# Roots of a Single-Variable equations | Lecture №2

1) Iteration procedure
2) Convergence
3) Newton's method
4) Secant method

$\left.\begin{matrix}\text{3) Newton's method}\\ \text{4) Secant method}\end{matrix}\right\}$ ~~single~~ one point method

5) Bisection method (Bracketing)
6) Regular falsi method (secant + Bracketing)

$\left.\begin{matrix}\text{5) Bisection method}\\ \text{6) Regular falsi method}\end{matrix}\right\}$ two point methods

**DF** If $f$ is a real-valued function defined on a closed interval $I=[a,b]$, then a number $\xi \in I$ is called a <u>zero</u> or a <u>root</u> of $f$, or a <u>solution</u> of the eq. $f(x)=0$, if $\boxed{f(\xi)=0}$.

**DF** if $f$ is algebraic polynomial of the form

$$f(x) \equiv P_n(x) = \sum_{j\geq 0}^{n} a_j x^j, \text{ with } a_j \in \mathbb{R},\ a_n \neq 0, \text{ then}$$

eq. $f(x)=0$ is called <u>algebraic</u> of degree <u>$n$</u>.

**DF:** Every eqs. which is not <u>algebraic</u> is called <u>transcendental</u>.

EX: $\ln x - 1/x = 0$

Comment:

The problem of root finding is related to the <u>minimization problem</u>:

$$\min_{x\in\mathbb{R}} h(x)\ ; \text{ if } h'(x) \text{ exists} \Rightarrow \min h \text{ is given by}$$

roots of $\boxed{h'(x)=0}$.

## Constructing an Iterative process

It's often usefull to represent our non-linear eq $f(x)=0$ in the form: $\boxed{\varphi(x)=x}$, where $\varphi$ is a real-valued continuos function on $I=[a,b]$.

fixed point eq.

**Lemma:** if $f$ and $g$ are continuous on $I$, and $g(x)\neq 0\ \forall x\in I$, than eqs. $\boxed{f(x)=0}$ and $\boxed{\varphi(x)=x}$; with $\boxed{\varphi(x)\equiv x - f(x)g(x)}$ have the same solutions on $I$.

□

$$\varphi(x) = x - f(x)g(x) = x$$

$$x - x = f(x)g(x)$$

$$\frac{0}{g(x)} = f(x) \iff f(x)=0 \qquad \square$$

**DF:** Solutions of fixed point eq. $\varphi(x)=x$, $\xi$, is called fixed point of $\varphi$ if $\boxed{\varphi(\xi)=\xi}$.

Fixed point iteration: chose an initial value of $x_0 \in I$, then the sequence $\{x_\nu\}$ can be constructed as

$$\boxed{x_\nu = \varphi(x_{\nu-1})}, \quad \nu = 1, 2, \ldots$$

Theorem:

If the $\{x_\nu\}$ converges to some real $\xi$, i.e., if $\boxed{\lim_{\nu\to\infty} x_\nu = \xi \in I}$, then $\xi$ is ~~to the fixed point~~ a solution of eq. $\varphi(x) = x$.

□

$$\xi = \lim_{\nu\to\infty} x_\nu = \lim_{\nu\to\infty} x_{\nu+1} = \lim_{\nu\to\infty} \varphi(x_\nu) = \varphi(\lim_{\nu\to\infty} x_\nu) = \varphi(\xi)$$

$$\xi = \varphi(\xi) \qquad \varphi\text{-continuos.}$$

□

Algorithm: for a given eq. $f(x) = 0$

1) suitable iteration function should be found.
2) initial value $x_0$ should be shosen
3) $x_0 \equiv$ initial guess
   $x_1 = \varphi(x_0)$
   $\vdots$
   $x_N = \varphi(x_{N-1})$
   if $|x_\nu - x_{\nu-1}| \leq \delta |x_\nu|$ relative iteration error stop.

Ex. Trivial example:

$$f(x) = \cos(x) - x = 0$$

$$\varphi(x) = \cos(x)$$

$$\boxed{x_i = \cos(x_{i-1})} \Leftarrow x = \varphi(x) \Rightarrow x = \cos(x)$$

Theorem (Existence and Uniqueness) a) if $\varphi \in C[a, b]$, and $\varphi(x) \in I \ \forall x \in I$, than $\varphi(x)$ has a fixed point on $I = [a, b]$

b) if $\varphi \in C^1[a, b]$, i.e., $g'(x)$ exists on $I$, $\exists\, 0 < k < 1$ with

$$|\varphi'(x)| \leq k < 1 \quad \forall x \in I$$

than the fixed point is unique.

