Program.

1) Linear systems: direct methods (moderate size matrices)
↳ all are factorization methods.

(L1)
- a) Gauss
- b) LU-decomposition (LR)
- c) inversion, determinant
- d) Error, refinement

Special matrices

(L2)
- a) Cholesky decomposition - symmetric, positively def matrices
- b) Singular Value decom. - singular matrices - underdet. sys.
- c) QR-dec. - $m\times n$ - over/ determined matrices

(L3)
- a) Sparse matrices
  1) tridiagonal → } eigenvalue problem.
  2) banddiagonal →
  3) Vandermonde matrices → polynomials fitting
  4) Toeplitz → signal processing.

2) Linear systems: iterative methods (large matrices)

(L4)
- a) Jacobi method
- b) Gauss-Seidel method
- c) Relaxation — " —
- d) conjugate gradient method

Systems of Linear Equations: direct methods.

$$(1)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{cases}$$

→ m linear eqs. of n unknown quantities x_i satisfied simultaneously.

$\boxed{a_{ik}\in\mathbb{R};\ x_i\in\mathbb{R},\ b_i\in\mathbb{R}}$

if $\hat{A} = (a_{ik}) = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$; coefficient matrix

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

then

$(1) \Leftrightarrow (2)\quad \boxed{\hat{A}\vec{x} = \vec{b}}$

DF: if $\vec{b}=0 \Rightarrow$ system $\hat{A}\vec{x}=0$ is called homogeneous; $\vec{b}\neq 0 \Rightarrow$ inhomogeneous.

DF: $\vec{x}$ is called a solution of linear system, if $\vec{x}$ solves (1) or (2).

↑ Real system of equations

- $\boxed{m>n}$ - overdetermined system
- $\boxed{m=n}$ → "square system"
- $\boxed{m<n}$ - underdetermined system

Remark:

If one deals with complex system of equations, i.e., in $\hat{A}\vec{x}=\vec{b}$; a_{ik}, x_i, or b_i are complex numbers, then (i) all methods can be used with complex arithmetic instead of real arithmetics, or (ii) the real system twice the original size can be solved:

$$(\hat{A}+i\hat{C})(\vec{x}+i\vec{y})=(\vec{b}+i\vec{d}) \Rightarrow$$

$$\hat{A}\vec{x}+i\hat{C}\vec{x}+i\hat{A}\vec{y}-\hat{C}\vec{y}=\vec{b}+i\vec{d} \Rightarrow$$

$$\begin{cases}\hat{A}\vec{x}-\hat{C}\vec{y}=\vec{b}\\ \hat{C}\vec{x}+\hat{A}\vec{y}=\vec{d}\end{cases} \Leftrightarrow \begin{pmatrix}\hat{A} & -\hat{C}\\ \hat{C} & \hat{A}\end{pmatrix}\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}=\begin{pmatrix}\vec{b}\\ \vec{d}\end{pmatrix}$$

↗ Block matrix of $2n\times 2n$ size.

Definitions 1

DF1: For $k\le n$, the (k,k) matrix $\hat{A}_k$, formed by deleting rows and columns $k+1,\dots,n$ from $\hat{A}_{nn}$ is called leading principal submatrix of order k. Its determinant is called the k^{th} leading principal minor of A

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1k} & \cdots a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots a_{2n}\\ \vdots & & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots a_{kn}\\ \hline a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots a_{nn}\end{array}\right)$$

→ order k principal submatrix.

DF2: a) A matrix $\hat{A}_{nn}$ is called nonsingular, if $\det(\hat{A})\neq 0$.

b) $\hat{A}_{nn}$ is strongly nonsingular, if all leading principal minors are nonzero, i.e. $\det(\hat{A}_k)\neq 0$, $\forall k=1,\dots n$.

c) $\hat{A}_{nn}$ is singular, if $\det(\hat{A}_{nn})=0$.

d) The rank of a matrix $\hat{A}_{nn}$ is defined as the size of the largest square submatrix of $\hat{A}$ that is nonsingular.

Solvability condition

1. Homogeneous system $\hat{A}\vec{x}=\vec{0}$ is always solvable. It has $[n-\operatorname{rank}\hat{A}]$ linearly independent solutions $\vec{y}_i$. Any linear combination of $\vec{y}_i$ is a solution of homogeneous system.

