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Physikalisches Institut
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http://www.thp.uni-bonn.de/kroha/

Condensed Matter Field Theory — WS09/10

Exercise 2

(Please return your solutions before Fr. 13.11. 12:00h)

2.1. Green's functions for non-interacting electrons (15 points)

In the lecture the retarded single-electron Green's function was defined as:

$$G^R_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\big\langle \big[a_{\mathbf{k}\sigma}(t), a^\dagger_{\mathbf{k}\sigma}(t')\big]_+ \big\rangle_0 \tag{1}$$

We will consider a system of non-interacting electrons

$$\mathcal{H}_0 = H_0 - \mu N = \sum_{\mathbf{k}\sigma} (\epsilon(\mathbf{k}) - \mu) a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

a) Determine the time-dependence of $a_{\mathbf{k}\sigma}(t)$ and $a^\dagger_{\mathbf{k}\sigma}(t')$ for the non-interacting system $\mathcal{H}_0$ by using the equation of motion for a Heisenberg operator.

b) Compute the retarded Green's function (1) for the non-interacting system using the results of a).

$$G^{R,0}_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\epsilon(\mathbf{k})-\mu)(t-t')} = G^{R,0}_{\mathbf{k}\sigma}(t-t') \tag{2}$$

c) Derive the Fourier transform

$$G^{R,0}_{\mathbf{k}\sigma}(E) = \int_{-\infty}^{\infty} d(t-t')\, G^{R,0}_{\mathbf{k}\sigma}(t-t')\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}E(t-t')} = \frac{\hbar}{E - (\epsilon(\mathbf{k}) - \mu) + \mathrm{i}0^+} \tag{3}$$

Hint: Use the residue theorem to show first that

$$\Theta(t-t') \;=\; \mathrm{i}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\, \frac{\mathrm{e}^{-\mathrm{i}E(t-t')}}{E + \mathrm{i}0^+}.$$

holds.

The Green's function can also be defined as the resolvent of a wave operator. In our case this is the Schrödinger operator:

$$(\mathrm{i}\hbar\partial_t - \mathcal{H}_0)G^{R,0}(t-t') = \delta(t-t')$$

d) Show that this equation is fullfilled by (2) by writing it in momentum space representation. What is the corresponding equation in energy space? Show that (3) is the resolvent of the Schrödinger operator in energy space.
Hint: $\langle \mathbf{k}\sigma | G^{R,0}(t-t') | \mathbf{k}\sigma\rangle = G^{R,0}_{\mathbf{k}\sigma}(t-t')$

In general, the retarded Green's function contains a non-infinitesimal imaginary part in the denominator

$$G^R_{\mathbf{k}\sigma}(E) = \frac{\hbar}{E - (\epsilon(\mathbf{k}) - \mu) + \mathrm{i}\tau^{-1}}$$

e) Use the residue theorem to calculate the time-dependent Green's function by Fourier transform. How does the Green's function behave for large $(t - t')$? How can one interprete τ? Try to explain why a finite $\tau > 0$ might occur.

2.2. Green's functions: general properties (15 points)

In the previous exercise, you calculated explicitely the retarded Green's function G for the special case of a diagonal Hamiltonian. However, in most cases the system is much more complex, e.g., in the presence of interactions. Nevertheless some analytical properties of G will always hold. We will discuss some of them in this exercise.

a) Normalization:
Use the explicit definition of the spectral function $S_{\mathbf{k}\sigma}(E)$ (see lecture),

$$\begin{aligned} S_{\mathbf{k}\sigma}(E) &= \frac{\hbar}{Z_G} \sum_{n,m} \langle E_n | a^\dagger_{\mathbf{k}\sigma} | E_m \rangle \langle E_m | a_{\mathbf{k}\sigma} | E_n \rangle \\ &\cdot \; e^{-\beta E_n} \left(e^{\beta E} - \epsilon \right) \delta(E - (E_n - E_m)), \end{aligned}$$

to show that $S_{\mathbf{k}\sigma}(E)$ is normalized,

$$\int_{-\infty}^{\infty} dE \, S_{\mathbf{k}\sigma}(E) = \hbar.$$

Use the spectral representation of $G^{\mathrm{R}}_{\mathbf{k}\sigma}(E)$ (see lecture),

$$G^{\mathrm{R}}_{\mathbf{k}\sigma}(E) = \int_{-\infty}^{\infty} dE' \, \frac{S_{\mathbf{k}\sigma}(E')}{E - E' + \mathrm{i}0^+},$$

to find the relation between $\mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(E)$ and $S_{\mathbf{k}\sigma}(E)$. Calculate

$$\int_{-\infty}^{\infty} dE \, \mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(E).$$

Hint: Use the Dirac identity:

$$\frac{1}{x - x_0 \pm \mathrm{i}0^+} = \mathcal{P}\frac{1}{x - x_0} \mp \mathrm{i}\pi\delta(x - x_0).$$

b) Asymptotic behavior:
We can assume that $S_{\mathbf{k}\sigma}(E) \equiv 0$ if $|E| > E_{max}$ for some $0 < E_{max} < \infty$. Show

$$\lim_{E \to \pm\infty} E \cdot G^{\mathrm{R}}_{\mathbf{k}\sigma}(E) = \hbar,$$

i.e., $G^{\mathrm{R}}_{\mathbf{k}\sigma}(E) \approx \hbar/E$ for large energies.