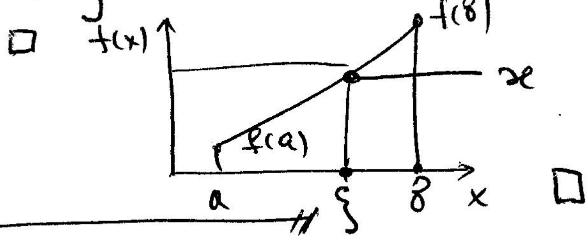
□ 1) if $\varphi(a) > a$; $\varphi(b) < b$; $f(x) = \varphi(x) - x$, $\Rightarrow f \in C[a, b] \Rightarrow$ $f(a) = \varphi(a) - a > 0$; $f(b) = \varphi(b) - b < 0 \Rightarrow$ (~~intermediate~~ mean value theorem) $\Rightarrow \exists\, \xi \in I$, $f(\xi) = 0 \Rightarrow \varphi(\xi) - \xi = 0 \Rightarrow \xi$ - fixed point.

□

Reminder:

~~Intermediate~~ Mean Value theorem: $f(x) \in C[a,b]$, $\varkappa$ is any number between $f(a)$ and $f(b)$ $\Rightarrow$ $\exists$ a number $\xi$ such that $f(\xi) = \varkappa$.

□




□

D2) $|\varphi'(x)| \leq k < 1 \quad \forall x \in I$; say $\xi_1$ and $\xi_2$ $(\xi_1 \neq \xi_2) \in I$ are both fixed points.

By "mean value theorem" a number exists between $\xi_1$ and $\xi_2$ such that $\dfrac{\varphi(\xi_1) - \varphi(\xi_2)}{\xi_1 - \xi_2} = \varphi'(\xi)$, then

fixed points

$$|\xi_1 - \xi_2| = |\varphi(\xi_1) - \varphi(\xi_2)| = |\varphi'(\xi)||\xi_1 - \xi_2| \leq k|\xi_1 - \xi_2| < |\xi_1 - \xi_2| \Rightarrow$$

contradiction $\Rightarrow$ $\xi_1 = \xi_2$

□

DF: A sequence $\{x_\nu\}$ is convergent if and only if for $\forall\, \varepsilon > 0$, there exists an integer $N$ such that
$$|x_\mu - x_\nu| < \varepsilon \text{ for all } \mu, \nu > N.$$

DF: if $\xi$ is a fixed point of $\varphi(x)$ and $\{x_\nu\}$ is generated by iteration function $x_{\nu+1} = \varphi(x_\nu)$, and $\boxed{|x_{\nu+1} - \xi| \leq C|x_\nu - \xi|^p}$ holds for all $\nu$, where $C < 1$ if $p = 1$ $\Rightarrow$ the iteration method is said to be a method of at least pth order for finding $\xi$.

Theorem: Each method of at least pth order for determining fixed point $\xi$ is convergent, in a sense that there is a neighborhood $N(\xi)$ of $\xi$ with the property that for all initial $x_0 \in N(\xi)$, the sequence $\{x_\nu\}$ generated by $\varphi(x)$ converges to $\xi$.

Ex.

□

$$x_{\nu+1} = \varphi(x_\nu) = \underbrace{\varphi(\xi)}_{=\xi} + \frac{(x_\nu - \xi)^p}{p!}\varphi^{(p)}(\xi) + O(|x_\nu - \xi|^{p+1}) \Longrightarrow$$

Taylor series expansion: $\varphi^{(k)}(\xi) = 0$; $k = 1, .., p-1$; $\varphi^{(p)}(\xi) \neq 0$