$\Rightarrow$ a) $\det\hat{A}\neq 0 \Rightarrow n-\operatorname{rank}\hat{A}=n-n=0 \Rightarrow$ trivial solution $\vec{x}=0$

b) $\det\hat{A}=0$, $\operatorname{rank}\hat{A}=r \Rightarrow$ $n-r$ linear independent solutions.

2. Inhomogeneous system $\hat{A}\vec{x}=\vec{b}\neq 0$ is solvable if and only if the rank of matrix $(\hat{A},\vec{b})_{m,n+1}$ is the same as rank of matrix $\hat{A}_{mn}$. Any solution $\vec{x}$ of $\hat{A}\vec{x}=\vec{b}$ is a sum of any solution of $\hat{A}\vec{x}=\vec{0}$ and a solution $\vec{x}_i$ of inhom. system.

For $m=n \Rightarrow$

a) if $\det\hat{A}\neq 0$, then $\exists$ one solution to $\hat{A}\vec{x}=\vec{b}$, namely $\vec{x}=\hat{A}^{-1}\vec{b}$ $(\operatorname{rank}(\hat{A},\vec{b})=\operatorname{rank}\hat{A}=n)$.

b) if $\det\hat{A}=0$; and $\operatorname{rank}(\hat{A},\vec{b})=\operatorname{rank}(\hat{A})=r<n \Rightarrow$ solution is not unique. Any linear combination of the $n-r$ linearly independent solutions of $\hat{A}\vec{x}=0$, gives some solution of $\hat{A}\vec{x}=\vec{b}$.

(Side note)

Gauss Algorithm with column pivot search

- Idea to transform a linear system with $m=n$ into triangular system: $\hat{A}\vec{x}=\vec{b}$ $\longrightarrow$ $\hat{U}\vec{x}=\hat{c}$

$$(3)\quad \begin{cases} r_{11}x_1 + r_{12}x_2 \cdots r_{1n}x_n = c_1 \\ \qquad r_{22}x_2 \cdots r_{2n}x_n = c_2 \\ \qquad \vdots \qquad\qquad \vdots \\ \qquad\qquad r_{nn}x_n = c_n \end{cases} \iff \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

upper triangular matrix.

if $r_{11}r_{12}\cdots r_{nn} \neq 0$, one can solve system (3) recursively with Backsubstitution:

$$(*)\quad \begin{cases} x_n = \dfrac{c_n}{r_{nn}} \\ x_{n-1} = \dfrac{1}{r_{n-1\,n-1}}\left(c_{n-1} - r_{n-1\,n}x_n\right), \text{ as } \boxed{r_{n-1\,n-1}x_{n-1} + r_{n-1\,n}x_n = c_{n-1}} \\ x_j = \dfrac{1}{r_{jj}}\left(c_j - \displaystyle\sum_{j=i+1}^{n} r_{ij}x_j\right), \text{ as } \boxed{r_{jj}x_j + \displaystyle\sum_{j=i+1}^{n} r_{ij}x_j = c_j} \end{cases}$$

Algorithm

step 1a Interchanging equations in the linear system (rows in matrix $\hat{A}$) does not change solution.

- Interchange rows in $\hat{A}$ such that the largest element in the first column appears at position $(1,1)$ [pivoting]

$$\begin{cases} a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 \cdots a_{1n}^{(0)}x_n = b_1^{(0)} \\ a_{n1}^{(0)}x_1 + a_{n2}^{(0)}x_2 \cdots a_{nn}^{(0)}x_n = b_n^{(0)} \end{cases} \Rightarrow \hat{A}^{(0)}\vec{x} = \vec{b}^{(0)}$$

step 1b eliminating x_1 from rows $i=2,\dots n$.

- multiply first equation in $\hat{A}^{(0)}\vec{x}=\vec{b}^{(0)}$ by $-\dfrac{a_{i1}^{(0)}}{a_{11}^{(0)}}$ and add to the i^{th} equation successively for $i=2,\dots n$.

