m\nabla_{\mathbf{p}}(G_p^{R(A)})^{-1}\ , \tag{4.9}$$

we readily verify the cancellation of the diverging term on the left-hand side of Eq. (4.7) by the last two terms on the right-hand side. We are thus left with the expression

$$\sigma(\omega)=\frac{2e^2}{dm^2}\int(dE)\sum_{\mathbf{p},\mathbf{p}'}\mathbf{p}\cdot\mathbf{p}'\frac{f_E^0-f_{E+\omega}^0}{\omega}\times\langle G^R(\mathbf{p},\mathbf{p}',E+\omega)G^A(\mathbf{p}',\mathbf{p},E)\rangle\ , \tag{4.10}$$

which forms a useful starting point for calculations of the conductivity.

Before considering the quantum correction, we show how the classical conductivity given by the diagram in Fig. 10 is obtained. From Eq. (4.10), it follows that

$$\sigma(\omega)=\frac{2e^2}{dm^2}\int(dp)\frac{f_E^0-f_{E+\omega}^0}{\omega}\mathbf{p}^2G_{p+q}^RG_p^A\ . \tag{4.11}$$

The retarded and advanced impurity-averaged Green's functions are given by (Abrikosov *et al.*, 1965)

$$G_{\mathbf{k},E}^{R(A)}=\frac{1}{E-\xi_{k(-)}^{+}\frac{i}{2\tau}}\ , \tag{4.12}$$

where $1/\tau$ is the impurity scattering rate

$$\frac{1}{\tau}=2\pi u^2N_d(0)\ . \tag{4.13}$$

In Eq. (4.13), $N_d(0)$ denotes the density of states for a single spin at the Fermi energy in d dimensions.

In the limit $\omega\ll\mu$ that we consider, the main contribution to the momentum integration in Eq. (4.11) comes from the region near the Fermi surface. We perform the momentum integration and then the energy integration, and obtain the Drude result

$$\sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}\ , \tag{4.14}$$

where the static conductivity σ_0 is

$$\sigma_0=\frac{ne^2\tau}{m}\ . \tag{4.15}$$

In the s-wave approximation we are considering, there is no difference between the transport time occurring in the conductivity and the lifetime τ occurring in the Green's function (4.12), since the contribution to the conductivity from the particle-hole ladder gives zero in this case (see, for example, Abrikosov *et al.*, 1965).

Langer and Neal (1966) found in their study of the dependence of resistivity on impurity concentration that the leading contribution at each order in the concentration is attributed to the maximally crossed diagrams of the type shown in Fig. 11. In two and three dimensions, they observed that the perturbation expansion breaks down in each order larger than or equal to d. In contrast to the analysis by Langer and Neal (1966), one should

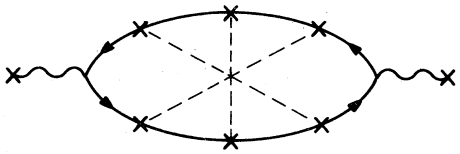


FIG. 11. A maximally crossed diagram.

sum the whole class of maximally crossed diagrams (Abrahams *et al.*, 1979; Gorkov *et al.*, 1979). Twisting the hole line in the diagram of Fig. 11, we observe that a maximally crossed diagram in the particle-hole channel becomes a ladder-type diagram in the particle-particle channel, as exhibited in Fig. 12.

Due to phase space constraints, crossed diagrams are generally of order $(k_Fl)^{-1}$, where $l=v_F\tau$ is the mean free path, relative to those that do not contain crossed impurity lines (Abrikosov *et al.*, 1965). The first quantum correction, as defined by the expansion in the disorder parameter $(k_Fl)^{-1}$, is thus given by the diagram in Fig. 13, where the particle-particle impurity ladder satisfies the equation depicted in Fig. 14. The particle-particle impurity ladder C is often called the Cooperon. To explain its properties, we first discuss the diffusive behavior characterizing the particle-hole ladder in the limit of low frequencies and long wavelengths.

The particle-hole impurity ladder satisfies the integral equation illustrated in Fig. 15. The top and bottom horizontal lines denote retarded (R) and advanced (A) propagators, respectively. To solve the integral equation, we shall need the function $\zeta=\zeta(\mathbf{q},\omega)$, defined from the appropriate insertion in the diagram of Fig. 15, and corresponding to the expression

$$\zeta(\mathbf{q},\omega)=u^2\int\frac{d\mathbf{k}}{(2\pi)^d}G_{\mathbf{k},E}^RG_{\mathbf{k}-\mathbf{q},E-\omega}^A\ , \tag{4.16}$$

where we have introduced the dimensionality d (=3, 2, or 1, corresponding to three, two, or one dimension). Using Eq. (4.12) for $G^{R(A)}$ and performing the integral over $\mathbf{k}$, one gets for $q\ll k_F$ and $d=3$

$$\zeta=\frac{i}{2ql}\ln\frac{ql+\omega\tau+i}{-ql+\omega\tau+i}\ . \tag{4.17}$$

Generally, in the limit $ql\ll1$, $\omega\tau\ll1$, we get that $\zeta\simeq1+i\omega\tau-D\tau q^2$, where the diffusion coefficient $D=(1/d)v_F^2\tau$ has been introduced. The particle-hole ladder is proportional to the geometric series $(1-\zeta)^{-1}$, which in the limit of long wavelengths and low frequencies becomes

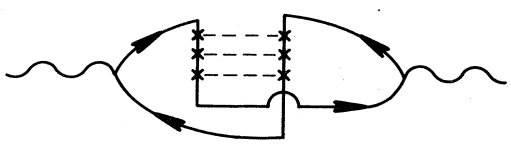


FIG. 12. The diagram of Fig. 11, displayed in the particle-particle channel.

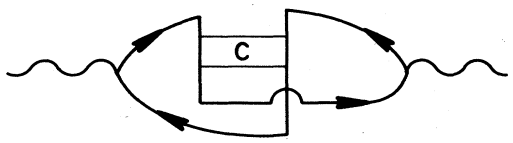


FIG. 13. The quantum correction to the conductivity determined by the Cooperon C.

which in the limit of long wavelengths and low frequencies becomes

$$\frac{1}{1-\xi}=\frac{1}{\tau}\frac{1}{-i\omega+Dq^2}\ . \tag{4.18}$$

The diffusive pole [Eq. (4.18)] in the particle-hole propagator is a consequence of the conservation of the number of particles and is thus a general property of the system in the limit of low frequencies and long wavelengths.

We are now ready to determine the form of the particle-particle impurity ladder C for small total momentum, since we see by comparing Figs. 16 and 17 that time-reversing the lower line in Fig. 16, for instance, brings us to the particle-hole impurity ladder, the hole line having reversed momenta. If time-reversal invariance holds, the values of the diagrams in Figs. 16 and 17 are therefore identical. The particle-hole impurity ladder has the diffusion pole exhibited in Eq. (4.18) for small energy and momentum transfer, and the Cooperon thus has a diffusion-type pole for small total momentum and small energy transfer:

$$C_\omega(\mathbf{p}+\mathbf{p}')=\frac{u^2}{\tau[-i\omega+D(\mathbf{p}+\mathbf{p}')^2]}\ ,$$
$$|\mathbf{p}+\mathbf{p}'|\,l\ll 1,\ \omega\tau\ll 1\ . \tag{4.19}$$

We therefore have the following contribution $\delta\sigma$ to the conductivity from the diagram depicted in Fig. 13:

$$\delta\sigma(\omega)=\frac{e^2}{\pi dm^2}\sum_{\mathbf{pp}'}\mathbf{p}\cdot\mathbf{p}'G_p^RG_p^RG_p^AG_{p'}^AC_\omega(\mathbf{p}+\mathbf{p}')\ . \tag{4.20}$$

In Eq. (4.20) we have performed the integration over E, since the momentum integrations leave Eq. (4.20) independent of E, and used $\omega\tau\ll 1$ [the label p' thus denotes $(\mathbf{p}',E)$]. The important contribution to the integral in Eq. (4.20) comes from regions of small total momentum $\mathbf{p}+\mathbf{p}'$. We therefore introduce the variable $\mathbf{Q}=\mathbf{p}+\mathbf{p}'$ and replace the $\mathbf{p}\cdot\mathbf{p}'$ factor by $-p_F^2$, since the important contribution comes from momenta Q, satisfying $Q\ll l^{-1}\ll k_F$. Similarly, we may replace $\mathbf{p}'$ by $-\mathbf{p}$

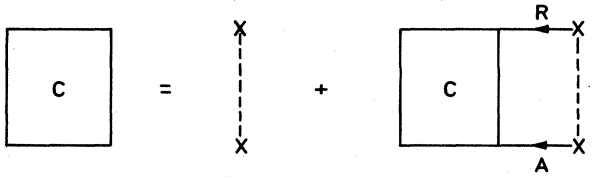


FIG. 14. The ladder equation for the Cooperon.

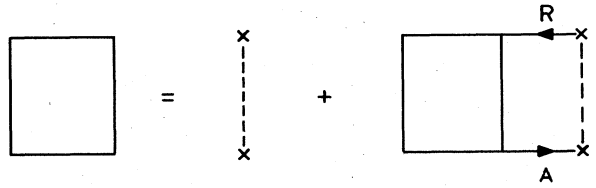


FIG. 15. The particle-hole ladder equation.

in the Green's functions $G_{p'}^{R(A)}$ that are given by Eq. (4.12). After turning the integral over $\mathbf{p}$ into an integral over ξ, using the peaked character of the integrand at the Fermi surface, we easily perform the ξ integral by use of the residue method. The result for $\delta\sigma$ is

$$\frac{\delta\sigma}{\sigma_0}=-\frac{1}{\pi N_d}\int(d\mathbf{Q})\frac{1}{-i\omega+DQ^2}\ . \tag{4.21}$$

In two dimensions we find, by evaluating Eq. (4.21), cutting off the integral over Q at the upper limit l^{-1}, that the first quantum correction to the conductivity is

$$\frac{\delta\sigma}{\sigma_0}=\frac{1}{\pi k_F l}\ln\omega\tau\ , \tag{4.22}$$

in agreement with Gorkov *et al.* (1979).

It is worthwhile to emphasize the crucial role played by time-reversal invariance, which is needed to ensure that the value of the diagram exhibited in Fig. 17 is unchanged when the hole lines are turned into electron lines with opposite momentum. If the time-reversal invariance is violated by perturbations such as a magnetic field or magnetic impurities, the correspondence between the two diagrams no longer holds. Consequently, these perturbations exert a marked influence on the weak localization effects in the electrical conductivity. By now there exists a large body of experimental results in evidence of such effects (see, for example, Bergmann, 1984 and Lee and Ramakrishnan, 1985). The significance of the weak localization effect as a tool for investigating material properties rests on the fact that it is sensitive to variations in, for example, a magnetic field on a scale much smaller than that which determines the classical magnetoresistance effects.

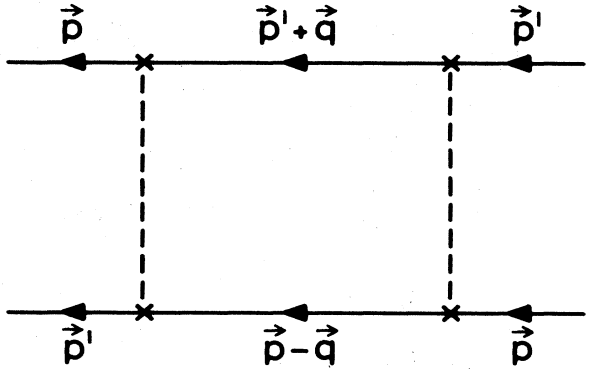


FIG. 16. A particle-particle channel diagram.

