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Physikalisches Institut
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Condensed Matter Field Theory — WS09/10

Exercise 5

(Please return your solutions before Fr. 8.1., 12:00h)

5.1 Matsubara method (15 points)

The Matsubara method is based on the assumption, that the time t is a completely imaginary parameter. Therefore one defines a real parameter by $\tau = it$.
An operator in Heisenberg representation then looks like

$$A(\tau) = e^{\frac{1}{\hbar}H\tau} A(0) e^{-\frac{1}{\hbar}H\tau}$$

and the equation of motion reads

$$-\hbar \frac{\partial}{\partial \tau} A(\tau) = [A(\tau), H]_- \, .$$

The thermal Green's (or Matsubara) function is defined as

$$G^M_{AB}(\tau, \tau') = -\langle T_\tau (A(\tau) B(\tau')) \rangle \, .$$

(a) Derive the equation of motion for $G^M_{AB}(\tau, \tau')$.

(b) Use the cyclic invariance of the trace to show that also $G^M_{AB}(\tau, \tau')$ depends only on time differences, i.e.

$$G^M_{AB}(\tau, \tau') = G^M_{AB}(\tau - \tau', 0) = G^M_{AB}(0, \tau' - \tau) \, .$$

(c) Use also the cyclic invariance of the trace to show the periodicity of the thermal function:

$$G^M_{AB}(\tau - \tau' + n\hbar\beta) = \varepsilon G^M_{AB}(\tau - \tau' + (n-1)\hbar\beta)$$

for $\hbar\beta > \tau - \tau' + n\hbar\beta > 0$ and $n \in \mathbb{Z}$.

In particular, for $n = 1$ we find

$$G^M_{AB}(\tau - \tau' + \hbar\beta) = \varepsilon G^M_{AB}(\tau - \tau') \, ,$$

when $-\hbar\beta < \tau - \tau' < 0$. The thermal Green's function is thus periodic with a periodicity intervall of $2\hbar\beta$.
Because of this periodicity we can make use of a Fourier expansion for the thermal Green's function:

$$\begin{aligned}
G^M(\tau) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{\hbar\beta}\tau + b_n \sin \frac{n\pi}{\hbar\beta}\tau \right] \\
a_n &= \frac{1}{\hbar\beta} \int_{-\hbar\beta}^{+\hbar\beta} d\tau \, G^M(\tau) \cos \frac{n\pi}{\hbar\beta}\tau \\
b_n &= \frac{1}{\hbar\beta} \int_{-\hbar\beta}^{+\hbar\beta} d\tau \, G^M(\tau) \sin \frac{n\pi}{\hbar\beta}\tau
\end{aligned}$$

With the definitions $E_n = \frac{n\pi}{\beta}$ and $G^M(E_n) = \frac{1}{2}\hbar\beta(a_n + \mathrm{i}b_n)$ we can then write:

$$\begin{aligned} G^M(\tau) &= \frac{1}{\hbar\beta}\sum_{n=-\infty}^{+\infty} e^{-\frac{\mathrm{i}}{\hbar}E_n\tau} G^M(E_n) \\ G^M(E_n) &= \frac{1}{2}\int_{-\hbar\beta}^{+\hbar\beta} d\tau\, G^M(\tau) e^{\frac{\mathrm{i}}{\hbar}E_n\tau} \end{aligned}$$

(d) The expression for $G^M(E_n)$ can be further simplified. Show that

$$\begin{aligned} G^M(E_n) &= \left[1 + \varepsilon e^{-\mathrm{i}\beta E_n}\right] \frac{1}{2}\int_0^{\hbar\beta} d\tau\, G^M(\tau) e^{\frac{\mathrm{i}}{\hbar}E_n\tau} \\ &= \int_0^{\hbar\beta} d\tau\, G^M(\tau) e^{\frac{\mathrm{i}}{\hbar}E_n\tau} \end{aligned}$$

holds and conclude that

$$E_n = \begin{cases} 2n\pi/\beta: & \text{Bosonen} \\ (2n+1)\pi/\beta: & \text{Fermionen} \end{cases} .$$

These are the so called Matsubara frequencies.

