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Condensed Matter Field Theory — WS09/10

Exercise 3

(Please return your solutions before Fr. 27.11. 12:00h)

3.1 Schrödinger, Heisenberg and Dirac picture (15 points)

To describe time dependences of physical systems one can choose between three equivalent pictures. In this exercise we want to study these three pictures and we will derive the equations of motions for the operators and the states depending on the representation. In the end we will derive an expression for the time evolution operator in the Dirac picture which will serve as a starting point for the diagrammatical treatment of many body physics.

In the **Schrödinger picture** the states evolve with time, while the operators are time-independent. The equation of motion reads

$$i\hbar\partial_t|\Psi_S(t)\rangle = H_t|\Psi_S(t)\rangle \,.$$

where the subindex t denotes the explicit time dependence of the Hamiltonian. The time-evolution operator is defined by

$$|\Psi_S(t)\rangle = U_S(t,t_0)|\Psi_S(t_0)\rangle \,,$$

with U_S unitary, $U_S(t_0,t_0) = 1$ and $U_S(t,t_0) = U_S(t,t')U_S(t',t_0)$.

(a) Derive the equation of motion for the time-evolution operator and solve it to show that

$$U_S(t,t_0) = 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, H_{t_1}\ldots H_{t_n}$$

holds. Where $t_0 \le t_n \le t_{n-1} \le \ldots \le t_1 \le t$. What does the time evolution operator look like for a closed system, i.e. $\frac{\partial H}{\partial t} = 0$?

In order to rewrite the expression for $U_S(t,t_0)$ we introduce Dyson's time-ordering operator:

$$T_D(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2)\,, & t_1 > t_2 \\ B(t_2)A(t_1)\,, & t_2 > t_1 \end{cases}$$

With the help of figure 1 one can see that:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{t_1}H_{t_2} = \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1\, H_{t_1}H_{t_2} \,. \tag{1}$$

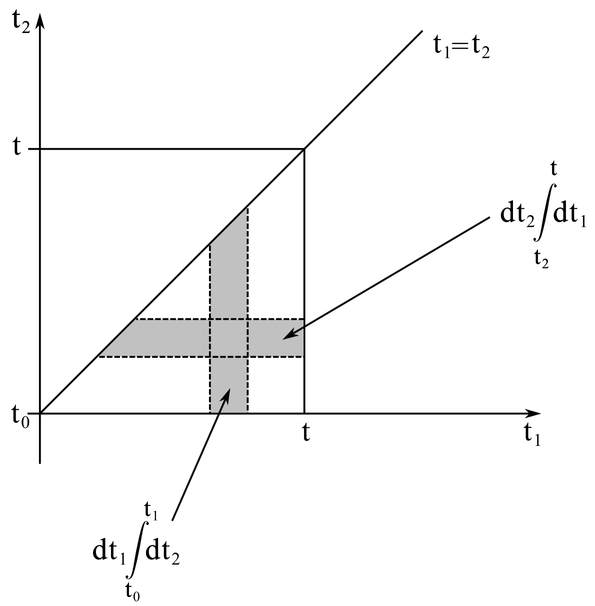

Figure 1: Transformation of the time ordering operator.

(b) Use relation (1) to show that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{t_1} H_{t_2} = \frac{1}{2!} \int_{t_0}^{t} \int_{t_0}^{t} dt_1 dt_2\, T_D(H_{t_1} H_{t_2})$$

holds and generalize this to

$$U_S(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int\limits_{t_0}^{t} dt_1 \int\limits_{t_0}^{t} dt_2 \cdots \int\limits_{t_0}^{t} dt_n T_D \left(H_{t_1} \ldots H_{t_n}\right).$$

In the **Heisenberg picture** the time-dependence is shifted to the operators and the states are constant in time. The state in Heisenberg representation is defined by

$$|\Psi_H\rangle = |\Psi_S(t_0)\rangle\,,$$

t_0 arbitrarily fixed.
Since $|\Psi_S(t)\rangle = U_S(t, t_0)|\Psi_S(t_0)\rangle$ it holds that $|\Psi_H\rangle = U_S^{-1}(t, t_0)|\Psi_S(t)\rangle$.