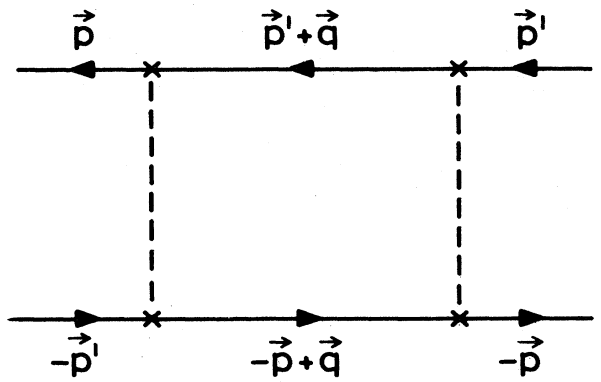


FIG. 17. The diagram of Fig. 16, with time-reserved lower line.

It is therefore possible to separate the weak localization effect by measuring magnetoresistance, and from such measurements extract values for characteristic scattering rates in different materials.

2. The kinetic equation

To derive a kinetic equation that includes the effects of weak localization, we must consider the effect of the Cooperon on the self-energy $\underline{\Sigma}$. The maximally crossed self-energy diagrams $\underline{\Sigma}_{\rm mc}$ are given by a series consisting of terms of the type shown in Fig. 18. We write

$$\underline{\Sigma}_{\rm mc}=\sum_{n=2}^{\infty}\underline{\Sigma}^{n}_{\rm mc}\,, \tag{4.23}$$

where $\underline{\Sigma}^{n}_{\rm mc}$ contains n impurity correlators (Fig. 18). A

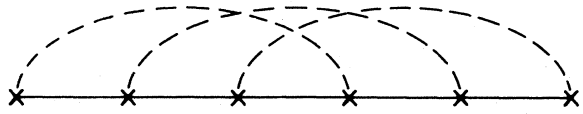


FIG. 18. A maximally crossed self-energy diagram.

self-energy diagram with n impurity correlators has the structure

$$\underline{\Sigma}^{n}_{\rm mc}=\text{const}\times\begin{bmatrix}(G^R)^{2n-1} & \sum_{p=0}^{2n-2}(G^R)^pG^K(G^A)^{2(n-1)-p}\\ 0 & (G^A)^{2n-1}\end{bmatrix} \tag{4.24}$$

provided $n\geq 2$. The structure (4.24) may be verified by raising the $\underline{G}$ matrix to the power $2n-1$. The Keldysh component of $\underline{\Sigma}^{n}_{\rm mc}$ contains the Cooperon coupled to the Keldysh component of the Green's function as illustrated diagrammatically in Fig. 19, which shows the contribution of the maximally crossed diagrams corresponding to the off-diagonal term in Eq. (4.24), with an equal number of retarded and advanced Green's functions ($p=n-1$). Algebraically, we thus have the weak localization contribution

$$\Sigma^K_{\rm wl}(1,1')=u^4\int dt_2\int dt_{2'}\tilde{C}(1,2\,|\,1',2')G^K(2,2')\,. \tag{4.25}$$

The coefficient u^4 in Eq. (4.25) arises from the impurity correlators at the ends of the self-energy diagram (Fig. 19). The quantity $\tilde{C}$ satisfies the integral equation depicted diagrammatically in Fig. 20.[25] In algebraic form, the integral equation is

$$\tilde{C}(1,2\,|\,2',1')=\int d\mathbf{x}_3\int dt_3\int dt_{3'}G^R(1,3)G^A(2',3')[\delta(t_3-t_2)\delta(t_{3'}-t_{1'})\delta(\mathbf{x}_3-\mathbf{x}_{1'})+u^2\tilde{C}(3,2\,|\,3',1')]\,, \tag{4.26}$$

with

$$\begin{aligned}&1=(\mathbf{x}_1,t_1),\quad 1'=(\mathbf{x}_{1'},t_{1'})\,,\\ &2=(\mathbf{x}_{1'},t_2),\quad 2'=(\mathbf{x}_1,t_{2'})\,,\\ &3=(\mathbf{x}_3,t_3),\quad 3'=(\mathbf{x}_3,t_{3'})\,.\end{aligned} \tag{4.27}$$

For the purpose of considering the kinetic equation we transform to a mixed representation with respect to time,

$$\begin{aligned}&T=\frac{t_1+t_{1'}}{2},\quad t=t_1-t_{1'}\,,\\ &T'=\frac{t_2+t_{2'}}{2},\quad t'=t_2-t_{2'}\,.\end{aligned} \tag{4.28}$$

The expression, Eq. (4.25), for the Keldysh component of the self-energy with $\tilde{C}$ given by Eq. (4.26) is equivalent to the summation of the maximally crossed conductivity diagrams shown in Figs. 13 and 14.

Equation (4.25) is transformed to the mixed representation according to

$$\Sigma^K_{\rm wl}(E,\mathbf{p},\mathbf{R},T)=u^4\int(d\mathbf{k}\,dE')dT'\tilde{C}(\mathbf{k},\mathbf{R},E,E',T,T')\times G^K(\mathbf{k}-\mathbf{p},\mathbf{R},E',T')\,, \tag{4.29}$$

where the energy variables E,E' enter by Fourier-transforming the relative time variables t,t' introduced in Eq. (4.28).

In order to determine the contributions of $\Sigma^K_{\rm wl}$ to the conductivity, we first consider the kinetic equation (2.60),

$$[G_0^{-1}-{\rm Re}\Sigma,G^K]_p-[\Sigma^K,{\rm Re}G]_p=i\Sigma^K(G^R-G^A)-i(\Sigma^R-\Sigma^A)G^K\,, \tag{4.30}$$

where we have performed the gradient expansion given by Eqs. (2.69) and (2.70). Instead of introducing a distribution function in the usual manner, we may solve the kinetic equation directly by treating $\Sigma^K_{\rm wl}$ as a perturbation in comparison to the classical impurity contribution to the

[25] Thus C is obtained from $\tilde{C}$ by removing the external legs.

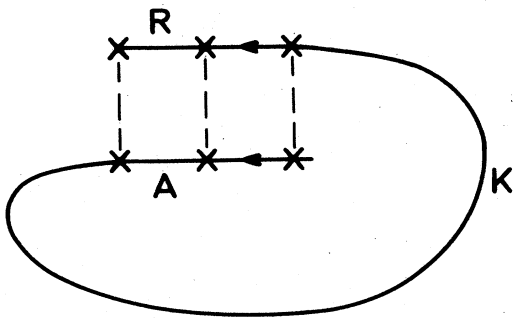


FIG. 19. The Keldysh component of the self-energy diagram, Fig. 18, containing the Cooperon.

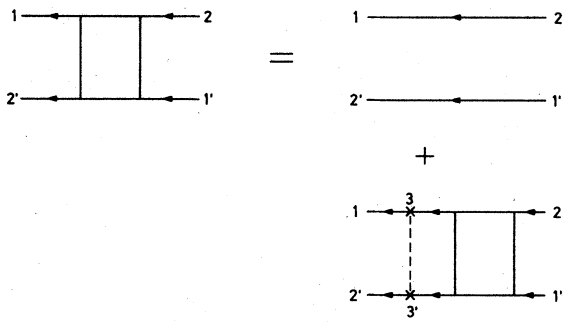


FIG. 20. The Cooperon equation for $\tilde{C}$.

self-energy $\Sigma_{\rm imp}$, which is given by the diagram shown in Fig. 7. When the kinetic equation (4.30) is linearized in the electric field, we obtain the perturbation solution by setting the change of the right-hand side due to the localization contribution equal to zero. The perturbation result thus becomes

$$G^K_{\rm wl}=\frac{(G^R-G^A)_{\rm imp}}{(\Sigma^R-\Sigma^A)_{\rm imp}}\Sigma^K_{\rm wl}$$
$$=i\tau(G^R-G^A)_{\rm imp}\Sigma^K_{\rm wl}\,, \tag{4.31}$$

where the subscript imp denotes quantities containing only the self-energy diagram, Fig. 7, corresponding to the solution of the usual impurity problem based on Eq. (2.72). The quantum correction $\delta\mathbf{j}$ to the current density is

$$\delta\mathbf{j}=-ie\int(dp)\mathbf{v}G^K_{\rm wl}$$
$$=\tau e\int(dp)\mathbf{v}(G^R-G^A)_{\rm imp}$$
$$\times u^4\int(d\mathbf{k})dT'\tilde{C}_{\mathbf{k}}(T,T')G^K_{\rm imp}(\mathbf{k}-\mathbf{p},E,T')\,, \tag{4.32}$$

where

$$G^K_{\rm imp}(\mathbf{p},E,T)=G^K_0-(G^R-G^A)_{\rm imp}\tau\frac{e}{m}\mathbf{E}\cdot\mathbf{p}\frac{\partial h_0}{\partial E} \tag{4.33}$$

is the Keldysh component of $\underline{G}$ without the weak localization contribution.

In writing Eq. (4.32), we have used the fact that the dominant contribution occurs for small frequencies and long wavelengths and that the Cooperon in this range has the form

$$\tilde{C}(\mathbf{k},T,T',t,t')=\delta(t-t')\tilde{C}_{\mathbf{k}}(T,T')\,, \tag{4.34}$$

where $\tilde{C}_{\mathbf{k}}(T,T')$ satisfies

$$[\tfrac{1}{2}(\partial_T-\partial_{T'})+Dk^2]\tilde{C}_{\mathbf{k}}(T,T')=\frac{1}{u^2\tau}\delta(T-T')\,. \tag{4.35}$$

Equations (4.34) and (4.35) may be proven as follows. To obtain Eq. (4.35), we Fourier transform with respect to the space variables in Eq. (4.27), and introduce the time variables in Eq. (4.28) in addition to the new internal variables $\tilde{t}=t_3-t_1$ and $\tilde{t}'=t_2-t_{3'}$.

In linear response, we neglect the influence of the time-dependent electric field on the Cooperon and obtain

$$\tilde{C}(\mathbf{k},T,T',t,t')=\int(d\mathbf{p}\,dE_1dE_2)G^R_{\mathbf{p},E_1}G^A_{\mathbf{k}-\mathbf{p},E_2}$$
$$\times\Big[\exp\{-i[(E_1-E_2)(T-T')+\tfrac{1}{2}(E_1+E_2)(t-t')]\}$$
$$+u^2\int d\tilde{t}\int d\tilde{t}'e^{i(E_1\tilde{t}-E_2\tilde{t}')}\tilde{C}(\mathbf{k},T+\tilde{t}/2,T'-\tilde{t}'/2,t+\tilde{t},t'-\tilde{t}')\Big]\,. \tag{4.36}$$

Note that as a result of performing the integration over the momentum variable $\mathbf{p}$, Eq. (4.36) contains the function $\zeta(\mathbf{k},E_1-E_2)$ defined in Eq. (4.16). In the diffusive region where $\zeta(\mathbf{k},\omega)\simeq 1+i\omega\tau-Dk^2\tau$, the dominant contribution to the first term on the right-hand side of Eq. (4.36), provided $|T-T'|>\tau$ and $kl<1$, is obtained by replacing ζ by unity, leading to an expression proportional to $\delta(t-t')\delta(T-T')$. From this we conclude that the solution to Eq. (4.36) has the form of Eq. (4.34). In the second term on the right-hand side of Eq. (4.36), the corresponding dominant contribution cancels against the left-hand side. The remaining terms that originate in the expansion of ζ are seen to yield the left-hand side of Eq. (4.35).

Since the first term in Eq. (4.33) contains the odd function Eq. (2.68), it does not contribute to the electric current as expected. We thus get $\delta\mathbf{j}=\delta\sigma\mathbf{E}$, where

$$\delta\sigma=-2\tau^2u^4\sigma_0\int(d\mathbf{k})\tilde{C}_k\,, \tag{4.37}$$

where $\tilde{C}_{\mathbf{k}}=u^{-4}C_\omega(\mathbf{k})$, with C_ω given by Eq. (4.19). The

result [Eq. (4.37)] is thus seen to be identical with Eq. (4.21).

