Universität Bonn 19.01.2006
Physikalisches Institut
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Condensed Matter Theory I — WS05/06

Exercise 6

(Please return your solutions before 31.01., 13:00 h)

6.1 Diagramatical Calculation of the BCS Equation (10 points)

In the lecture the BCS Hamiltonian and the BCS ground state were introduced. We will now derive the BCS equations from a diagramatic perturbation theory calculation. Since in the superconducting phase there are *pairs* of electrons with opposite momenta, it is convenient to work explicitly in the Hilbert subspace of particles and holes with opposite momenta (*Nambu representation*). In the Nambu representation the BCS Hamiltonian is given by (compare lecture)

$$H_{BCS} = \sum_k \Psi_k^\dagger h_k \Psi_k$$

with:

$$h_k = \begin{pmatrix} \varepsilon_k & \Delta \\ \Delta & -\varepsilon_k \end{pmatrix} \qquad \Psi_k = \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \qquad \Psi_k^\dagger = \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix}$$

a) On the one hand, the full (matrix) Green's function $G(E,k)$ is the resolvent

$$(E-H)\cdot G = \mathbb{1}$$

Use this relation to calculate $G(E,k)$. Calculate also the free ($\Delta = 0$) Green's function $G_0(E,k)$ and the self energy Σ.

b) On the other hand, the (time-ordered) Green's function at $T=0$ in time space is given by

$$\begin{aligned} G(t,k) &= -i\langle \Psi_{BCS}|T\{\Psi_k(t)\,\Psi_k^\dagger(0)\}|\Psi_{BCS}\rangle \\ &= -i\begin{pmatrix} \langle \Psi_{BCS}|T\{c_{k\uparrow}(t)\,c^\dagger_{k\uparrow}(0)\}|\Psi_{BCS}\rangle & \langle \Psi_{BCS}|T\{c_{k\uparrow}(t)\,c_{-k\downarrow}(0)\}|\Psi_{BCS}\rangle \\ \langle \Psi_{BCS}|T\{c^\dagger_{-k\downarrow}(t)\,c^\dagger_{k\uparrow}(0)\}|\Psi_{BCS}\rangle & \langle \Psi_{BCS}|T\{c^\dagger_{-k\downarrow}(t)\,c_{-k\downarrow}(0)\}|\Psi_{BCS}\rangle \end{pmatrix} \end{aligned}$$

$|\Psi_{BCS}\rangle$ denotes the BCS ground state.
The perturbation expansion with these objects is very similar to the usual procedure. Especially the same diagram rules can be used. Then each full line

denotes a factor $iG(E,k)$, each bare line a factor $iG_0(E,k)$. Additionally to the usual rules each interaction line has to be multiplied with factor τ_z (Pauli matrix) at each end. Finally each closed fermion loop denotes $(-1) \cdot tr(\tau_z G)$. Calculate with these rules the self energy Σ in Hartree-Fock approximation (Fig.1).

c) Obtain from b) the self consistent BCS equation for Δ and from a) the quasi-particle excitation energies.

6.2 Pair Correlation Function (6 points)

Calculate first the particle number expectation value (here $|\Psi_{BCS}\rangle$ again denotes the BCS ground state)

$$\langle \Psi_{BCS} | c^\dagger_{k\sigma} c_{k\sigma} | \Psi_{BCS} \rangle$$

and use the result to calculate the pair expectation value, i.e. the "pair correlation function"

$$\langle \Psi_{BCS} | c^\dagger_{-k,-\sigma} c^\dagger_{k\sigma} c_{k\sigma} c_{-k,-\sigma} | \Psi_{BCS} \rangle$$

What is the probability that a pair is created from electrons well below the energy gap ($\varepsilon_k \ll -\Delta$) and from electrons inside the gap ($-\Delta \leq \varepsilon_k \leq \varepsilon_F = 0$)?

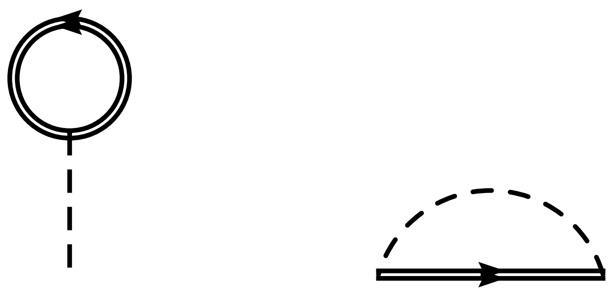

Figure 1: Hartree-Fock Self Energy Diagrams