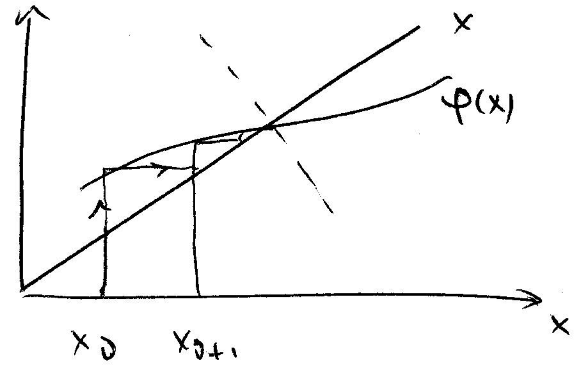
$$(x_{\nu+1} - \xi) = (x_\nu - \xi)^p \frac{\varphi^{(p)}(\xi)}{p!} \quad \nu \to \infty \quad \Rightarrow a)\ |x_{\nu+1} - \xi| = C|x_\nu - \xi|^p \text{ and}$$

this sequence is precisely order $p$ if $p \geq 2$.

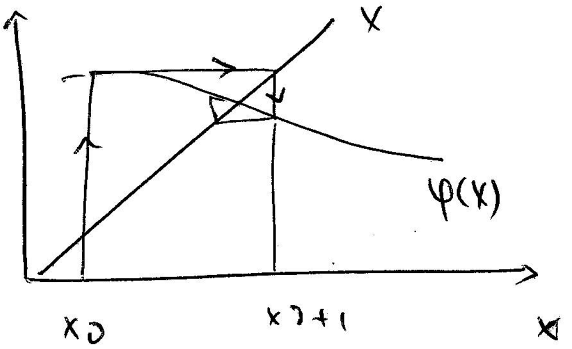
8) is $p = 1 \Rightarrow |x_{\nu+1} - \xi| < |x_\nu - \xi||\varphi'(\xi)|$ only if $|\varphi'(\xi)| < 1$ $\square$

linear convergence.

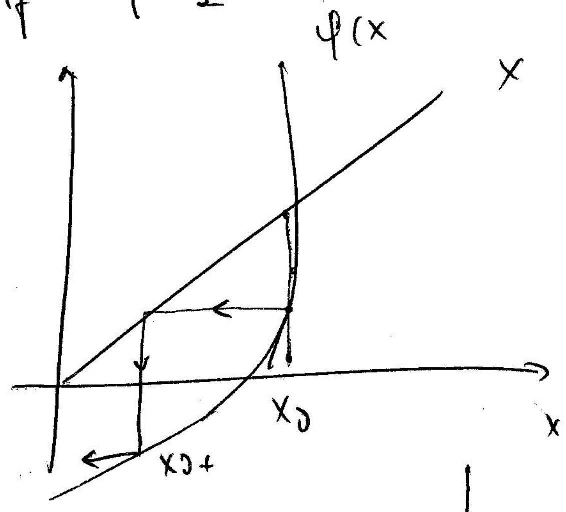
if $0 < \varphi'(\xi) < 1 \Rightarrow$ monotonically convergent sequence



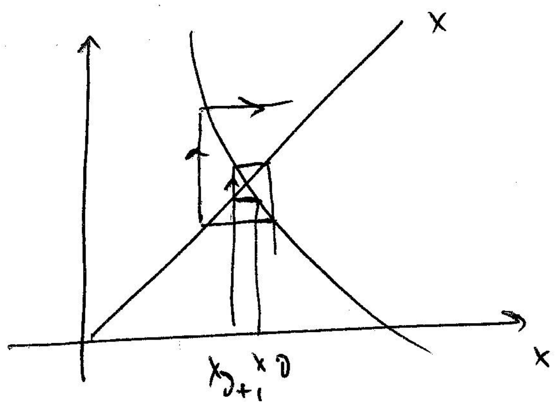

if $-1 < \varphi'(x) < 0 \Rightarrow$ alternate convergent sequence




if $\varphi' > 1$




if $\varphi' < -1$




$\downarrow$ $\downarrow$

divergent cases.

$$|x_{\nu+1} - \xi| \leq c|x_\nu - \xi|$$

$c < 1$ if $|\varphi'| > 1$ ; contradiction.

Theorem: (Fixed point theorem) $\varphi(x) \in C[a,b]$;

$\varphi(x) \in I, \forall x \in I$, $\varphi'(x) \in C[a,b]$ with $|\varphi'(x)| \le k < 1 \ \forall x \in I$ (Existence and Uniqueness), if $\varphi'(\xi) \neq 0$, then for any $x_0 \in I$ sequence $x_n = \varphi(x_{n-1})$, $n \ge 1$ converges only linearly to unique fixed point on $[a,b]$.