The weak localization effect has a simple physical interpretation, which is suggested by the structure of the maximally crossed diagrams. The retarded Green's function represents the quantum-mechanical amplitude for the electron to be scattered through a sequence of impurity collisions forming a closed loop, whereas the advanced Green's function represents the complex conjugate of the quantum-mechanical amplitude for the electron to suffer the opposite sequence of impurity collisions. The Cooperon thus describes the quantum-mechanical interference between the electronic alternatives of traversing a closed loop in opposite directions of time. For a presentation of the weak localization effect based on a quasiclassical description that avoids diagrams, we refer the reader to Chakravarty and Schmid (1986).

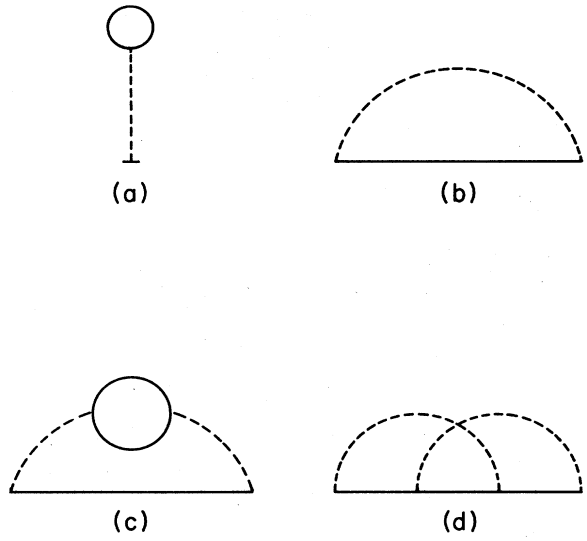


FIG. 21. Lowest-order self-energy diagrams.

B. Electron-electron scattering

Having considered the weak localization due to impurity scattering, we now turn to a discussion of the effects of electron-electron scattering. First we shall demonstrate in this section how the collision integral appropriate to two-particle scattering is derived in the simplest possible model corresponding to an instantaneous short-range two-particle interaction. In the next section (IV.C), we then include the impurity scattering along with the Coulomb interaction.

The retarded and advanced components of the instantaneous two-particle interaction are identical, leading to a vanishing Keldysh component,

$$U^R=U^A=V(\mathbf{x}_1-\mathbf{x}_{1'})\delta(t_1-t_{1'})\,,\qquad U^K=0\,, \tag{4.38}$$

where $V(\mathbf{x}_1-\mathbf{x}_{1'})$ denotes the interaction. In Fig. 21 we illustrate some low-order self-energy diagrams. The instantaneous two-particle interaction is denoted by a broken line. The Hartree-Fock diagrams [Figs. 21(a) and 21(b)] do not contribute to the collision integral due to the instantaneous nature of the interaction. We take the Fourier transform of the interaction $V(\mathbf{q})$ to be independent of $\mathbf{q}$, $V(\mathbf{q})=V_0$, consistent with our assumption of short-range scattering in the dilute Fermi gas.[26] Since V_0 is independent of momentum, diagram (c) in Fig. 21 contributes (-2) times as much as diagram (d), due to the internal spin sum and the closed fermion loop. We therefore focus our attention on diagram (d), and simply change its sign to get the total contribution of (c) and (d).

Using the Feynman rules and identities of the type

$$G^R(t_1,t_{1'})G^A(t_1,t_{1'})=G^R(t_1,t_{1'})G^R(t_{1'},t_1)=0\,, \tag{4.39}$$

one may easily show that the contribution of Fig. 21(d) is

$$\Sigma^R(p)=i^2\left[\frac{1}{\sqrt{2}}\right]^4 V_0^2\int(dp_1)\int(dp_2)(G^R_{p+p_2-p_1}G^K_{p_2}G^K_{p_1}+G^K_{p+p_2-p_1}G^A_{p_2}G^K_{p_1}+G^K_{p+p_2-p_1}G^K_{p_2}G^R_{p_1}+G^R_{p+p_2-p_1}G^A_{p_2}G^R_{p_1}) \tag{4.40}$$

and

$$\Sigma^K(p)=i^2\left[\frac{1}{\sqrt{2}}\right]^4 V_0^2\int(dp_1)\int(dp_2)[(G^R-G^A)_{p+p_2-p_1}G^K_{p_2}(G^R-G^A)_{p_1}-G^K_{p+p_2-p_1}(G^R-G^A)_{p_2}(G^R-G^A)_{p_1}$$
$$-(G^R-G^A)_{p+p_2-p_1}(G^R-G^A)_{p_2}G^K_{p_1}+G^K_{p+p_2-p_1}G^K_{p_2}G^K_{p_1}]\,, \tag{4.41}$$

where the energy and momentum arguments in the Green's functions have been indicated by subscripts.

To derive the collision integral in a form in which it may be compared to the quasiparticle Boltzmann equation, we introduce the quasiparticle approximations

[26]The kinetic equation of a dilute Fermi gas is also derived by Lifshitz and Pitaevskii (1981) using the representation of Eqs. (2.18) and (2.19).

$$G^R-G^A=\frac{2\pi}{i}\delta(E-\xi-U) \tag{4.42}$$

and

$$G^K=(G^R-G^A)h_{\mathbf{p}}=\frac{2\pi}{i}\delta(E-\xi-U)h_{\mathbf{p}} . \tag{4.43}$$

An external potential $U=U(\mathbf{R},T)$ has been included. The dependence on $\mathbf{R},T$ has been suppressed in the expressions for Σ^R and Σ^K given above. Using $\Sigma^R=(\Sigma^A)^*$, we obtain the collision integral from the integral

$$I=\int dE(i\Sigma^K A-i\Gamma G^K) \tag{4.44}$$

[cf. Eqs. (2.60) and (2.72)]. The quasiparticle collision integral equals $I_{\rm qp}=-\frac{1}{2}I$, since the distribution function $h_{\mathbf{p}}$ is related to the quasiparticle distribution $f_{\mathbf{p}}$ by $h_{\mathbf{p}}=(1-2f_{\mathbf{p}})$. We then get

$$I_{\rm qp}=-V_0^2(2\pi)^4\int(d\mathbf{p}_1)\int(d\mathbf{p}_2)\int(d\mathbf{p}_3)\delta(\xi_{\mathbf{p}}+\xi_{\mathbf{p}_2}-\xi_{\mathbf{p}_1}-\xi_{\mathbf{p}_3})\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_1-\mathbf{p}_3)$$
$$\times[f_{\mathbf{p}}f_{\mathbf{p}_2}(1-f_{\mathbf{p}_1})(1-f_{\mathbf{p}_3})-(1-f_{\mathbf{p}})(1-f_{\mathbf{p}_2})f_{\mathbf{p}_1}f_{\mathbf{p}_3}] . \tag{4.45}$$

To cast the collision integral in the form of Eq. (4.45), it is necessary to add and subtract the terms $f_{\mathbf{p}}f_{\mathbf{p}_1}f_{\mathbf{p}_2}f_{\mathbf{p}_3}$, which do not appear explicitly in I. As written, the collision integral agrees with the one used in Fermi-liquid theory (see, for example, Baym and Pethick, 1978) when the singlet part of the scattering amplitude is equal to a constant and the triplet part is equal to zero.

C. Electron-electron and electron-impurity scattering

The magnitudes of the electronic transport coefficients of normal metals are determined at low temperatures by the scattering of electrons from impurities and imperfections in the crystal, whereas the temperature dependence is determined by inelastic scattering. At sufficiently low temperatures, $T\ll 1/\tau$, there occurs the possibility of quantum interference between elastic and inelastic scattering mechanisms. In this section we describe how such effects are included in the kinetic equation. To avoid excessive detail we shall focus primarily on the inelastic collision rate. The interference effects were first studied on a microscopic basis by Schmid (1973,1974). Here we shall follow the work of Altshuler (1978) and Altshuler and Aronov (1978a,1979a,1979c). The inelastic collision rate may also be obtained rather more directly by the method of exact impurity eigenstates (Abrahams *et al.*, 1981).

1. The model of disorder

The disorder will be represented by randomly distributed static impurities and the inelastic scattering by a screened Coulomb interaction. We consider the impurity and the exchange self-energy[27] depicted in Fig. 22,

$$\underline{\Sigma}=\underline{\Sigma}_{\rm imp}+\underline{\Sigma}_{\rm Coul} , \tag{4.46}$$

where the wiggly line in the figure represents the screened Coulomb propagator U and the triangle the impurity-renormalized vertex Γ.[28]

To lowest order in $1/k_F l$, the vertex Γ is given by the impurity ladder diagrams in the particle-hole channel shown in Fig. 23.

In the general case the self-energy $\underline{\Sigma}_{\rm Coul}$ is of a more complex nature than those we have previously encountered. Since the translational invariance in space and time is broken in a general nonequilibrium state, we no longer have a simple diagrammatic technique in the energy-momentum representation. The self-energy in the mixed representation will therefore contain Poisson brackets terms. Splitting these terms off, we represent the self-energy as

$$\underline{\Sigma}=\underline{\Sigma}^0+\delta\underline{\Sigma} , \tag{4.47}$$

where $\underline{\Sigma}^0$ does not contain Poisson brackets. The change $\delta\underline{\Sigma}$ in the self-energy must be included if we consider the electrical conductivity (Altshuler, 1978; Altshuler and Aronov, 1978a). Since, in the following, we calculate only the inelastic collision rate, we may neglect $\delta\underline{\Sigma}$ throughout. The kinetic equation thus acquires the form of Eq. (3.84) with

$$I[h]=i\Sigma^K-ih(\Sigma^R-\Sigma^A) , \tag{4.48}$$

where we now and henceforth drop the superscript 0, since it is understood that the Poisson brackets terms are discarded. The solution of the coupled integral equations corresponding to the diagram shown in Fig. 23 is described in the Appendix. The resulting self-energies $\Sigma^{R,A,K}$ yield the expression (A21) for the inelastic part $I^{\rm inel}$ of the collision integral, which vanishes only if the distribution function h is the equilibrium one.

The collision integral (A21) may be written in the con-

[27] We assume that the contribution from the impurity ladder diagrams in the particle-particle channel (weak localization effect) have been suppressed. The Hartree term is not taken into account. This is justified when the inverse screening length is smaller than the Fermi wave vector.

[28] In this section we shall not use the symbol Γ for the linewidth function, so no confusion should arise.

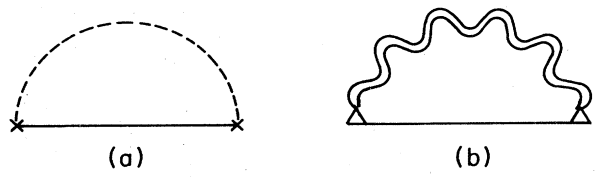


FIG. 22. Impurity and exchange self-energy diagrams.

venient form for a momentum-independent distribution function

$$I^{\text{inel}}=\text{Im}\int (dq)\frac{1}{1-\zeta}G^A_{p-q}\times[2\,\text{Im}U^R_q(1-h_Eh_{E-\omega})-iU^K_q(h_{E-\omega}-h_E)]\,, \tag{4.49}$$

where ζ was introduced in Eq. (4.16). We have used the properties $G^R=(G^A)^*$, $U^R=(U^A)^*$, U^K being purely imaginary together with the usual representation $G^K=(G^R-G^A)h$.