$$\begin{cases} a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 \cdots a_{1n}^{(0)}x_n = b_1^{(0)} \\ \boxed{\begin{matrix} \tilde{a}_{22}^{(1)}x_2 + \cdots \tilde{a}_{2n}^{(1)}x_n = \tilde{b}_2^{(0)} \\ \vdots \\ \tilde{a}_{n2}^{(1)}x_2 + \cdots \tilde{a}_{nn}^{(1)}x_n = \tilde{b}_n^{(1)} \end{matrix}} \end{cases} \qquad \begin{pmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ \vdots \\ 0 & \times & \times & \cdots & \times \end{pmatrix}$$

here
$$\tilde{a}_{ik}^{(1)} = \begin{cases} 0 \; ; & \text{for } k=1, \; i=2,\dots,n \\ a_{ik}^{(0)} - a_{1k}^{(0)} \dfrac{a_{i1}^{(0)}}{a_{11}^{(0)}} \; ; \end{cases}$$
$$\tilde{b}_i^{(1)} = b_i^{(0)} - b_1^{(0)} \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} \; ; \; i=2 \dots n$$

Ex:
$$a_{21}^{(0)} x_1 + a_{22}^{(0)} x_2 + \dots + a_{n2}^{(0)} x_n = b_2^{(0)}$$
$$+ \; -\frac{a_{21}^{(0)}}{\cancel{a_{11}^{(0)}}} \cancel{a_{11}^{(0)}} x_1 - \frac{a_{21}^{(0)}}{a_{11}^{(0)}} a_{21}^{(0)} x_2 - \dots - \frac{a_{21}^{(0)}}{a_{11}^{(0)}} a_{n1}^{(0)} x_n = -b_1^{(0)} \frac{a_{21}^{(0)}}{a_{11}^{(0)}}$$
$$0 \quad \left(a_{22}^{(0)} - \frac{a_{21}^{(0)\,2}}{a_{11}^{(0)}}\right) x_2 + \dots + \left(a_{n2}^{(0)} - \frac{a_{21}^{(0)}}{a_{11}^{(0)}} a_{n1}^{(0)}\right) x_n = \left(b_2^{(0)} - b_1^{(0)} \frac{a_{21}^{(0)}}{a_{11}^{(0)}}\right)$$

Step 2: apply steps 1a)–1b) to $n-1$ system.

step j:
$$\begin{cases} \tilde{a}_{ik}^{(j)} = \begin{cases} 0 \; ; & k=1,j, \; i=j+1 \dots n \\ a_{ik}^{(j-1)} - a_{jk}^{(j-1)} \dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}} \end{cases} \\ \tilde{b}_i^{(j)} = b_i^{(j-1)} - b_j^{(j-1)} \dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}} \; ; \; \forall \, i=j+1 \dots n \end{cases}$$

system has the form

$$\begin{cases} a_{11}^{(0)} x_1 + a_{12}^{(0)} x_2 + \dots \; a_{1n}^{(0)} x_n = b_1^{(0)} \\ 0 \qquad a_{jj}^{(j-1)} x_j + \dots a_{jn}^{(j-1)} x_n = b_j^{(j-1)} \\ \qquad \tilde{a}_{j+1,j+1}^{(j)} x_{j+1} + \dots \tilde{a}_{j+1\,n}^{(j)} x_n = \tilde{b}_{j+1}^{(j)} \\ \qquad \tilde{a}_{j+2,j+1}^{(j+1)} \underline{x_{j+1}} + \dots \tilde{a}_{j+2,n}^{(j+1)} x_n = \tilde{b}_{j+2}^{(j+1)} \\ \qquad \vdots \\ \qquad \tilde{a}_{n,j+1}^{(j+1)} \underline{x_{j+1}} \dots \tilde{a}_{nn}^{(j+1)} x_n = \tilde{b}_n^{(j+1)} \end{cases}$$

step $j=n-1$ apply steps 1a–1b to the rest 2×2 system.

$$\boxed{\hat{A}\vec{x} = \vec{b} \Rightarrow \hat{U}\vec{x} = \vec{c}} \quad \text{with} \quad \boxed{u_{ik} = a_{ik}^{(i-1)} \; ; \; c_i = b_i^{(i-1)}}$$

Number of operation. $\boxed{\sum_{i=0}^{n-1}(a+bi)=\frac{n}{2}(2a+(n-1)b)}$

1) backsubstitution: n times $(n-i)$ multiplications and one division $\Rightarrow$

$$N_1=\sum_{i=1}^{n}(\underline{n-i+1})=\sum_{i'=0}^{n-1}(n+1-i'-1)=\sum_{i'=0}^{n-1}(n+(-1)i')=\frac{n}{2}(2n+(n-1)(-1))=$$

