Condensed Matter Theory I — WS09/10

Exercise 6

(Please return your solutions before Fr., 15.1., 12:00h)

6.1 Phonons in a metal: Kohn Anomaly   (10 points)

In metals one expects that the harmonic interactions between the ions of a lattice
are not only between nearest neighbors, but are long-ranged. The physical origin is
a displacement of an ionic charge that induces an electric charge polarization in the
electron sea which is long-ranged and oscillatory in space. This oscillatory polariza-
tion acts on the other ions in the lattice. The oscillations are induced by the fact that
the electronic states are filled up to the Fermi wave number k_F and the wave number
of the resulting oscillatory ion-ion potential is $2k_F$ (Friedel oscillations).

We now consider a one-dimensional linear chain of ions in an electron sea with the
lattice constant a. The force constant between an ion on lattice site i and an ion on
lattice site j is

$$\kappa_{ij} = \kappa_0 \frac{\sin 2k_F a(i-j)}{k_F a(i-j)} \ .$$

(a) Write down the Lagrange function for the system of ions and derive the equa-
tions of motion.

(b) Make an ansatz of plane wave solutions for the ion displacements:

$$q_j(t) = q_0 e^{i(kx - \omega t)}, \ x_j = j \cdot a$$

Derive an expression for $\omega(k)^2$ and $\frac{\partial \omega^2}{\partial k}$ and show that $\omega(k)^2$ has a cusp (diver-
gent slope) at $k = 2k_F$. This effect is called Kohn anomaly after Walter Kohn,
who predicted it in 1959.

6.2 Optical phonons   (10 points)

Consider a one-dimensional harmonic chain with two different atoms in the primitive
cell,
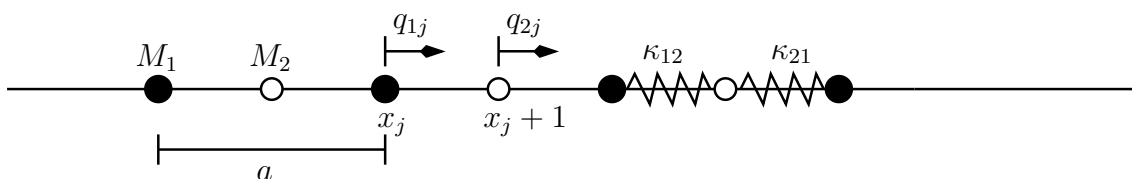
where $x_j = a \cdot j$ is the position of the jth primitive cell and q_{sj} is the elongation of
the sth atom in the jth cell.
The Hamiltonian of the system is given by

$$H = \sum_{js} \frac{1}{2} M_s \dot{q}_s(x_j) + \frac{1}{2} \sum_j \kappa_{12}(q_{2j} - q_{1j})^2 + \kappa_{21}(q_{1j} - q_{2j-1})^2) \ .$$

For simplicity we assume $\kappa_{12} = \kappa_{21} = \kappa$.

(a) Write down the equations of motion. Use the ansatz

$$q_{sj}(t) = q_s^0 e^{i(kaj - \omega t)} , \quad s = 1, 2$$

to derive the following equation for the frequencies of the eigenmodes (phonons):

$$\omega^2 = \frac{\kappa}{M^*} \pm \sqrt{\left(\frac{\kappa}{M^*}\right)^2 - 2\frac{\kappa^2}{M_1 M_2}(1 - \cos ka)}$$

$M^* = \frac{M_1 M_2}{M_1 + M_2}$ is the reduced mass.

(b) Calculate $\omega(k)^2$ for $k \to 0$ and $k \to \pm\frac{\pi}{a}$ and sketch its behavior for the full Brillouin zone $[-\frac{\pi}{a}, \frac{\pi}{a}]$. Identify the acoustic and the optical branch.

(c) Discuss the limit $M_1 = M_2$ and show that this limit results in the acoustic phonon dispersion with lattice constant $\frac{a}{2}$.

6.3 Meissner effect $\hfill$ (10 points)
The London equations of superconductivity are

$$\frac{\partial}{\partial t}\left(\Lambda \vec{j}\right) = \vec{E} \qquad \vec{\nabla} \times \left(\Lambda \vec{j}\right) = -\vec{H} \qquad \Lambda = \frac{m}{2n_s e^2}.$$

Consider a thin superconducting slab of thickness d, infinitely extended in y and z direction, in a uniform, static magnetic field $\vec{H}_0 = H_0 \hat{e}_z$ parallel to the slab surface.

a) Use the Maxwell equations to calculate the magnetic field $\vec{H}(\vec{x})$ inside the slab. Show that the field enters the slab only on a length scale λ_L (London penetration depth). Draw the result.

b) Calculate also the current density inside the slab and draw the result.

c) Calculate the average magnetization of the slab,

$$\langle H(x) \rangle = \frac{1}{d} \int\limits_{-d/2}^{d/2} dx\, H(x),$$

and draw it as a function of d.
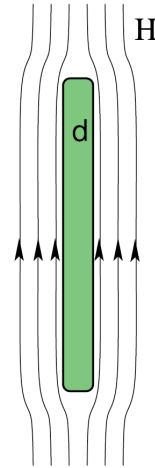
Consider now a current flow through an infinitely long cylindrical superconducting wire of radius R ($R \gg \lambda_L$).

d) Use the continuity equation to show that the current density distribution $\vec{j}(r)$ obeys an ordinary differential equation of the form (*modified Bessel differential equation*)

$$x^2 \frac{d^2 j}{dx^2} + x \frac{dj}{dx} - x^2 j = 0$$

Hint: Laplace operator in cylindrical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$
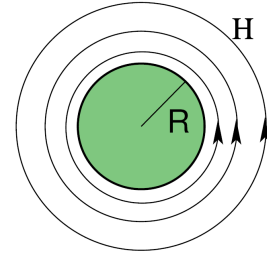
e) Draw the solution of d) (*modified Bessel function of the first kind*) and show that the current flows only in a small layer underneath the surface of the wire.

f) Use Biot-Savart's law to calculate the magnetic field outside the wire induced by the current. If the magnetic field at the surface exceeds a critical value H_c, the superconductivity will collapse. What is the corresponding critical current I_c?