2. Electron-electron interaction in weakly disordered conductors

We now specify the screened Coulomb propagator by adapting the random-phase approximation with the impurities included to leading order in $(k_Fl)^{-1}$ as illustrated in Fig. 24. The structure of the equation in Keldysh space is

$$U_{k_1k_{1'}}=U^0_{k_1k_{1'}}+U^0_{k_1k_2}\pi^{k_2k_{2'}}U_{k_{2'}k_{1'}}\,, \tag{4.50}$$

where

$$U^0_{k_1k_{1'}}=\delta_{k_1k_{1'}}V^0\,. \tag{4.51}$$

V^0 is the bare Coulomb interaction, and the polarization is given by

$$\pi^{k_2k_{2'}}=-2i\tilde{\Gamma}^{k_2}_{ii'}G_{i'j'}G_{ji}\gamma^{k_{2'}}_{j'j}\,. \tag{4.52}$$

It is readily verified that

$$\underline{\pi}=\begin{bmatrix}\pi^R & \pi^K\\ 0 & \pi^A\end{bmatrix}, \tag{4.53}$$

with $\pi^{R,A,K}$ given by Eqs. (A24)–(A26). For $\underline{U}$ we have

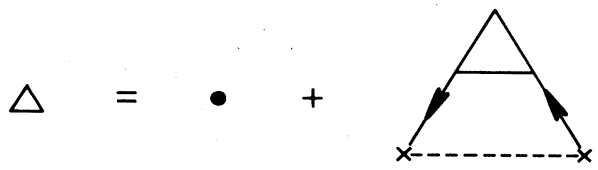


FIG. 23. Impurity-renormalized vertex equation.

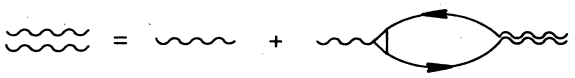


FIG. 24. Equation for the impurity-renormalized screened Coulomb propagator.

$$\underline{U}=\begin{bmatrix}U^R & U^K\\ 0 & U^A\end{bmatrix}, \tag{4.54}$$

with

$$U^{R(A)}=\frac{V^0}{\varepsilon^{R(A)}_q} \tag{4.55}$$

and

$$U^K=\frac{(V^0)^2}{\varepsilon^R_q\varepsilon^A_q}\pi^K\,, \tag{4.56}$$

where

$$\varepsilon^{R(A)}_q=1-V^0\pi^{R(A)}\,. \tag{4.57}$$

In the following, we need only the expression for the screened Coulomb propagator in equilibrium. We then have

$$\pi^A_q=(\pi^R_q)^*=-i\int (dp)\left[G^K_pG^R_{p-q}+G^A_pG^K_{p-q}+\frac{h_{E-\omega}-h_E}{1-\zeta^*}\zeta^*G^A_pG^R_{p-q}\right], \tag{4.58}$$

$$\pi^K_q=-2iu^{-2}\int (dE)(1-h_Eh_{E-\omega})\text{Re}\frac{\zeta}{1-\zeta}\,,$$
$$\text{Im}\varepsilon_q=V^0u^{-2}\int (dE)(h_{E-\omega}-h_E)\text{Re}\frac{\zeta}{1-\zeta}\,, \tag{4.59}$$

where all quantities in Eqs. (4.58) and (4.59) are taken to be their equilibrium values. We have introduced the usual dielectric function ε_q, given by

$$\varepsilon_q\equiv\varepsilon^R_q=(\varepsilon^A_q)^*\,. \tag{4.60}$$

3. Electron-electron collision rate

As an example of the use of the theory introduced above we calculate the inelastic collision rate due to Coulomb interactions. A momentum-dependent distribution function will, in the course of the short time τ, relax to a momentum-independent distribution function $\bar{h}_E$. The energy relaxation will now proceed, governed by the inelastic collision integral. The kinetic equation for $\bar{h}$ is

$$\partial_T\bar{h}=I^{\text{inel}}[\bar{h}]\,, \tag{4.61}$$

since other terms arising from the left-hand side of Eq. (3.84) vanish upon averaging over momentum.

The appropriate momentum average is performed by multiplying both sides of the kinetic equation with the

spectral weight $A(p)$ and integrating over $\mathbf{p}$. We assume here that the width of the spectral weight is determined by the impurity scattering and therefore use the impurity Green's functions (4.12) everywhere in Eq. (4.61). This requires that the impurity scattering rate $1/\tau$ be much larger than the electron-electron collision rate, which is certainly satisfied if E and T are much less than $1/\tau$, as the following calculations will show.

The kinetic equation thus reduces to

$$\partial_T n_E = -\int d\omega \int dE' P(\omega) R^{\omega}_{E,E'} , \tag{4.62}$$

where

$$R^{\omega}_{E,E'} = n_E n_{E'-\omega}(1-n_{E-\omega})(1-n_{E'}) - n_{E-\omega} n_{E'}(1-n_E)(1-n_{E'-\omega}) , \tag{4.63}$$

and we have introduced the distribution function

$$n_E = \tfrac{1}{2}(1-\bar{h}_E) \tag{4.64}$$

and the transition probability[29]

$$P(\omega) = P_d(\omega) = \frac{2N_d(0)\tau^2}{\pi} \int (d\mathbf{q}) \left[\frac{V^0}{|\varepsilon_q|} \mathrm{Re} \frac{\zeta}{1-\zeta} \right]^2 . \tag{4.65}$$

Let us for a start consider the energy dependence of the collision rate at temperature $T=0$. From Eq. (4.62) we get

$$\frac{1}{\tau_E} = \int_0^E d\omega\, \omega P_d(\omega) . \tag{4.66}$$

To proceed further we must calculate the transition probability. From Eq. (4.17) we observe that (in two and three dimensions)

$$\zeta = \begin{cases} \dfrac{\pi}{2ql}, & ql \gg \omega\tau, ql \gg 1, \\ 1 + i\omega\tau - D\tau q^2, & ql, \omega\tau \ll 1 , \end{cases} \tag{4.67}$$

where D is the diffusion constant $D=(1/d)v_F l$. In the region $ql, \omega\tau \ll 1$, $(1-\zeta)^{-1}$ thus displays the diffusion pole, as noted below Eq. (4.17).

In calculating the dielectric function, we may interchange the order of integration using the standard subtraction method (Abrikosov *et al.*, 1965); we obtain

$$\varepsilon_q = \begin{cases} 1 + \dfrac{D\kappa_d^{d-1} q^{3-d}}{-i\omega + Dq^2}, & ql, \omega\tau \ll 1 \\ 1 + \dfrac{\kappa_d^{d-1}}{q^{d-1}}, & ql \gg \omega\tau,\ ql \gg 1 , \end{cases} \tag{4.68}$$

where κ_d is the inverse screening length

$$\kappa_d = \begin{cases} [8\pi e^2 N_3(0)]^{1/2}, & d=3, \\ 4\pi e^2 N_2(0), & d=2 . \end{cases} \tag{4.69}$$

The bare Coulomb potential is[30]

$$V^0 = \begin{cases} \dfrac{4\pi e^2}{q^2}, & d=3, \\ \dfrac{2\pi e^2}{q}, & d=2 . \end{cases} \tag{4.70}$$

In the three-dimensional case we obtain

$$P_3(\omega) = \frac{\omega^{-1/2}}{8\sqrt{2}\pi^2 N_3(0) D^{3/2}}, \quad \omega\tau \ll 1 , \tag{4.71}$$

and in the energy range $\omega \gg 1/\tau$

$$P_3(\omega) = \begin{cases} \dfrac{\pi e^2}{4v_F^2 \kappa_3}, & \kappa_3 \ll k_F, \\ \dfrac{\pi}{8\mu}, & \kappa_3 \gg k_F . \end{cases} \tag{4.72}$$

The right-hand side of Eq. (4.71) stems from the region of phase space where $q \sim \sqrt{\omega/D}$ and $ql, \omega\tau \ll 1$, while the right-hand side of Eq. (4.72) comes from the part of phase space where $ql \gg \omega\tau$, $ql \gg 1$.

Thus we obtain for the collision rate at zero temperature at energies $E \ll 1/\tau$

$$\frac{1}{\tau_E} = \frac{\sqrt{6}}{4} \frac{\tau^{1/2}}{(k_F l)^2} E^{3/2} . \tag{4.73}$$

At energies $E \gg 1/\tau$, one obtains the well-known results in the clean limit,

$$\frac{1}{\tau_E} = \begin{cases} \dfrac{\pi^2}{64} \dfrac{\kappa_3}{k_F} \dfrac{E^2}{\mu}, & \kappa_3 \ll k_F, \\ \dfrac{\pi}{16} \dfrac{E^2}{\mu}, & \kappa_3 \gg k_F . \end{cases} \tag{4.74}$$

Note that the result of Eq. (4.45) agrees with this, since $V_0 = [2N_3(0)]^{-1}$ corresponds to the limit $\kappa_3 \gg k_F$.

Next we turn to the calculation of the temperature dependence of the collision rate at the Fermi energy. From Eq. (4.62) we obtain

$$\frac{1}{\tau_T} = \int_{-\infty}^{\infty} d\omega\, P_d(\omega) \frac{\omega}{\sinh \dfrac{\omega}{T}} . \tag{4.75}$$

In three dimensions we get at temperatures $T \ll 1/\tau$

[29] Notice that we can write the even function P_d as

$$P_d(\omega) = \frac{\tau}{\pi\omega} \mathrm{Im} \int (d\mathbf{q}) U_q^R \mathrm{Re} \frac{\zeta}{1-\zeta} .$$

[30] The $1/q$ behavior of the bare Coulomb potential in two dimensions arises from constricting to two dimensions the movement of electrons interacting via a $1/r$ potential.

$$\frac{1}{\tau_T}=\frac{3\sqrt{3\pi}}{16}\zeta(\tfrac{3}{2})(\sqrt{8}-1)\frac{\tau^{1/2}}{k_F l}\frac{T^{3/2}}{\mu\tau}\ , \tag{4.76}$$

where ζ is the Riemann zeta function.

At temperatures $T\gg 1/\tau$, we have the results of the clean limit,

$$\frac{1}{\tau_T}=\begin{cases}\dfrac{\pi^3 e^2}{8v_F^2\kappa_3}T^2, & \kappa_3\ll k_F,\\ \dfrac{\pi^3}{16\mu}T^2, & \kappa_3\gg k_F\ .\end{cases} \tag{4.77}$$

Calculating the same quantities for the two-dimensional case we obtain[31]

$$P_2(\omega)=\frac{1}{8\mu\tau}\frac{1}{\omega},\quad \omega\tau\ll 1\ . \tag{4.78}$$

The inelastic collision rate at energies $E\ll 1/\tau$ is thus given by

$$\frac{1}{\tau_E}=\frac{E}{4k_F l}\ . \tag{4.79}$$

This result is in agreement with Abrahams *et al.* (1981) to within a factor of 2. We note that a missing factor of 2 has also been reported by Lopes dos Santos (1983), who considered the temperature dependence of the inelastic collision rate.[32]

When calculating the transition probability at energies $\omega\gg 1/\tau$, we encounter the logarithmically divergent expressions

$$P_2(\omega)=\frac{\pi^2 N_2(0)e^4}{v_F^2}\begin{cases}\displaystyle\int_0^{2k_F}dq\frac{1}{q\kappa_2^2}, & \kappa_2\gg k_F,\\ \displaystyle\int_0^{\infty}dq\frac{1}{q(q+\kappa_2)^2}, & \kappa_2\ll k_F\ .\end{cases} \tag{4.80}$$

The reason for the appearance of this singularity in the forward scattering is our use of momentum states that are determined only to within the inverse impurity mean free path. Instead, we approach the problem from the clean limit and obtain the Boltzmann equation for two-particle scattering. We then refer to Hodges *et al.* (1971), who calculated the inelastic collision rate. Their result is

$$\frac{1}{\tau_E}=\frac{1}{8\pi^2\mu}E^2\left|\ln\frac{E}{\mu}\right|\ . \tag{4.81}$$

Turning to the temperature dependence of the inelastic collision rate, we obtain at temperatures $T\ll 1/\tau$

$$\frac{1}{\tau_T}=\frac{1}{2k_F l}\int_0^{\infty}d\omega\frac{1}{\sinh\frac{\omega}{T}}\ . \tag{4.82}$$

To cure this infrared divergence one should consider terms of next order in the coupling constant e^2 as noted by Abrahams *et al.* (1981). This leads to a lower cutoff at $\omega_0=4T^3/T_1^2$, where $T_1=(2mD)^2D\kappa_2^2$ and an inelastic collision rate

$$\frac{1}{\tau_T}=\frac{T}{2mD}\ln\frac{T_1}{T}\ . \tag{4.83}$$

At temperatures $T\gg 1/\tau$, the same state of affairs prevails as in the result stated in Eq. (4.81). We state the result by Hodges *et al.* (1971) for completeness

$$\frac{1}{\tau_T}=\frac{T^2}{2\pi\mu}\ln\frac{\mu}{T}\ . \tag{4.84}$$

In conclusion, we note that the results of this section show that diffusion renormalization enhances the electron-electron interaction. The interpretation of this enhancement is given in terms of the breaking of translational invariance due to the presence of the disorder. The violation of momentum conservation gives more phase space for final states.