$i'=i-1 \Rightarrow i=i'+1$

$$=\underline{\frac{n}{2}(n+1)}\approx\frac{n^2}{2}$$

2) updating $\vec{b}$ to $\vec{c}$: $(n-1)$ times $(n-i)$ multiplications and $(n-i)$ divisions.

$$N_2=\sum_{i=1}^{n-1}2(n-i)=\begin{Bmatrix}i'=i-1\\ n'=n-1\end{Bmatrix}=2\sum_{i'=0}^{n'-1}(n-(i'+1))=2\sum_{i'=0}^{n'-1}((n-1)+(-1)i')=$$

$$=2\,\frac{n'}{2}(2(n-1)-(n'-1))=2\frac{(n-1)}{2}(2n-\cancel{2}-n+\cancel{2})=\underline{\underline{(n-1)n}}\approx n^2$$

3) updating $\hat{A}$ to $\hat{U}$: requires $N_3=\frac{n(n^2-1)}{3}\approx\frac{n^3}{3}$

$$N_{total}=N_1+N_2+N_3\approx\frac{n^3}{3}-\frac{n}{3}+\frac{3}{2}n^2\approx\frac{n^3}{3}+O(n^2)$$

!! Number of operation scales like n^3, where n is the system size.

LU (LR) decomposition

Suppose $\hat{A}$ can be written as a product of two matrices (*) $\boxed{\hat{L}\cdot\hat{U}=\hat{A}}$; where $\hat{L}$ is lower triangular, $\hat{U}$ is upper —//—

DF: Triangular matrix; $n\times n$ matrix $\hat{L}$ is called lower triangular if $l_{ik}=0$, for $k>i$; and $n\times n$ matrix is called upper triangular if $u_{ik}=0$ for $i>k$.

$$(*)\begin{pmatrix}l_{11} & 0 & 0 & \cdots & 0\\ l_{21} & l_{22} & 0 & \cdots & 0\\ & & & & \\ l_{n1} & l_{n2} & \cdots & & l_{nn}\end{pmatrix}\begin{pmatrix}u_{11} & u_{12} & \cdots & u_{1n}\\ 0 & u_{22} & \cdots & u_{2n}\\ & & & \\ 0 & 0 & \cdots & u_{nn}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

Theorem: Strongly non-singular $n\times n$ $\hat{A}$ can be factored uniquely into a product $A=LR$, where L unit lower tr. m.

Appl. 1: For $\boxed{\hat{A}\vec{x} = \vec{b}}$ =>

$$\hat{A}\hat{x} = (\hat{L}\cdot\hat{U})\vec{x} = \hat{L}\cdot(\hat{U}\cdot\vec{x}) = \vec{b} \Rightarrow \begin{cases} \hat{L}\vec{y} = \vec{b} & (1) \\ \hat{U}\cdot\vec{x} = \vec{y} & (2) \end{cases}$$

Eq. (1) can be solved by forward substitution:

$$\begin{cases} l_{11}y_1 = b_1 \\ l_{21}y_1 + l_{22}y_2 = b_2 \\ \vdots \\ l_{n1}y_1 + l_{n2}y_2 + \dots l_{nn}y_n = b_n \end{cases} \Rightarrow \begin{cases} y_1 = \dfrac{b_1}{l_{11}} \\ \\ y_i = \dfrac{1}{l_{ii}}\left(b_i - \sum\limits_{j=1}^{i-1} l_{ij}y_j\right) \end{cases}$$

while Eq. (2) can be solved by backward substitution.

$$\begin{cases} x_n = \dfrac{y_n}{u_{nn}} \\ x_i = \dfrac{1}{u_{ii}}\left(y_i - \sum\limits_{j=i+1}^{n} u_{ij}x_j\right) \end{cases}$$

Appl 2: $\boxed{\hat{A}\vec{x} = \vec{b}_i}$ can be solved by one forward and one backward substitution $\forall\ \vec{b}_i$ if $\hat{L}$ and $\hat{U}$ determined.

Appl 3 $\hat{A} = \hat{L}\hat{U} \Rightarrow \det\hat{A} = \det\hat{L}\det\hat{U} = 1\cdot\det\hat{U} = \prod\limits_{i=1}^{n} u_{ii}$ determinant of the matrix $\hat{A}$.