(c) On condition that $\langle\Psi_H|A_H(t)|\Psi_H\rangle \stackrel{!}{=} \langle\Psi_S(t)|A_S|\Psi_S(t)\rangle$ holds for an arbitrary operator A deduce the relation between $A_H(t)$ and A_S. Derive the equation of motion for $A_H(t)$.

The **Dirac picture** is a mixed representation. Starting point is the Hamiltonian

$$H = H_0 + V_t\,,$$

where H_0 is for the free system and V_t describes a (in general) time-dependent interaction. For this picture we make the following ansatz:

$$\begin{aligned} |\Psi_D(t_0)\rangle &= |\Psi_S(t_0)\rangle = |\Psi_H\rangle \\ |\Psi_D(t)\rangle &= U_D(t, t')|\Psi_D(t')\rangle \\ |\Psi_D(t)\rangle &= e^{\frac{i}{\hbar}H_0(t-t')}|\Psi_S(t)\rangle \end{aligned}$$

Note that without interaction ($V_t = 0$) the Dirac picture becomes the Heisenberg picture.

(d) Derive the relation between $U_D(t,t')$ and $U_S(t,t')$.

(e) Provided that $\langle\Psi_D(t)|A_D(t)|\Psi_D(t)\rangle \stackrel{!}{=} \langle\Psi_S(t)|A_S|\Psi_S(t)\rangle$ derive $A_D(t)$ in dependence of A_S and write down the equation of motion for A_H. You will see that the dynamics of the operators is completely determined by H_0.

(f) Derive the equation of motion for the state in Dirac representation and show that the dynamics of the states is given by V_t.

(g) Finally, derive the equation of motion for $U_D(t,t')$ and solve it to show that

$$U_D(t,t') = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{t'}^{t} \dots \int_{t'}^{t} dt_1 \dots dt_n \, T_D(V^D_{t_1} \dots V^D_{t_n}) \tag{2}$$

holds.

3.2 Perturbative expansion of the causal Green's function (15 points)

The time evolution operator (2) is ordered by powers of the interaction V. In the following we will use it to construct the ground state of the interacting system out of the unique groundstate of the free system $|\eta_0\rangle$. To do so, we will introduce an artificial time dependence to switch on the interaction adiabatically

$$\mathcal{H}_\alpha = \mathcal{H}_0 + V e^{-\alpha|t|} \quad ; \quad \alpha > 0,$$

and perform the limit $\alpha \to 0$ in the end. The time evolution operator becomes

$$U^D_\alpha(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n \int_{t_0}^{t} \dots \int_{t_0}^{t} dt_1 \cdots dt_n e^{-\alpha(|t_1|+\cdots|t_n|)} T_D\{V^D(t_1)\cdots V^D(t_n)\}. \tag{3}$$

The eigenstate of the interacting system will be time independent in the limit $\alpha \to 0$ and $t \to -\infty$. For $T = 0$ this state will differ form $|\eta_0\rangle$ only by a phase factor which can be set to 1. The *Gell-Mann-Low-Theorem*[1] now states, that if the state

$$|\tilde{E}_0\rangle = \lim_{\alpha\to 0} \frac{U^D_\alpha(0,-\infty)|\eta_0\rangle}{\langle\eta_0|U^D_\alpha(0,-\infty)|\eta_0\rangle}, \tag{4}$$

exists in all orders of perturbation theory it is an exact eigenstate of $\mathcal{H}$.

(a)* Prove the Gell-Mann-Low-Theorem.

It is not for sure that there will be no 'crossover' of the states during evolution in time. Therefore we make the additional assumption that there will be no crossover and $|E_0\rangle$ is indeed the *groundstate* of the interacting system.

[1] W. Nolting, Grundkurs Theoretische Physik 7

(b) The state

$$|\tilde{E}_0'\rangle = \lim_{\alpha\to 0} \frac{U_\alpha^D(0,+\infty)|\eta_0\rangle}{\langle\eta_0|U_\alpha^D(0,+\infty)|\eta_0\rangle},$$

which evolves from the free groundstate backward in time is identical to (4). Why is this so?

The Green's function for $T = 0$ is an expectation value of Heisenberg operators in the normalized groundstate $|E_0\rangle$.