V. KINETIC EQUATIONS FOR SUPERCONDUCTING METALS

In this section we shall give an account of the quasiclassical description of transport phenomena in superconductors. The development of the quasiclassical method to describe transport phenomena in superfluids is due primarily to Eilenberger (1968), Larkin and Ovchinnikov (1968,1975,1977), Eliashberg (1971), and Schmid and Schön (1975a).

A. The quasiclassical description

The basic feature of the superconducting state, the pairing interaction, is made compatible with the standard Feynman diagrammatics by introducing the Nambu pseudospinor field

$$\psi=\begin{bmatrix}\psi_\uparrow\\ \psi_\downarrow^\dagger\end{bmatrix}\ . \tag{5.1}$$

Then the only additional feature of the Feynman rules is the interpretation of the components of the electronic Green's function in Keldysh space as (2×2)-matrix Green's functions in particle-hole (or Nambu) space (Nambu, 1960; Schrieffer, 1964).

We introduce the basic Green's functions in particle-

[31] We note that the Coulomb interaction in a weakly disordered metal can be considered two dimensional when the sample thickness a satisfies $D/a^2\max\{E,T\}\gg 1$.

[32] Notice that Lopes dos Santos (1983) used a definition of the inverse lifetime differing from the inelastic collision rate by a factor of 2.

hole space[33]

$$G^{>}(1,1')=-i\tau^3\begin{bmatrix}\langle\psi_\uparrow(1)\psi_\uparrow^\dagger(1')\rangle & \langle\psi_\uparrow(1)\psi_\downarrow(1')\rangle\\ \langle\psi_\downarrow^\dagger(1)\psi_\uparrow^\dagger(1')\rangle & \langle\psi_\downarrow^\dagger(1)\psi_\downarrow(1')\rangle\end{bmatrix}, \tag{5.2}$$

$$G^{<}(1,1')=i\tau^3\begin{bmatrix}\langle\psi_\uparrow^\dagger(1')\psi_\uparrow(1)\rangle & \langle\psi_\downarrow(1')\psi_\uparrow(1)\rangle\\ \langle\psi_\uparrow^\dagger(1')\psi_\downarrow^\dagger(1)\rangle & \langle\psi_\downarrow(1')\psi_\downarrow^\dagger(1)\rangle\end{bmatrix}. \tag{5.3}$$

The results of Sec. II are immediately taken over and we obtain the equation of motion

$$\{\underline{G}_0^{-1}-\underline{\Sigma}\}\otimes\underline{G}=\delta(1-1'), \tag{5.4}$$

where

$$\underline{G}_0^{-1}(1,1')=\underline{G}_0^{-1}(1)\delta(1-1'), \tag{5.5}$$

$$\underline{G}_0^{-1}(1)=i\tau^3\partial_{t_1}-\varepsilon(1), \tag{5.6}$$

$$\varepsilon(1)=-\frac{1}{2m}\underline{\boldsymbol{\partial}}^2+e\varphi(1)-\mu, \tag{5.7}$$

$$\underline{\boldsymbol{\partial}}(1)=\nabla_{\mathbf{x}_1}-ie\tau^3\mathbf{A}(1). \tag{5.8}$$

The matrix structure is the tensorial product of Keldysh and particle-hole space (unit matrices are suppressed). Thus τ^3 appearing in Eqs. (5.6) and (5.8) denotes

$$\begin{bmatrix}\tau^3 & 0\\ 0 & \tau^3\end{bmatrix}.$$

Similarly, we have the conjugate equation

$$\underline{G}\otimes\{\underline{G}_0^{-1}-\underline{\Sigma}\}=\delta(1-1'). \tag{5.9}$$

The arguments that bring us from the above equations of motion to the quasiclassical ones are identical with those we used for the normal state. Subtracting the two Dyson equations (5.4) and (5.9) we can, as in the normal state, introduce the quasiclassical Green's function satisfying

$$[\underline{g}_0^{-1}+i\underline{\sigma}\circ\underline{g}]_-=0, \tag{5.10}$$

$$\underline{g}_0^{-1}(\hat{p},\mathbf{R},t_1,t_{1'})=\underline{g}_0^{-1}(\hat{p},\mathbf{R},t_1)\delta(t_1-t_{1'}), \tag{5.11}$$

$$\underline{g}_0^{-1}(\hat{p},\mathbf{R},t_1)=\tau^3\partial_{t_1}+\mathbf{v}_F\cdot[\nabla_{\mathbf{R}}-ie\tau^3\mathbf{A}(\mathbf{R},t_1)]+ie\varphi(\mathbf{R},t_1). \tag{5.12}$$

For the case of inelastic interactions, Eq. (5.10) was first obtained by Eliashberg (1971).

In the superconducting state, the components of the retarded and advanced Green's functions in particle-hole space will not reduce to numbers as in the normal state [cf. Eq. (3.12)]. Equation (5.10) thus constitutes a complicated set of equations describing the coupling between states ($g^{R(A)}$) and the occupation (g^K) of these states. An important simplifying feature, when it comes to reducing Eq. (5.10) to a more solvable form, is the existence of a normalization condition

$$\underline{g}\circ\underline{g}=\delta(t_1-t_{1'}). \tag{5.13}$$

The important feature, that the coefficient on the right-hand side of Eq. (5.13) is independent of the space coordinates, was established by Eilenberger (1968) and Larkin and Ovchinnikov (1968) for the static case. It was noted by Larkin and Ovchinnikov (1975), by taking the spatial derivative of Eq. (5.13), that the normalization condition is in general compatible with the equation of motion. The normalization condition for the general, nonstationary case was discussed recently by Shelankov (1985).

The normalization condition allows us to represent[34] the Keldysh component in terms of a distribution matrix h,

$$g^K=g^R\circ h-h\circ g^A. \tag{5.14}$$

Equation (5.10) consists of three coupled equations,

$$[g_0^{-1}+i\sigma^{R(A)}\circ g^{R(A)}]_-=0, \tag{5.15}$$

$$g^R\circ B-B\circ g^A=0, \tag{5.16}$$

and

$$B=\sigma^K+h\circ\sigma^A-\sigma^R\circ h+[g_0^{-1}\circ h]_-. \tag{5.17}$$

The equation for $g^{R(A)}$ determines the states of the system, while the equation for h determines the occupation of these states. In obtaining Eq. (5.16) from the Keldysh component of Eq. (5.10) we have utilized the equations for $g^{R(A)}$ and the fact that the composition $\circ$ is associative.

To get a closed set of equations we must specify the self-energy. Apart from the inherent electron-phonon interaction, we shall allow for impurity scattering in the Born approximation, including spin flip. The self-energy $\underline{\sigma}$ is given by

$$\underline{\sigma}=\underline{\sigma}^{e\text{-ph}}+\underline{\sigma}^{\text{imp}}+\underline{\sigma}^{\text{sf}} \tag{5.18}$$

The expressions for the components of $\underline{\sigma}^{e\text{-ph}}$ are identical in form to Eq. (3.8),

$$\sigma_{e\text{-ph}}^{R(A)}=\frac{\lambda}{8}\int d\hat{p}'(g^K D^{R(A)}+g^{R(A)}D^K),$$
$$\sigma_{e\text{-ph}}^{K}=\frac{\lambda}{8}\int d\hat{p}'[(g^R-g^A)(D^R-D^A)+g^K D^K]. \tag{5.19}$$

Furthermore

$$\underline{\sigma}^{\text{imp}}=-i\pi n_i N_0\int\frac{d\hat{p}'}{4\pi}|v(\hat{p}\cdot\hat{p}')|^2\underline{g}(\mathbf{R},\hat{p}',t_1,t_{1'}), \tag{5.20}$$

$$\underline{\sigma}^{\text{sf}}=-\frac{i}{2\tau_s}\int\frac{d\hat{p}'}{4\pi}\tau^3\underline{g}(\mathbf{R},\hat{p}',t_1,t_{1'})\tau^3, \tag{5.21}$$

[33]The third Pauli matrix, operating in particle-hole space, is denoted by τ^3. No confusion with our notation in Keldysh space should arise.

[34]This choice is not unique. A choice making the resemblance between the Boltzmann equation and Eq. (5.16) immediate in the quasiparticle approximation has been introduced by Shelankov (1980).

where we have treated the spin-flip scattering in the s-wave approximation corresponding to the spin-flip scattering rate

$$\frac{1}{\tau_s}=2\pi n_{\rm sf}N_0S(S+1)\int\frac{d\hat{p}'}{4\pi}|v_{\rm sf}(\hat{p}\cdot\hat{p}')|^2 . \quad (5.22)$$

Here $n_{\rm sf}$ is the concentration of spin-flip impurities, S their spin and $v_{\rm sf}(\hat{p}\cdot\hat{p}')$ the interaction. For later use we introduce the normal impurity scattering rate given by

$$\frac{1}{\tau}=2\pi n_iN_0\int\frac{d\hat{p}'}{4\pi}|v(\hat{p}\cdot\hat{p}')|^2 . \quad (5.23)$$

To elucidate the information contained in Eq. (5.15), we solve it in equilibrium and take the BCS limit, obtaining

$$g^{R(A)}=\alpha^{R(A)}\tau^3+\beta^{R(A)}\tau^1 , \quad (5.24)$$

$$\alpha^{R(A)}=(\pm)N_1(E)+iR_1(E) , \quad (5.25)$$

$$\beta^{R(A)}=N_2(E)(\pm)iR_2(E) , \quad (5.26)$$

where

$$N_1(E)=\frac{|E|}{(E^2-\Delta^2)^{1/2}}\theta(E^2-\Delta^2) , \quad (5.27)$$

$$N_2(E)=\frac{\Delta}{(\Delta^2-E^2)^{1/2}}\theta(\Delta^2-E^2) , \quad (5.28)$$

$$R_2(E)=\frac{\Delta}{E}N_1(E) , \quad (5.29)$$

$$R_1(E)=-\frac{|E|}{\Delta}N_2(E) , \quad (5.30)$$

with Δ being the BCS energy gap.

In the static case it follows from Eq. (5.15) that $g^{R(A)}$ is traceless, so that

$$g^{R(A)}=\alpha^{R(A)}\tau^3+\beta^{R(A)}\tau^1+\gamma^{R(A)}\tau^2 . \quad (5.31)$$

In applications to time-dependent phenomena, the traceless form in Eq. (5.31) has so far proved to be sufficiently general.

The quantities α, β, and γ denote the generalized densities of states, and we recognize N_1 in the BCS limit as the usual normalized BCS density of states.

We only have to consider one set of generalized densities of states, since from the general equality ($*$ denotes complex conjugation and $\dagger$ denotes Hermitian conjugation)

$$G^R(1,1')=\tau^3(G^A(1',1))^\dagger\tau^3 \quad (5.32)$$

it follows that

$$\alpha^A=-(\alpha^R)^*,\quad \beta^A=(\beta^R)^*,\quad \gamma^A=(\gamma^R)^* . \quad (5.33)$$

We shall not consider Eq. (5.16) in its full generality, but derive the collision integral for the case of a dirty superconductor.