Appl 4. Matrix inversion

A^{-1} is inverse matrix, if $AA^{-1} = I$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad B = A^{-1} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

$$A\vec{b}_1 = \begin{pmatrix} a_{12} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{e}_1$$

$\hat{A}\vec{b}_i = \vec{e}_i \Rightarrow$ having $\hat{L}\cdot\hat{U}$ one construct $B = A^{-1}$

By solving n - ~~linear~~ linear systems of the type $\boxed{\hat{A}\vec{b}_i = \vec{e}_i}$. (8)

Constructing $\hat{L}\hat{U}$ - decomposition

Step 1 of Gauss elimination:

$$\begin{cases} a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 + \dots a_{1n}^{(0)}x_n = b_1^{(0)} \\ \boxed{\begin{matrix} \tilde{a}_{22}^{(1)}x_2 + \dots \tilde{a}_{2n}^{(1)}x_n = \tilde{b}_2^{(1)} \\ \vdots \\ \tilde{a}_{n2}^{(1)}x_2 + \dots \tilde{a}_{nn}^{(1)}x_n = \tilde{b}_n^{(1)} \end{matrix}} \end{cases}$$

can be formally written using "elementary lower triangular matrix"

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & \\ & & 1 & \\ \vdots & 0 & 0 \dots & 0 \\ -l_{n1} & 0 & 0 & 1 \end{pmatrix}$$

with $l_{i1} = \frac{a_{i1}^{\prime(0)}}{a_{11}^{\prime(0)}}$, and $a_{ij}^{\prime(0)}$ elements obtained after rows permutation.

and "permutation matrix"

$$P_{ik} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \dots 1 \\ & & & 1 \dots 0 \end{pmatrix};$$

in ~~making~~ the identity matrix one interchange i and k rows.

Note: $P_{ik}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_k \\ x_{i+1} \\ \vdots \\ x_{k-1} \\ x_i \\ x_{k+1} \\ \vdots \end{pmatrix}$ changes i and k rows.

as follows:

$$\boxed{\begin{matrix} (\hat{A}, \vec{b}) \rightarrow (\hat{A}', \vec{b}') \rightarrow (\tilde{A}, \tilde{\vec{b}}) \\ (\hat{A}', \vec{b}') = P_1(\hat{A}, \vec{b}) \; ; \; (\tilde{A}, \tilde{\vec{b}}) = L_1 P_1 (\hat{A}, \vec{b}) \end{matrix}}$$

As $P_1^{-1} = P_1$; $L_1^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ \ell_{21} & 0 & 1 & \\ \ell_{n1} & & 0 & 1 \end{pmatrix} \Rightarrow$

$A x = \vec{b}$ and $\tilde{A} x = \tilde{\vec{b}}$ have the same roots.

Δ

$L_1 P_1 \hat{A} x = \underbrace{\tilde{A} x = \tilde{\vec{b}}} = L_1 P_1 \vec{b}$ $\quad | \quad P_1^{-1} L_1^{-1} \tilde{A} x = \underbrace{\hat{A} x = \vec{b}} = P_1^{-1} L_1^{-1} \tilde{\vec{b}}$.

Step 2: After step 1 the resulting matrix has the form:

$$(\tilde{A}, \tilde{b}) = \left(\begin{array}{c|c|c} a_{11}^{(0)} & a_{12}^{(0)} \cdots a_{1n}^{(0)} & b_1^{(0)} \\ \hline 0 & \underbrace{\tilde{\tilde{A}}}_{(n-1)\times(n-1) \text{ matrix}} & \underbrace{\tilde{\tilde{\vec{b}}}}_{(n-1) \text{ vectors}} \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{c|c|c} a_{11}^{(0)} & a_{12}^{(0)} \cdots a_{1n}^{(0)} & b_1^{(0)} \\ \hline 0 & L_2 P_2 \tilde{\tilde{A}} & L_2 P_2 \tilde{\tilde{\vec{b}}} \end{array} \right) =$$

$$= \begin{pmatrix} 1 & 0 \cdots & 0 \\ \vdots & & \\ 0 & L_2 & \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \\ 0 & P_2 & \end{pmatrix} (\tilde{A}, \tilde{\vec{b}}) = L_2 P_2 (\tilde{A}, \tilde{\vec{b}}) \Rightarrow$$

step j: $\quad (A^{(j)}; \vec{b}^{(j)}) = L_j P_j (A^{(j-1)}, \vec{b}^{(j-1)}) \Rightarrow$

