Solution 3.2

We start with the free electron Green's function

$$G_E^{ret,\,av}(\mathbf{k}) = \frac{1}{E - \frac{k^2}{2m} \pm i0^+}\,.$$

We want to derive the associated space-dependent Green's function by Fourier transformation.

$$\begin{aligned}
G_E(\mathbf{r}) &= \int \frac{d^3k}{(2\pi)^3}\, e^{ikr\cos\Theta} \frac{1}{E - \frac{k^2}{2m} \pm i0^+} \\
&= \frac{1}{(2\pi)^3} \underbrace{2\pi}_{\varphi-\text{integration}} \int_{-1}^{1} dcos\Theta \int_0^\infty dk\, k^2 e^{ikr\cos\Theta} \frac{1}{E - \frac{k^2}{2m} \pm i0^+} \\
&= \frac{1}{(2\pi)^2} \int_0^\infty dk\, k^2 \frac{1}{E - \frac{k^2}{2m} \pm i0^+} \left(\frac{1}{ikr} \left(e^{ikr} - e^{-ikr} \right) \right) \\
&= \frac{1}{(2\pi)^2} \frac{1}{ir} \int_0^\infty dk\, k \frac{1}{\frac{k_0^2}{2m} - \frac{k^2}{2m} \pm i0^+} \left(e^{ikr} - e^{-ikr} \right) \\
&= \frac{1}{(2\pi)^2} \frac{2m}{ir} \int_0^\infty dk\, k \frac{1}{k_0^2 - k^2 + i0^+} \left(e^{ikr} - e^{-ikr} \right) \\
&= \frac{1}{(2\pi)^2} \frac{2m}{ir} \int_0^\infty dk\, k \frac{e^{ikr} - e^{-ikr}}{(k_0 - k \pm i0^+)(k_0 + k \pm i0^+)} \\
&= \frac{1}{(2\pi)^2} \frac{2m}{ir} \int_{-\infty}^\infty dk\, k \frac{e^{ikr}}{(k_0 - k \pm i0^+)(k_0 + k \pm i0^+)} \\
&= -\frac{1}{(2\pi)^2} \frac{2m}{ir} \int_{-\infty}^\infty dk\, k \frac{e^{ikr}}{(k - k_0 \mp i0^+)(k + k_0 \pm i0^+)} \\
&= -\frac{1}{(2\pi)^2} \frac{2m}{ir} \left(2\pi i\, k_0 \frac{e^{\pm ik_0 r}}{2k_0} \right) \\
&= -\frac{2m}{4\pi} \frac{e^{\pm ik_0 r}}{r}
\end{aligned}$$

with $k_0 = \sqrt{2mE}$.