B. The dirty limit

In the dirty limit, $\tau T_c\ll 1$, we can reduce the integral equation with respect to the non-spin-flip impurity scattering to a differential equation of diffusion type. In the dirty limit the Green's function will be almost isotropic. We therefore make an expansion in spherical harmonics keeping only the s- and p-wave parts (Usadel, 1970),

$$\underline{g}(\hat{p},\mathbf{R},t_1,t_{1'})=\underline{g}_s(\mathbf{R},t_1,t_{1'})+\hat{p}\cdot\underline{\mathbf{g}}_p(\mathbf{R},t_1,t_{1'}) . \quad (5.34)$$

The self-energy

$$\underline{\sigma}(\hat{p},\mathbf{R},t_1,t_{1'})=\underline{\sigma}_s(\mathbf{R},t_1,t_{1'})+\hat{p}\cdot\underline{\boldsymbol{\sigma}}_p(\mathbf{R},t_1,t_{1'}) \quad (5.35)$$

is given by[35]

$$\underline{\sigma}_s=-\frac{i}{2\tau}\underline{g}_s+\underline{\sigma}'_s , \quad (5.36)$$

$$\underline{\sigma}'_s=-\frac{i}{2\tau_s}\tau^3\underline{g}_s\tau^3+\underline{\sigma}_s^{e\text{-ph}} , \quad (5.37)$$

and, since the non-spin-flip impurity scattering is assumed to be dominant,

$$\hat{p}\cdot\underline{\boldsymbol{\sigma}}_p=-i\pi n_iN_0\int\frac{d\hat{p}'}{4\pi}|v(\hat{p}\cdot\hat{p}')|^2\hat{p}'\cdot\underline{\mathbf{g}}_p . \quad (5.38)$$

Performing the angular average on the right-hand side of Eq. (5.38) gives

$$\underline{\boldsymbol{\sigma}}_p=-\frac{i}{2}\left[\frac{1}{\tau}-\frac{1}{\tau_{\rm tr}}\right]\underline{\mathbf{g}}_p , \quad (5.39)$$

where $\tau_{\rm tr}$ is the impurity transport lifetime,

$$\frac{1}{\tau_{\rm tr}}=2\pi n_iN_0\int\frac{d\hat{p}'}{4\pi}|v(\hat{p}\cdot\hat{p}')|^2[1-(\hat{p}\cdot\hat{p}')] . \quad (5.40)$$

We also rewrite (without approximations) the inverse propagator $\underline{g}_0^{-1}$ in s- and p-wave form,

$$\underline{g}_0^{-1}=\underline{g}_{0_s}^{-1}+\hat{p}\cdot\underline{\mathbf{g}}_{0_p}^{-1} , \quad (5.41)$$

with

$$\underline{g}_{0_s}^{-1}=(\tau^3\partial_{t_1}+ie\varphi)\delta(t_1-t_{1'}) , \quad (5.42)$$

$$\underline{\mathbf{g}}_{0_p}=v_F\underline{\boldsymbol{\partial}} , \quad (5.43)$$

$$\underline{\boldsymbol{\partial}}=(\nabla_{\mathbf{R}}-ie\tau^3\mathbf{A})\delta(t_1-t_{1'}) . \quad (5.44)$$

1. The equation of motion

In the dirty limit we can split the equation of motion (5.10) into an even and odd part with respect to $\hat{p}$,

$$[\underline{g}_{0_s}^{-1}+i\underline{\sigma}'_s\circ\underline{g}_s]_-+\tfrac{1}{3}v_F[\underline{\boldsymbol{\partial}}\circ\underline{\mathbf{g}}_p]_-=0 , \quad (5.45)$$

[35]The subscript on $\underline{\sigma}_s^{e\text{-ph}}$ designates that it is only a functional of $\underline{g}_s$.

$$\frac{1}{2\tau_{\rm tr}}[\underline{g}_s \circ \underline{\mathbf{g}}_p]_- + v_F[\underline{\partial} \circ \underline{g}_s]_- = 0 \ . \tag{5.46}$$

Using the s- and p-wave parts of the normalization condition,

$$\underline{g}_s \circ \underline{g}_s = \delta(t_1 - t_{1'}) \ , \tag{5.47}$$

$$[\underline{g}_s \circ \underline{\mathbf{g}}_p]_+ = \mathbf{0} \ , \tag{5.48}$$

we can solve Eq. (5.46) for $\underline{\mathbf{g}}_p$,

$$\underline{\mathbf{g}}_p = -l\underline{g}_s \circ [\underline{\partial} \circ \underline{g}_s]_- \ , \tag{5.49}$$

where l is the impurity mean free path,

$$l = v_F \tau_{\rm tr} \ . \tag{5.50}$$

Inserting Eq. (5.49) into Eq. (5.45) we obtain for the s-wave part

$$[\underline{g}_{0_s}^{-1} + i\underline{\sigma}'_s - D\underline{\partial} \circ \underline{g}_s \circ \underline{\partial} \circ \underline{g}_s]_- = 0 \ , \tag{5.51}$$

where D is the diffusion constant,

$$D = \tfrac{1}{3} v_F l \ . \tag{5.52}$$

Equation (5.51) is the starting point for considering general nonequilibrium phenomena in a dirty superconductor.

2. Generalized densities of states in the static limit

In this section we shall further reduce the diagonal components of Eq. (5.51) by considering the static limit. We shall treat the pairing effect (contained in $\mathrm{Re}\sigma^{R(A)}$) in the BCS approximation and approximate the electronic damping by the equilibrium expression. Then the retarded (advanced) electron-phonon self-energy reduces to

$$\sigma^{R(A)}_{e\text{-ph}} = \left[-\lambda E_{(+)}^{-} \frac{i}{2\tau_{\rm in}} \right] \tau^3 - i\hat{\Delta} \ , \tag{5.53}$$

with the gap matrix

$$\hat{\Delta} = \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} , \tag{5.54}$$

where[36] (from now on we drop the s-wave subscript)

$$\Delta = -\frac{i\lambda}{8(1+\lambda)} \int_{-\omega_D}^{\omega_D} dE \, \mathrm{Tr}[(\tau^1 - i\tau^2) g^K] \tag{5.55}$$

is the order parameter, and ω_D is the Debye frequency. The renormalization factor is $(1+\lambda)$ and the inelastic scattering rate is denoted by $1/\tau_{\rm in}$.

We now perform a Taylor expansion to lowest order in the equation

$$[g_0^{-1} + i\sigma'^{R(A)} - D\partial \circ g^{R(A)} \circ \partial \circ g^{R(A)}]_- = 0 \tag{5.56}$$

and obtain for the off-diagonal components of Eq. (5.56)

$$\tfrac{1}{2} D[\alpha \mathscr{D}^2(\beta - i\gamma) - (\beta - i\gamma)\nabla_{\mathbf{R}}^2 \alpha]^{R(A)} = \left[\left[-iE_{(-)}^{+} \frac{1}{2\tau_{\rm in}} \right] (\beta - i\gamma) - \Delta\alpha + \frac{1}{\tau_s}\alpha(\beta - i\gamma) \right]^{R(A)} , \tag{5.57}$$

$$\tfrac{1}{2} D[\alpha \mathscr{D}^{*2}(\beta + i\gamma) - (\beta + i\gamma)\nabla_{\mathbf{R}}^2 \alpha]^{R(A)} = \left[\left[-iE_{(-)}^{+} \frac{1}{2\tau_{\rm in}} \right] (\beta + i\gamma) - \Delta^*\alpha + \frac{1}{\tau_s}\alpha(\beta + i\gamma) \right]^{R(A)} , \tag{5.58}$$

where we have neglected terms involving time derivatives, since we consider only the static limit. We have furthermore introduced the gauge-invariant derivative

$$\mathscr{D} = \nabla_{\mathbf{R}} - 2ie\mathbf{A} \ . \tag{5.59}$$

Together with the normalization condition

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \ , \tag{5.60}$$

Eqs. (5.57) and (5.58) determine the generalized densities of states in the static limit. For brevity we have in Eq. (5.60) and the following denoted α^R by α, β^R by β, and γ^R by γ.

3. Dirty-limit kinetic equation in the static limit

To obtain a kinetic equation we take the Keldysh component of Eq. (5.51). Using the equation of motion for $g^{R(A)}$ and the fact that the composition $\circ$ is associative, we can bring it to the form

$$g^R \circ B - B \circ g^A = 0 \tag{5.61}$$

with

$$B[h] = (g_0^R)^{-1} \circ h - h \circ (g_0^A)^{-1} - i\sigma^K_{e\text{-ph}} - D\partial \circ g^R \circ [\partial \circ h]_- - D[\partial \circ h]_- \circ g^A \circ \partial \ , \tag{5.62}$$

where[37]

$$(g_0^{R(A)})^{-1} = -iE\tau^3 + ie\varphi + \hat{\Delta} + i\sigma'^{R(A)}_{e\text{-ph}} + \frac{1}{2\tau_s}\tau^3 g^{R(A)} \tau^3 \ . \tag{5.63}$$

In the static limit considered we can choose the distribution matrix h to be diagonal,

[36] Tr denotes trace in particle-hole space.

[37] The prime on the electron-phonon self-energy indicates that it does not include the order-parameter term.

$$h=h_1\tau^0+h_2\tau^3 . \tag{5.64}$$

We now perform a Taylor expansion in Eq. (5.61) and linearize the equation with respect to h_1-h_0 and h_2. Subsequently, we take the trace of this equation and the trace of the equation multiplied by τ^3. For the phenomena considered below, we may set the γ term in Eq. (5.31) equal to zero. This ansatz is valid in many physical situations (see Schmid, 1981 for further discussion). In addition, we choose a gauge in which Δ is real:

$$-D\nabla_{\mathbf{R}}\cdot M_1(E,E)\nabla_{\mathbf{R}}h_1+D\nabla_{\mathbf{R}}^2h_2 +4N_2R_2e\mathbf{A}\cdot\nabla_{\mathbf{R}}h_2=K_1[h_1] \tag{5.65}$$

and

$$-D\nabla_{\mathbf{R}}\cdot M_2(E,E)\nabla_{\mathbf{R}}h_2+2\Delta N_2h_2 +4DN_2R_2e\mathbf{A}\cdot\nabla_{\mathbf{R}}h_1=K_2[h_2] , \tag{5.66}$$

where

$$K_i[h_i]=-\frac{\pi}{1+\lambda}\int dE'\mu(E-E')M_i(E,E')\frac{h_i(E)\cosh^2\frac{E}{2T}-h_i(E')\cosh^2\frac{E'}{2T}}{\sinh\frac{E-E'}{2T}\cosh\frac{E}{2T}\cosh\frac{E'}{2T}} . \tag{5.67}$$

Here μ is the Fermi-surface average of the function μ occurring in (3.18) and

$$M_i(E,E')=\begin{cases}N_1(E)N_1(E')-R_2(E)R_2(E'), & i=1 ,\\ N_1(E)N_1(E')+N_2(E)N_2(E'), & i=2 .\end{cases} \tag{5.68}$$

Together with the expressions for the charge and current density,

$$\rho=-eN_0\left[2e\varphi+\tfrac{1}{4}\int dE\,\mathrm{Tr}g^K\right] , \tag{5.69}$$

$$\mathbf{j}=-\frac{eN_0v_F}{12}\int dE\,\mathrm{Tr}(\tau^3\mathbf{g}_p^K) , \tag{5.70}$$

the kinetic equations (5.65) and (5.66) supplemented with Eqs. (5.57) and (5.58) for the generalized densities of states, the gap equation (5.55) and Maxwell's equations, constitute a complete description of a dirty superconductor in the static limit.

C. An example: Impurity scattering in the presence of superflow and a temperature gradient

As shown in the previous section, the dirty limit presents a particularly simple case, since the Green's functions are nearly isotropic due to the dominance of impurity scattering. This makes it possible to obtain an expansion in s and p waves. The resulting kinetic equations are not necessarily restricted to the dirty limit for validity. When pair breaking is negligible, the kinetic equations (5.65) and (5.66) reduce to the quasiparticle Boltzmann equation, as demonstrated in detail by Beyer Nielsen *et al.* (1982). In the presence of superflow, the anisotropy associated with the moving condensate becomes important, and one must investigate in detail the meaning of the dirty limit and the validity of the quasiparticle Boltzmann equation. This section illustrates how the transition between these two extremes is accomplished for the case of impurity scattering in the presence of superflow.