(c) Show that

$$\langle E_0|A^H(t)|E_0\rangle = \lim_{\alpha\to 0} \frac{\langle\eta_0|U_\alpha^D(\infty,t)A^D(t)U_\alpha^D(t,-\infty)|\eta_0\rangle}{\langle\eta_0|U_\alpha^D(\infty,t),U_\alpha^D(t,-\infty)|\eta_0\rangle}, \tag{5}$$

holds and derive a similar expression for

$$\langle E_0|A^H(t)B^H(t')|E_0\rangle. \tag{6}$$

where $A^H(t)$ and $B^H(t')$ are operators in the Heisenberg picture.

In the definition of the causal Green's function (8) Wick's time ordering operator T_ϵ occurs. In contrast to Dyson's time ordering operator T_D, T_ϵ gives rise to an additional factor of $\epsilon = \pm 1$ for each commutation of two operators. Nevertheless, since the interaction operator for fermions always contains an even number of creation or annihilation operators, we can set

$$T_\epsilon = T_D.$$

(d) Plug the time evolution operator (3) into (5) and (6). Show that

$$\begin{aligned}\langle E_0|A^H(t)|E_0\rangle =& \lim_{\alpha\to 0} \frac{1}{\langle\eta_0|S_\alpha|\eta_0\rangle} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_n e^{-\alpha(|t_1|+\cdots|t_n|)} \\ & \cdot \langle\eta_0|T_\epsilon\left\{V(t_1)\cdots V(t_n)A(t)\right\}|\eta_0\rangle \end{aligned} \tag{7}$$

holds and that the causal Green's function can be written as

$$\begin{aligned}\mathrm{i}G^c_{AB}(t,t') =& \langle E_0|T_\epsilon\left\{A^H(t)B^H(t')\right\}|E_0\rangle \\ =& \lim_{\alpha\to 0} \frac{1}{\langle\eta_0|S_\alpha|\eta_0\rangle} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_n e^{-\alpha(|t_1|+\cdots|t_n|)} \\ & \cdot \langle\eta_0|T_\epsilon\left\{V(t_1)\cdots V(t_n)A(t)B(t')\right\}|\eta_0\rangle. \end{aligned} \tag{8}$$

where we have dropped the index for operators in the Dirac picture for simplicity.

It will turn out that the expectation values of the free groundstate will lead to the *free* causal Green' function $G^{c,0}_{AB}(t,t')$ which is simple to calculate (see exercise 2.1). Hereby it is possible to express the full causal Green's function $G^c_{AB}(t,t')$ in a complicated way in terms of $G^{c,0}_{AB}(t,t')$ and the interaction potential V (see Wick's theorem, Feynman diagrams).

Hints:

(a)* Consider

$$(\mathcal{H}-\eta_0)U_\alpha^D(0,-\infty)|\eta_0\rangle = \left[\mathcal{H}, U_\alpha^D(0,-\infty)\right]_-|\eta_0\rangle$$

and use the equation of motion for a Dirac operator to convert the commutator into a time derivative. Then use partial integration to get rid of the time derivative. Extract a coupling constant λ out of the interaction $V(t) = \lambda\tilde{V}(t)$ and adjust the summation index in both summands stemming from the partial integration to show that

$$(\mathcal{H}-\eta_0)U_\alpha^D(0,-\infty)|\eta_0\rangle = \alpha i\hbar\lambda\frac{\partial}{\partial\lambda}U_\alpha^D(0,-\infty)|\eta_0\rangle$$

holds. Finally take the limit $\alpha \to 0$.

(c) Write $A^H(t)$ and $B^H(t')$ in Dirac representation and introduce the scattering matrix $S_\alpha = U_\alpha^D(\infty,-\infty)$.

(d) Use (7) as a starting point. Consider a time t with

$$t_1, t_2, \ldots, t_n > t$$
$$\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m < t$$

to split up

$$T_\epsilon\{\ldots\} = T_\epsilon\{V(t_1)\cdots V(t_n)\}\,A(t)\,T_\epsilon\{V(\bar{t}_1)\cdots V(\bar{t}_m)\}.$$