$\square$ $\{x_n\} \xrightarrow[n\to\infty]{} \xi$. Since $\varphi'$ $\exists$ by the mean value theorem $\Rightarrow$

$$x_{n+1} - \xi = \varphi(x_n) - \varphi(\xi) = \varphi'(\varkappa_n)(x_n - \xi)$$

$\varkappa_n$ lies between $x_n$ and $\xi$

$\{x_n\} \xrightarrow[n\to\infty]{} \xi$ and $\{\varkappa_n\} \xrightarrow[n\to\infty]{} \xi$ due to $\varphi' \in C[a,b] \Rightarrow$

$$\lim_{n\to\infty} \varphi'(\xi_n) = \varphi'(\lim_{n\to\infty} \xi_n) = \varphi'(\xi) \Rightarrow$$

$$\lim_{n\to\infty} \frac{x_{n+1} - \xi}{x_n - \xi} = \lim_{n\to\infty} \varphi'(\varkappa_n) = \varphi'(\xi) \Rightarrow \lim_{n\to\infty} \frac{|x_{n+1} - \xi|}{|x_n - \xi|} = |\varphi'(\xi)| < 1$$

linear convergence if $\varphi'(\xi) \neq 0$ $\square$

## A priori estimate of the error

① By mean value theorem

$$|x_n - \xi| = |\varphi(x_{n-1}) - \varphi(\xi)| = |\varphi'(\varkappa)|\,|x_{n-1} - \xi| \le k\,|x_{n-1} - \xi|$$

By induction:

$$|x_n - \xi| \le k\,|x_{n-1} - \xi| \le k^2 |x_{n-2} - \xi| \dots \le k^n |x_0 - \xi| \quad \square$$

① $|\Delta^n| = |x_n - \xi| \le k^n \operatorname{Max}\{x_0 - a,\ b - x_0\}$, since $\xi \in [a,b]$

② $n \ge 1$ $|x_{n+1} - x_n| = |\varphi(x_n) - \varphi(x_{n-1})| = |\varphi'(\varkappa)|\,|x_n - x_{n-1}|$

$\le k\,|x_n - x_{n-1}| \le k^n |x_1 - x_0|$

for $m > n \ge 1$ $|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} \dots + x_{n+1} - x_n| \le$

$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots |x_{n+1} - x_n| \le$

$\le k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \dots k^n |x_1 - x_0|$

since $\{x_m\} \xrightarrow[m\to\infty]{} \xi$ then

$$|\xi - x_n| = \lim_{m\to\infty} |x_m - x_n| \le K^n |x_1 - x_0| \sum_{i=0}^{\infty} K^i =$$

$$= \left\{ \sum_{i=0}^{\infty} K^i = \frac{1}{1-K} \right\} = \frac{K^n}{1-K} |x_1 - x_0| \Rightarrow$$

(2) $$\boxed{|\Delta^n| = |x_n - \xi| \le \frac{K^n}{1-K} |x_1 - x_0|}$$

$\square$

## Newton's Method:

If $\xi$ is a zero of $f(x)$, and $f(x) \in C^{(K)}[a,b]$ then the Taylor series expansion of $f(x)$ about some point $x_0 \in [a,b]$ is

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots O((x-x_0)^3)$$

if $x = \xi$ is zero of $f(x)$ $\Rightarrow$

and $|x_0 - \xi|$ is small $\Rightarrow$

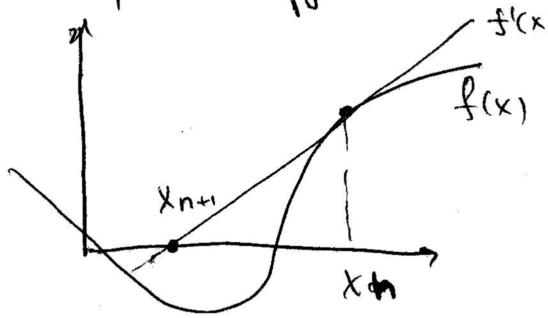
$$f(\xi) = 0 \approx f(x_0) - (\xi - x_0) f'(x_0) \Rightarrow$$

$\boxed{\xi \approx x_0 - \frac{f(x_0)}{f'(x_0)}}$ is approximation for $\xi$.