In the example discussed below we investigate the nonequilibrium distribution set up by a temperature gradient for the case when a supercurrent is present. To keep the discussion simple, we neglect electron-phonon scattering and treat only impurity scattering. The simultaneous presence of a temperature gradient and a supercurrent gives rise to a charge imbalance that may be observed by use of a tunneling probe (Clarke, 1972; Tinkham, 1972). This particular effect was first proposed by Pethick and Smith (1979) and subsequently observed by Clarke, Fjordbøge, and Lindelof (1979). The role of impurity scattering has been considered by Schmid and Schön (1979), Clarke and Tinkham (1980), and Beyer Nielsen *et al.* (1980).

The general equation of motion (5.10) contains the inverse of the noninteracting Green's function g_0^{-1} given in Eq. (3.3). In the present case, g_0^{-1} is a 2×2 matrix in Nambu space, which is given by Eq. (5.12),

$$g_0^{-1}=\tau^3(E-\mathbf{p}\cdot\mathbf{v}_s)+i\frac{\mathbf{p}}{m}\cdot\nabla-e\varphi . \tag{5.71}$$

Here, E is the energy variable, $\mathbf{p}=p_F\hat{p}$ the momentum, $\mathbf{v}_s$ the superfluid velocity, and φ the electrostatic potential. We neglect for simplicity the effects of phonon renormalization, although they may be included readily (Beyer Nielsen *et al.*, 1982). The gap parameter Δ occurring in Eq. (5.54) is taken to be real, in accordance with our neglect of lifetime effects. The retarded part σ^R of the self-energy is then

$$\sigma^R=-i\Delta\tau^1+\sigma^R_{\mathrm{imp}} , \tag{5.72}$$

where σ^R_{imp} is the retarded component of Eq. (5.20). To solve the equation of motion (5.10) we use the ansatz $\gamma=0$ in Eq. (5.31), corresponding to the normalization condition $\alpha^2+\beta^2=1$. The kinetic equation is obtained from the component of Eq. (5.10) which is off-diagonal in Keldysh space, and proportional to τ^3 in Nambu space,

$$\mathrm{Tr}\{\tau^3[(g_0^{-1}-\sigma^R)g^K-\sigma^Kg^A-g^K(g_0^{-1}-\sigma^A) +g^R\sigma^K]\}=0 . \tag{5.73}$$

When g_0^{-1} and σ are inserted from Eqs. (5.71) and (5.72), together with Eq. (5.14) for g^K, with h given by Eq. (5.64), the following kinetic equation results:

$$N_1\frac{\mathbf{p}}{m}\cdot\nabla f^L=-\frac{1}{\tau}[(N_1\langle N_1\rangle+N_2\langle N_2\rangle)f^T - N_1\langle N_1 f^T\rangle - N_2\langle N_2 f^T\rangle] - 2\Delta N_2 f^T , \tag{5.74}$$

where $\langle\ \rangle$ here denotes an average over the direction of the momentum $\mathbf{p}$. We have introduced the distribution functions f^L and f^T (Schmid and Schön, 1975a), which are connected with h_1 and h_2 by the definitions $h_1=1-2f^L$, $h_2=2f^T$. The integral equation (5.74) is separable and may readily be solved once $f^L\simeq f^0(E)$ is inserted on the left-hand side. When $\mathbf{v}_s$ is parallel to ∇T, Eq. (5.74) may be written as

$$c(E)N_1 x=(N_1\langle N_1\rangle+N_2\langle N_2\rangle)f^T - N_1\langle N_1 f^T\rangle - N_2\langle N_2 f^T\rangle+\gamma_0^{-1}N_2 f^T , \tag{5.75}$$

with $x=\hat{p}\cdot\hat{\nabla}T$, $\gamma_0=1/2\tau\Delta$ and $c(E)=v_F\tau\,|\nabla T|\,(E/T)(\partial f^0/\partial E)$. We can now find the averages $\langle N_1 f^T\rangle$ and $\langle N_1 f^T x\rangle$,

$$\langle N_1 f^T\rangle=c\frac{\langle N_1^{-1}\tilde{\tau}x\rangle+\gamma_0\langle N_1 x\rangle\langle N_1^{-2}N_2\tilde{\tau}\rangle}{1-\langle N_1^{-1}\tilde{\tau}\rangle} \tag{5.76}$$

and

$$\langle N_1 f^T x\rangle=c(\langle N_1^{-1}\tilde{\tau}x^2\rangle+\gamma_0\langle N_1 x\rangle\langle N_2 N_1^{-2}\tilde{\tau}x\rangle) + \langle N_1 f^T\rangle\langle N_1^{-1}\tilde{\tau}x\rangle , \tag{5.77}$$

where the quantity $\tilde{\tau}$ is defined as

$$\frac{1}{\tilde{\tau}}=N_1^{-3}(N_1\langle N_1\rangle+N_2\langle N_2\rangle+\gamma_0^{-1}N_2) . \tag{5.78}$$

The results of Eqs. (5.76)—(5.78) are obtained most easily by eliminating $\langle N_2 f^T\rangle$ by averaging Eq. (5.75) over angle. From the averages [Eqs. (5.76) and (5.77)] the charge imbalance Q^* and heat current $j_{\rm th}$ are determined according to

$$Q^*=2N_0\int dE\langle N_1 f^T\rangle \tag{5.79}$$

and

$$j_{\rm th}=2N_0 v_F\int dE\,E\langle N_1 f^T x\rangle . \tag{5.80}$$

As discussed by Schmid (1981), for example, the charge imbalance of Eq. (5.79) is directly observable in a tunneling experiment (Clarke, 1972,1981).

Although formally the problem is solved by these results, one must know N_1 and N_2 as functions of energy and angle in order to carry out the integrals over these two variables. The component of Eq. (5.10) diagonal in Keldysh space yields one equation which α and β must satisfy,

$$(E-p_F v_s\cos\theta)\beta-i\Delta\alpha+\frac{1}{2\tau}(\langle\alpha\rangle\beta-\langle\beta\rangle\alpha)=0 . \tag{5.81}$$

The functions α and β are obtained as solutions of Eq. (5.81) and the normalization condition

$$\alpha^2+\beta^2=1 . \tag{5.82}$$

The transition from the clean limit to the dirty limit has been discussed extensively by Schön (1981) and by Beyer Nielsen *et al.* (1982). The effect of impurities is found to depend strongly on energy. For the case of weak impurity scattering one finds that the charge relaxation rate for states in the pocket with energy $\Delta-p_F v_s\le E\le\Delta+p_F v_s$ is of the order $(1/\tau)(p_F v_s/\Delta)^{1/2}$. Pair breaking in the pocket region cannot be treated as small when the charge relaxation rate is comparable to a typical quasiparticle energy, measured with respect to the gap Δ. Thus for energies near Δ the transition between clean and dirty behavior occurs when

$$\frac{1}{\tau}\left[\frac{p_F v_s}{\Delta}\right]^{1/2}\sim p_F v_s ,$$

or

$$\frac{1}{\tau\Delta}\sim\left[\frac{p_F v_s}{\Delta}\right]^{1/2} . \tag{5.83}$$

For $|E|-\Delta-p_F v_s\gg p_F v_s$ Beyer Nielsen *et al.* (1982) found for the quantity of importance in the expression for the charge imbalance, Eq. (5.79),

$$\langle N_1 f^T\rangle=-\tau\mathbf{v}_s\cdot\nabla T\frac{E}{T}\frac{\partial f^0}{\partial E}v_F p_F \times\frac{\frac{4}{15}(E^2-\Delta^2)^2+2(E^2-\Delta^2)\Gamma^2+\frac{2}{3}\Gamma^4}{(E^2-\Delta^2+\Gamma^2)^2(E^2-\Delta^2)} , \tag{5.84}$$

where $\Gamma=1/2\tau$. This result shows that the transition between the clean and dirty limits occurs when $(E^2-\Delta^2)^{1/2}\simeq 1/2\tau$. Close to T_c, the region where $|E|-\Delta-p_F v_s\le p_F v_s$ contributes little to the charge imbalance, and to leading order in Δ the charge imbalance, in Eq. (5.79) may be obtained by integrating over E the high-energy expression (5.84) with Δ put equal to zero. One finds

$$Q^*=\tfrac{4}{15}N_0 p_F v_F\tau\mathbf{v}_s\cdot\frac{\nabla T}{T}I(\Gamma/T) , \tag{5.85}$$

where

$$I(\Gamma/T)=2+7y\psi'(\tfrac{1}{2}+y)+4y^2\psi''(\tfrac{1}{2}+y) , \tag{5.86}$$

where $y=\Gamma/(2\pi T)$, and ψ' and ψ'' are the trigamma function and its derivative. In the clean limit ($\Gamma/T\to 0$), $I=2$ and Q^* is given by

$$\tfrac{8}{15}N_0 v_F\tau\mathbf{v}_s\cdot\frac{\nabla T}{T} ,$$

while in the dirty limit $I=5$ and Q^* is given by

$$\tfrac{4}{3}N_0 v_F\tau\mathbf{v}_s\cdot\frac{\nabla T}{T} .$$

The transition between these two limiting behaviors is

given by the function $I(\Gamma/T)$, which increases monotonically with increasing Γ/T.

VI. CONCLUSIONS

The use of quantum field-theoretical methods in transport theory serves a dual purpose: it provides a microscopic justification for the use of a semiclassical Boltzmann equation, and it allows one to generalize this approach to take renormalization and lifetime effects into account in a systematic manner. The Keldysh formulation, which we have reviewed here, provides a convenient and general framework for deriving kinetic equations. In the context of certain applications, such as those involving localization and interaction effects in dirty metals, the use of a kinetic equation may be quite cumbersome. In such cases, the Keldysh technique may be used as a convenient starting point for linear-response theory, which allows answers to be obtained more directly. The advantage of the kinetic equation method over linear-response theory is that it allows one to consider situations where nonlinearities are important, and in addition provides a direct link to a description at the level of the semiclassical Boltzmann equation.

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APPENDIX: VERTICES, SELF-ENERGIES, AND THE COLLISION INTEGRAL

In this appendix we solve the algebraic equations for the vertex function Γ corresponding to Fig. 23. We then determine the self-energy Σ and derive the collision integral.

We first write out the self-energy $\Sigma_{\rm Coul}$ corresponding to Fig. 22(b),

$$\Sigma_{ij}^{\rm Coul}(1,1')=i\int d2\int d3\int d4\int d5\,\Gamma_{ii'}^{k}(5;1,3)U_{kk'}(5,4)\times G_{i'j'}(3,2)\tilde{\Gamma}_{j'j}^{k'}(4;2,1')\,. \tag{A1}$$

For calculating the inelastic collision rate, we can use the equilibrium value of the screened Coulomb propagator U, so that

$$U(5,4)=U(5-4)\,. \tag{A2}$$

We introduce the Fourier transformations

$$\Gamma(5;1,3)=\int (dq)e^{iq(x_5-x_3)}\Gamma(q;1,3)\,, \tag{A3}$$

$$\tilde{\Gamma}(4;2,1')=\int (dq)e^{iq(x_4-x_2)}\tilde{\Gamma}(q;2,1')\,, \tag{A4}$$

and obtain

$$\Sigma_{\rm Coul}(1,1')=i\int d2\int d3\int (dq)U(q)e^{iq(x_3-x_2)}\times\Gamma(-q;1,3)G(3,2)\times\tilde{\Gamma}(+q;2,1')\,. \tag{A5}$$

Transforming to the mixed representation and neglecting the gradient terms leads to

$$\Sigma_{ij}^{\rm Coul}(X,p)=i\int (dq)U_{kk'}(q)\Gamma_{ii'}^{k}(-q;p,X)\times G_{i'j'}(p-q,X)\tilde{\Gamma}_{j'j}^{k'}(q;p,X)\,. \tag{A6}$$

The absorption vertex Γ_{ij}^{k} is determined by the integral equation (cf. Fig. 23)

$$\Gamma_{ij}^{k}(-q;X,p)=\gamma_{ij}^{k}+\int (d\mathbf{k})\,|v(\mathbf{p}-\mathbf{k})|^2 G_{ii'}(X,k)\Gamma_{i'j'}^{k}(-q;X,k)G_{j'j}(X,k-q)\,, \tag{A7}$$

where $|v(\mathbf{p})|^2$ is the Fourier transform with respect to $\mathbf{r}-\mathbf{r}'$ of the correlator $\langle v(\mathbf{r})v(\mathbf{r}')\rangle$. A similar equation holds for the emission vertex $\tilde{\Gamma}$.