(e) Finally derive

$$G^M_{AB}(E_n) = \int_{-\infty}^{+\infty} dE' \, \frac{S_{AB}(E')}{\mathrm{i}E_n - E'} .$$

Hint: First, using the spectral representation and the definition of the spectral function (see exercise 2.2) show that

$$\langle A(\tau)B(0)\rangle = \frac{1}{\hbar}\int_{-\infty}^{+\infty} dE \, \frac{S_{AB}(E)}{1 - \varepsilon e^{-\beta E}} e^{-\frac{1}{\hbar}E\tau} .$$

Plug this into the result for $G^M(E_n)$ of (d) and use

$$\int_0^{\hbar\beta} d\tau\, e^{\frac{1}{\hbar}(\mathrm{i}E_n - E)\tau} = \frac{\hbar}{\mathrm{i}E_n - E}\left[\varepsilon e^{-\beta E} - 1\right] .$$

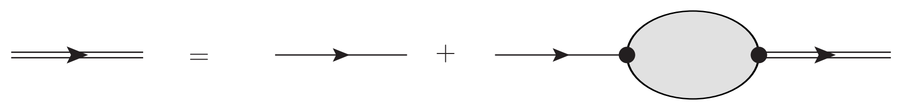
Therefore, the retarded (advanced) Green's function can be obtained from the thermal Green's function by the transition $\mathrm{i}E \to E \pm \mathrm{i}0^+$.

5.2 Selfenergy in 1st order perturbation theory ($T \neq 0$) (15 points)

The thermal Green's function is often denoted by

$$G^M_{\mathbf{k}\sigma}(E) = G_{\mathbf{k}\sigma}(\mathrm{i}E) .$$

Dyson's equation reads



$$G_{\mathbf{k}\sigma}(\mathrm{i}E) \qquad G^0_{\mathbf{k}\sigma}(\mathrm{i}E) \qquad G^0_{\mathbf{k}\sigma}(\mathrm{i}E)\left(\tfrac{1}{\hbar}\Sigma_{\mathbf{k}\sigma}(\mathrm{i}E)\right) G_{\mathbf{k}\sigma}(\mathrm{i}E)$$

where the free thermal Green's function is given by

$$G^0_{\mathbf{k}\sigma}(\mathrm{i}E) = \frac{\hbar}{\mathrm{i}E - \epsilon(\mathbf{k}) + \mu} \,.$$

The Feynman rules read ($k \equiv (\mathbf{k}, \sigma)$)

1. Vertex $\Leftrightarrow \frac{1}{\hbar\beta} v(kl;nm)\delta_{E_{n_k}+E_{n_l}, E_{n_m}+E_{n_n}}$.
2. Propagating and non-propagating line $\Leftrightarrow -G^0_k(\mathrm{i}E_{n_k})$.
3. Factor $\exp(\frac{\mathrm{i}}{\hbar}E_{n_k}0^+)$ for each non-propagating line.
4. Factor $(-1)^S \left(\frac{-1}{\hbar}\right)^n$, with S number of loops.
5. Summation/Integration over all internal wavenumbers, spins and energies.
6. External lines: $G^0_k(\mathrm{i}E_{n_k})$.

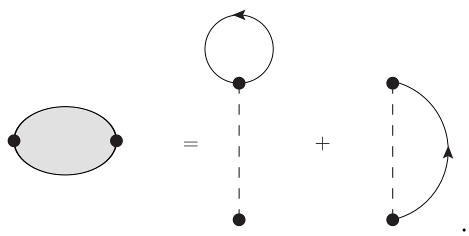
In the calculation of $T \neq 0$ diagrams there will occur sums over the Matsubara frequencies. These Matsubara sums can be converted into an integral around the poles of the function which is summed over. For a given function F with $\lim_{|z|\to\infty} F(z) = 0$

$$\frac{1}{\beta}\sum_{E_n} F(\mathrm{i}E_n) \;=\; -\oint_{C_1} \frac{dz}{2\pi\mathrm{i}}\, f(z)\, F(z) \;=\; \oint_{C_2} \frac{dz}{2\pi\mathrm{i}}\, f(z)\, F(z) \tag{1}$$

holds. C_1 encloses only the poles of $f(z)$ and C_2 only those of $F(z)$.

(a) Prove (1). Use that the fermionic Matsubara frequencies are the poles of the Fermi function $f(z)$ to show the first equality. Continue by inflating the integral contour to infinity by sparing out the poles of $F(z)$ to show the second equality.

(b) Use the above Feynman rules and (1) to calculate the selfenergy in 1^{st} order perturbation theory for a general interaction V.



(c) Now, consider a pair interaction $V(\mathbf{x}, \mathbf{y}) = V(|\mathbf{x} - \mathbf{y}|)$ for free electrons ($|k\rangle = |\mathbf{k}\sigma\rangle$ plane waves):

$$v(kl;nm) = \langle k, l|V|n, m\rangle = \delta_{\mathbf{k}+\mathbf{l},\mathbf{n}+\mathbf{m}}\delta_{\sigma_k\sigma_n}\delta_{\sigma_l\sigma_m}\tilde{v}(\mathbf{k} - \mathbf{n})$$

where $\tilde{v}(\mathbf{k} - \mathbf{n}) = \tilde{v}(\mathbf{q})$ is the Fourier transform of V to calculate the selfenergy in 1st order perturbation theory.