One has a iteration function $\boxed{\varphi(x) = x - \frac{f(x)}{f'(x)}}$; or

in term of relation $\boxed{\begin{array}{l} f(x) = 0 \\ \varphi(x) = x - f(x)\, g(x);\ g(x) = 1/f'(x) \end{array}}$

DF: Geometrically that means, ~~that f(x) is~~ instead of zeros of $f(x)$ one looks for zeros of the linear approximation of $f(x)$, which passes through $(x_n, f(x_n))$.



## Convergence:

$$|x_{n+1}-\xi| \le c\,|x_0-\xi|^p \Rightarrow$$

$$|x_{n+1}-\xi| = \frac{1}{p!}\varphi^{(p)}(\xi)\,|x_0-\xi|^p$$

For Newton method: $\varphi(x) = x - f(x)/f'(x)$, $f'(x) \neq 0 \Rightarrow$

$$\varphi(\xi) = \xi;\quad \varphi'(\xi) = \left.\frac{f(x)\,f''(x)}{(f'(x))^2}\right|_{x=\xi} = 0$$

$$\varphi''(\xi) = \frac{f''(\xi)}{f'(\xi)} \neq 0 \Rightarrow |x_{n+1}-\xi| = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}|x_0-\xi|^2 \Rightarrow$$

$p=2$, the method is at least second order convergent. □

So if $|\Delta_n|$ is small, and $\frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ is not large $\Rightarrow$

$|\Delta_{n+1}|$ is smaller then $|\Delta_n|$

Algorithm:
1) for given $f(x)$ to find $f'(x)$
2) starting with initial guess $x_0$
   $x_{n+1} = x_n - f(x_n)/f'(x_n)$
3) if $|x_{n+1}-x_n| \le \delta\,|x_n|$ stop
   $\xi \approx x_{n+1}$

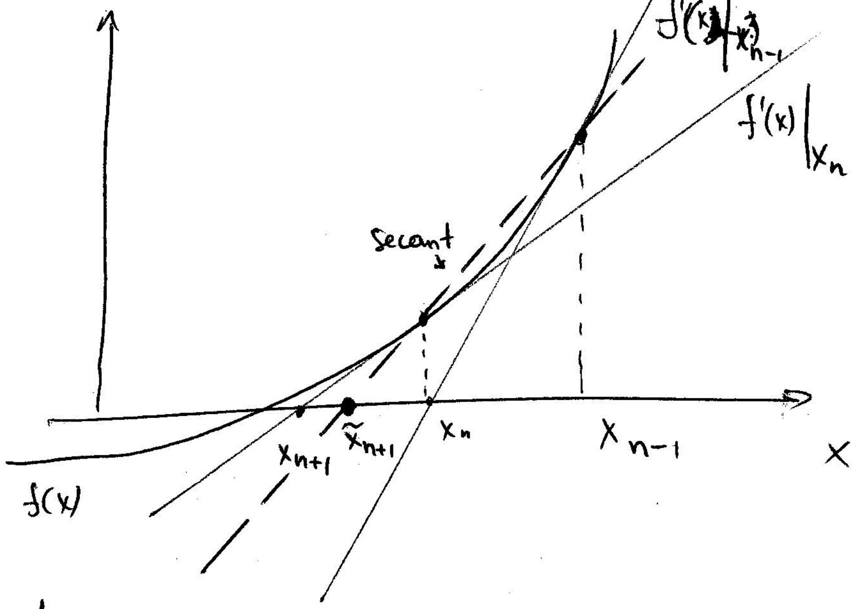
One need $f'(x) \longrightarrow$ Secant method:

$$f'(x_n) \approx \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}};\ \text{as}\ f'(x) = \lim_{h\to x}\frac{f(h)-f(x)}{h-x}. \Rightarrow$$

By using finite difference in Newton method $\Rightarrow$ Secant method

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n)-f(x_{n-1})}\,f(x_n)}$$

iteration function.




$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Algorithm:

1) $f(x)$
2) two nearby initial points $x_0, x_1$
3) $x_{n+1} = x_n - f(x_n) \dfrac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$
4) if $|x_{n+1} - x_n| \le \delta |x_n|$ stop with $\xi \approx x_{n+1}$