Exercises 2

Exercises on April 24th - 28th.

Exercise 2.1 Ideal Gas

If the entropy of a system is given as a function of the extensive state variables from a microscopic theory, then we can determine the state equations explicitly by means of the thermodynamic fundamental relations. We shall here use the converse way:
For an ideal gas let the two equations of states be experimentally given by

$$U = \frac{f}{2} N k_B T, \qquad pV = N k_B T,$$

where k_B is the Boltzmann constant and f is the number of degrees of freedom per molecule (for a one-atomic ideal gas for example $f = 3$). We shall then derive the different thermodynamic relations and the entropy.

a) Show that for an adiabatic change of state ($\mathrm{d}S = \frac{\delta Q}{T} = 0$) with constant particle number N the eqns.

$$pV^{(f+2)/f} = const. \qquad \text{and} \qquad VT^{f/2} = const.$$

hold.

Use the differential form of the fundamental relation $\mathrm{d}S = (1/T)\mathrm{d}U + (p/T)\mathrm{d}V - (\mu/T)\mathrm{d}N$, and eliminate the unknown variables.

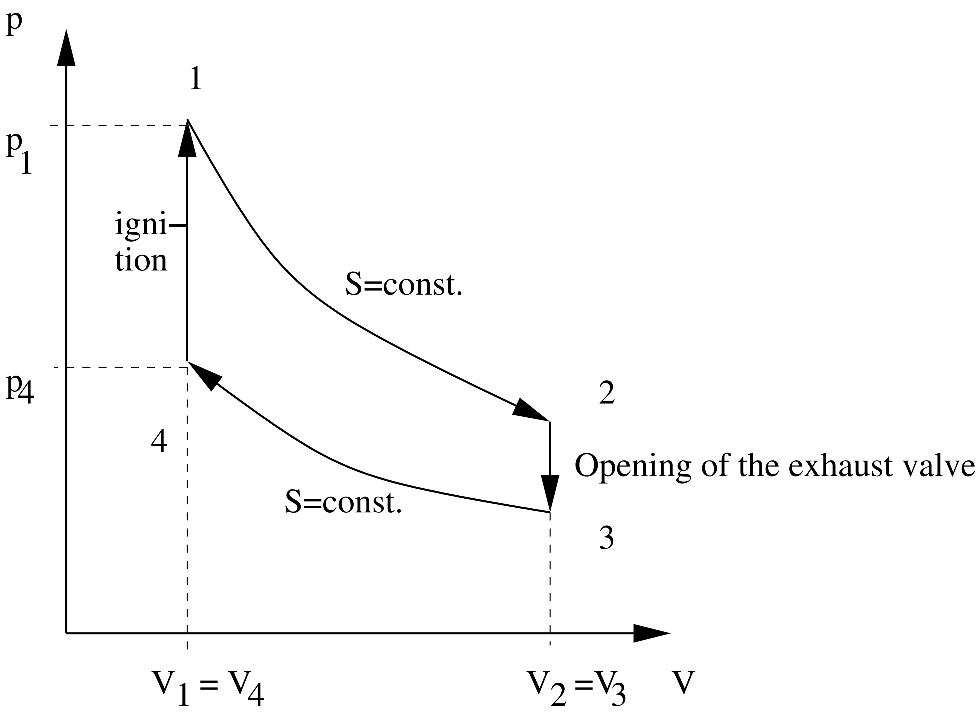
b) Show that the entropy of the ideal gas is given by

$$S(U,V,N) = S_o \frac{N}{N_o} + N k_B \Big[\frac{f}{2} \ln\Big(\frac{U}{U_o}\Big) + \ln\Big(\frac{V}{V_o}\Big) - \frac{f+2}{2} \ln\Big(\frac{N}{N_o}\Big) \Big],$$

where S_o, U_o, V_o, N_o are integration constants.
(Hint: Integrate the equation $\mathrm{d}s = (1/T)\mathrm{d}u + (p/T)\mathrm{d}v$, with $s = S/N$, $u = U/N$, $v = V/N$. How do you get this equation from the thermodynamic fundamental relation?)

Exercise 2.2 Otto Engine

We idealize an internal combustion engine by the circle process shown in the picture. The cylinder will be filled up with a gasoline/air mixture at point 3. We suppose that the engine uses an ideal gas. Calculate the efficiency factor $\eta = \Delta W / \Delta Q_{23}$ depending on the compression ratio $\epsilon = V_1/V_2$.



Exercise 2.3 Thermodynamic potentials and Legendre transformations

Let $f(x_1, ..., x_n)$ be a function of n variables x_i, $i = 1, \cdots, n$. By a Legendre transformation we exchange the variable x_i by the variable $y_i = \frac{\partial f}{\partial x_i}$. The Legendre transform is then $f - y_i x_i$.

a. Let $f(x) = a + bx^2$, $g(x) = -\frac{1}{x}$, $x > 0$. Calculate the Legendre transform of f and g. Show that the Legendre transformation is involutive.

b. Let $U(S, V, N)$ be the internal energy of a system. The thermodynamic potentials $H(S, P, N)$ (enthalpy), $F = F(T, V, N)$ (free energy), $G(T, P, N)$ (Gibbs enthalpy) can be expressed as the Legendre transforms of U with respect to S, V, S, and V. Calculate these Legendre transforms and give the relation between U, H, F, and G.