For simplicity we treat the impurity scattering in the s-wave approximation, so

$$|v(\mathbf{p})|=u \tag{A8}$$

[cf. Eq. (4.1)]. In that case the integral equations for the vertices will reduce to algebraic equations that are readily solved:

$$\Gamma^{1}(-q;X,E)=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & \dfrac{\eta^{RK}+\eta^{KA}}{1-\eta^{RA}}\\ 0 & 1\end{bmatrix}, \tag{A9}$$

$$\Gamma^{2}(-q;X,E)=\frac{1}{\sqrt{2}(1-\eta^{AR})}\begin{bmatrix}\eta^{KR} & \dfrac{1-\eta^{AR}+\eta^{KR}\eta^{RK}+\eta^{KK}+\eta^{KA}\eta^{AK}}{1-\eta^{RA}}\\ 1 & \eta^{AK}\end{bmatrix}, \tag{A10}$$

$$\tilde{\Gamma}^{1}(-q;X,E)=\Gamma^{2}(-q;X,E)\,|_{\eta\to\tilde{\eta}}\,, \tag{A11}$$

$$\tilde{\Gamma}^{2}(-q;X,E)=\Gamma^{1}(-q;X,E)\,|_{\eta\to\tilde{\eta}}\,, \tag{A12}$$

where (we suppress the spatial dependence)

$$\eta^{\alpha\beta}=\eta^{\alpha\beta}_{q,E}=u^{2}\int(d\mathbf{k})G^{\alpha}_{k}G^{\beta}_{k-q}\,, \tag{A13}$$

$$\tilde{\eta}^{\alpha\beta}=\tilde{\eta}^{\alpha\beta}_{q,E}=u^{2}\int(d\mathbf{k})G^{\alpha}_{k-q}G^{\beta}_{k}\,, \tag{A14}$$

and $\alpha,\beta=R,A,K$. These vertices have considerably more components in Keldysh space than the bare vertices, and the self-energies will accordingly be quite lengthy:

$$\Sigma^{R}_{\mathrm{Coul}}(X,p)=\frac{i}{2}\int(dq)\left[U^{R}_{q}\left[\frac{\tilde{\eta}^{KR}}{1-\tilde{\eta}^{AR}}G^{R}_{p-q}+\frac{1}{1-\tilde{\eta}^{AR}}G^{K}_{p-q}+\frac{\eta^{RK}}{(1-\eta^{RA})(1-\tilde{\eta}^{AR})}G^{A}_{p-q}\right]\right.$$
$$\left.+U^{K}_{q}G^{R}_{p-q}+U^{A}_{q}\frac{\eta^{KR}}{1-\eta^{AR}}G^{R}_{p-q}\right]\,, \tag{A15}$$

$$\Sigma^{A}_{\mathrm{Coul}}(X,p)=\frac{i}{2}\int(dq)\left[U^{A}_{q}\left[\frac{\eta^{AK}}{1-\eta^{AR}}G^{A}_{p-q}+\frac{1}{1-\eta^{AR}}G^{K}_{p-q}+\frac{\tilde{\eta}^{KA}}{(1-\eta^{AR})(1-\tilde{\eta}^{RA})}G^{R}_{p-q}\right]\right.$$
$$\left.+U^{K}_{q}G^{A}_{p-q}+U^{R}_{q}\frac{\tilde{\eta}^{AK}}{1-\tilde{\eta}^{AR}}G^{A}_{p-q}\right]\,, \tag{A16}$$

$$\Sigma^{K}_{\mathrm{Coul}}(X,p)=\frac{i}{2}\int(dq)\left[U^{R}_{q}G^{R}_{p-q}\frac{1-\tilde{\eta}^{AR}+\tilde{\eta}^{KR}\tilde{\eta}^{RK}+\tilde{\eta}^{KK}+\tilde{\eta}^{KA}\tilde{\eta}^{AK}}{(1-\tilde{\eta}^{AR})(1-\tilde{\eta}^{RA})}+U^{R}_{q}G^{K}_{p-q}\frac{\tilde{\eta}^{AK}}{1-\tilde{\eta}^{AR}}+\frac{\tilde{\eta}^{RK}+\tilde{\eta}^{KA}}{1-\tilde{\eta}^{RA}}U^{K}_{q}G^{R}_{p-q}\right.$$
$$+U^{K}_{q}G^{K}_{p-q}+\frac{\eta^{KR}}{1-\eta^{AR}}U^{A}_{q}G^{R}_{p-q}\frac{\tilde{\eta}^{KA}}{1-\tilde{\eta}^{RA}}+\frac{\eta^{KR}}{1-\eta^{AR}}U^{A}_{q}G^{K}_{p-q}+\frac{\eta^{RK}}{1-\eta^{RA}}U^{R}_{q}G^{A}_{p-q}\frac{\tilde{\eta}^{AK}}{1-\tilde{\eta}^{AR}}$$
$$\left.+\frac{\eta^{RK}+\eta^{KA}}{1-\eta^{RA}}U^{K}_{q}G^{A}_{p-q}+U^{A}_{q}G^{A}_{p-q}\frac{1-\eta^{AR}+\eta^{KR}\eta^{RK}+\eta^{KK}+\eta^{KA}\eta^{AK}}{(1-\eta^{AR})(1-\eta^{RA})}\right]\,. \tag{A17}$$

We can now write down the collision integral I, but beforehand we divide it into two parts, depending on the presence of the Coulomb propagator,

$$I=I_{\mathrm{imp}}+\delta_{\mathrm{Coul}}I\,, \tag{A18}$$

where I_{imp} and $\delta_{\mathrm{Coul}}I$ originate in $\underline{\Sigma}_{\mathrm{imp}}$ and $\underline{\Sigma}_{\mathrm{Coul}}$, respectively.

The impurity collision integral has the form

$$I_{\mathrm{imp}}=u^{2}\int(d\mathbf{k})i(G^{R}_{k}-G^{A}_{k})(h_{k}-h_{p})\,, \tag{A19}$$

as obtained from Eqs. (2.71) and (3.85). It vanishes when h is only a function of the energy, and therefore does not contribute to the inelastic collision integral.

The increment to the collision integral $\delta_{\mathrm{Coul}}I$ can be divided into an inelastic part I^{inel} that only vanishes if h is the equilibrium distribution, and an elastic part I^{el} that vanishes whenever h is only a function of the energy,

$$\delta_{\mathrm{Coul}}I=I^{\mathrm{el}}+I^{\mathrm{inel}}\,. \tag{A20}$$

As we shall see, this separation requires the addition and subtraction of the terms involving u^{2} explicitly in Eqs. (A21) and (A22) below.

Using Eqs. (A15)—(A17) we obtain

$$I^{\mathrm{inel}}=\tfrac{1}{2}\int(dq)\left\{U^{R}_{q}G^{R}_{p-q}+U^{A}_{q}G^{A}_{p-q}+U^{K}_{q}G^{K}_{p-q}-2h_{p}\mathrm{Re}(U^{R}_{q}G^{K}_{p-q}+U^{K}_{q}G^{R}_{p-q})+2\,\mathrm{Re}\left[U^{K}_{q}G^{R}_{p-q}\frac{\tilde{\eta}^{RK}+\tilde{\eta}^{KA}}{1-\zeta^{*}}\right]\right.$$
$$+2\,\mathrm{Re}\left[\frac{\zeta}{1-\zeta}U^{A}_{q}G^{A}_{p-q}\left[1-\frac{u^{2}}{\zeta}\int(d\mathbf{k})G^{R}_{k}G^{A}_{k-q}h_{k}h_{k-q})\right]\right]$$
$$\left.+2\,\mathrm{Re}\left[\frac{u^{2}}{1-\zeta}U^{R}_{q}G^{A}_{p-q}\int(d\mathbf{k})G^{R}_{k}G^{A}_{k-q}h_{k}h_{k-q}\right]\right\} \tag{A21}$$

and

$$I^{\mathrm{el}}=\int (dq)\Bigg[\mathrm{Re}\Bigg\{U_q^A G_{p-q}^A\Bigg[\frac{\eta^{KR}\eta^{RK}+\eta^{KK}+\eta^{KA}\eta^{AK}}{|1-\zeta|^2}-\frac{\eta^{KR}}{1-\zeta^*}h_{p-q}+\frac{u^2}{1-\zeta}\int (d\mathbf{k})G_k^R G_{k-q}^A h_k h_{k-q}$$
$$-h_p\Bigg[-\frac{\eta^{AK}}{1-\zeta^*}+\frac{\zeta^*}{1-\zeta^*}h_{p-q}\Bigg]\Bigg]\Bigg\}$$
$$+\mathrm{Re}\Bigg\{U_q^R G_{p-q}^A\Bigg[\frac{\eta^{RK}\tilde{\eta}^{AK}}{(1-\zeta)^2}-\frac{\tilde{\eta}^{AK}}{1-\zeta}h_{p-q}$$
$$-\Bigg[\frac{\eta^{AK}}{(1-\zeta)^2}-\frac{\zeta}{1-\zeta}h_{p-q}\Bigg]h_p-\frac{u^2}{1-\zeta}\int (d\mathbf{k})G_k^R G_{k-q}^A h_k h_{k-q}\Bigg]\Bigg\}\Bigg], \tag{A22}$$

where the expressions have been simplified by noting that

$$\tilde{\eta}^{\alpha\bar{\alpha}}=\eta^{\bar{\alpha}\alpha}=(\eta^{\alpha\bar{\alpha}})^*,\quad \alpha=R,A,\quad \bar{\alpha}=\begin{cases}R, & \alpha=A\\ A, & \alpha=R\end{cases}. \tag{A23}$$

This completes the derivation of the collision integrals. We shall in addition need the polarization given in Eq. (4.52). Using the Feynman rules we obtain for the components of Eq. (4.53) the result

$$\Pi_q^R=\frac{-i}{1-\zeta}\int (dp)(G_p^R G_{p-q}^K+G_p^K G_{p-q}^A+\tilde{\eta}^{KR}G_p^R G_{p-q}^R+\tilde{\eta}^{AK}G_p^A G_{p-q}^A), \tag{A24}$$

$$\Pi_q^A=-i\int (dp)(G_p^K G_{p-q}^R+G_p^A G_{p-q}^K+\frac{\tilde{\eta}^{RK}+\tilde{\eta}^{KA}}{1-\zeta^*}G_p^A G_{p-q}^R), \tag{A25}$$

$$\Pi_q^K=\frac{-i}{1-\zeta}\int (dp)\Bigg[G_p^R G_{p-q}^A+\tilde{\eta}^{KR}G_p^K G_{p-q}^R+G_p^K G_{p-q}^K+\tilde{\eta}^{AK}G_p^A G_{p-q}^K$$
$$+\frac{1-\tilde{\eta}^{AR}+\tilde{\eta}^{KR}\tilde{\eta}^{RK}+\tilde{\eta}^{KK}+\tilde{\eta}^{KA}\tilde{\eta}^{AK}}{1-\zeta^*}G_p^A G_{p-q}^R\Bigg]. \tag{A26}$$

Note that the various inter-relations between the expressions appearing in this appendix constitute a useful check on their correctness.

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