$(A, \vec{b}) = (A^{(0)}, \vec{b}^{(0)}) \rightarrow \cdots (A^{(n-1)}, \vec{b}^{(n-1)}) = (U, \vec{c}) \Rightarrow$

$\boxed{(U, \vec{c}) = L_{n-1} P_{n-1} \cdots L_1 P_1 (A, \vec{b}).}$ (1) • if no permutation are necessary, then $(U, \vec{c}) = L_{n-1} \cdots L_1 (A, \vec{b})$

$\hat{U} = L_{n-1} \cdots L_1 \hat{A}$

$\hat{A} = L_1^{-1} \cdots L_{n-1}^{-1} \hat{U} = \hat{L}\hat{U}$, with $L = L_1^{-1} \cdots L_{n-1}^{-1}$

here $L_i^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & & l_{i+1 i} & 1 & \\ 0 & & l_{ni} & 0 & \ddots & 1 \end{pmatrix}$; $\Rightarrow \hat{L}$ is unit lower triangular.

$\hat{U}$ is upper triangular by construction. □

(2) if permutations are nessecary.

reminder: $P_j = P_j^{-1} \Rightarrow P_j P_j = I \Rightarrow P_j L_{j+1} P_j = \tilde{L}_{j-1}$

makes permutations on matrix $(j-1)\times(j-1)$

L_{j+1} is $(j-1)\times(j-1)$ $\Big) \Rightarrow$

$\tilde{L}_{j+1}$ is $(j-1)\times(j-1)$.

$\hat{U} = L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots P_1 L_1 \hat{A}$

$\hat{U} = L_{n-1} \underbrace{P_{n-1} L_{n-2} P_{n-1}}_{\tilde{L}_{n-2}} P_{n-1} P_{n-2} \cdots P_1 L_1 \hat{A}$

$\hat{U} = L_{n-1} \tilde{L}_{n-2} P_{n-1} \underbrace{P_{n-2} L_{n-3} \underline{P_{n-2}}} P_{n-2} P_{n-3} + \cdots$

$\underbrace{P_{n-1} \tilde{L}_{n-3} \cdot \underline{P_{n-1}}} P_{n-1} P_{n-2} P_{n-3} + \cdots$

$\tilde{\tilde{L}}_{n-3}$

$\hat{U} = L_{n-1} \tilde{L}_{n-2} \tilde{\tilde{L}}_{n-3} P_{n-1} P_{n-2} P_{n-3} + \cdots P_1 L_1 \hat{A} \Rightarrow$

$\hat{U} = L'_{n-1} L'_{n-2} \cdots L'_1 \underbrace{P_{n-1} \cdots P_1}_{\text{complite permutations matrix } P} \hat{A}$

$\hat{U} = L'_{n-1} \cdots L'_1 \underline{PA} \Rightarrow \boxed{PA = \hat{L}\hat{U}}$ so this procedure gives LU-decomposition of PA. □

Consequence: 1) $A\vec{x} = \vec{b}$; – given

2) found: $\hat{L}, \hat{U}, \hat{P} \Rightarrow$

$$PA\hat{x} = P\hat{b} \Rightarrow \hat{L}\hat{U}\vec{x} = P\vec{b} \Rightarrow \begin{cases} \hat{L}\vec{y} = P\vec{b} & \text{– forward} \\ \hat{U}\vec{x} = \vec{y} & \text{– backward.} \end{cases}$$

$$\left(\begin{array}{ccc|c} a_{11}^{(0)} x_1 & + \; a_{12}^{(0)} x_2 & \cdots & a_{1n}^{(0)} x_n \\ l_{j1} & l_{j2} & a_{jj}^{(j-1)} x_j & \cdots \; a_{jn}^{(j-1)} x_n \\ l_{j+1,1} & l_{j+1,2} & & \\ \cdots & \cdots & & \end{array} \right)$$

Algorithm.

$$a_{ik}^{(j)} = \begin{cases} 0 = l_{ij} = \dfrac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}}, & k = 1, j; \; i = j+1 \ldots n \\[2ex] a_{ik} - l_{ij}\, a_{jk}^{(j-1)}, & k = j+1, n; \; i = j+1, n \end{cases}$$

$a_{ij} \Rightarrow$ rest
