Exercises 4

Exercises on May 09 – May 13

4.1 Stirling Formula

Integrals of the shape $I_N = \int_b^a dz f(z) e^{-Ng(z)}$ frequently occur in statistical physics. N is a large positive integer; $z = z_s$ be the single minimum of $g(z)$ on the interval (a, b), and $f(z_s) \neq 0$.

In order to approximately calculate this integral, the so-called stationary phase approximation is used:

- The functions $f(z)$ and $g(z)$ are expanded about $z = z_s$; then, the substitution $x = N^{\frac{1}{2}}(z - z_s)$ is used.
- For $N \gg 1$, only the term $f(z_s)$ as well as the first two terms of the TAYLOR expansion of $g(z)$ are retained.

By means of those approximations, the integral I_N becomes a GAUSS integral.

a) Using the aforementioned steps, find a general approximation formula for I_N.
Hint: $\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$.

b) For all $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, the Gamma function is defined by $\Gamma(z+1) = \int_0^\infty dt e^{-t} t^z$. Show that $\Gamma(z+1) = z\Gamma(z)$ as well as $\Gamma(1) = 1$. Then show that $\Gamma(N+1) = N!$ for $N = 1, 2, 3....$

c) By means of the stationary phase approximation find the STIRLING formula:

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$$

where $N \gg 1$.

4.2 Joule-Thomsom Effect and Joule Cycle

The JOULE-THOMSON effect takes place in a system composed of two chambers which is perfectly isolated. In each chamber, the pressure is kept constant, $p_1 = \mathrm{const.}$, $p_2 = \mathrm{const.}$, but $p_1 > p_2$. The chambers are connected by a permeable wall, so the gas can adiabatically expand from the volume V_1 into the volume V_2. Here, we assume that the gas be an ideal gas.

a) Show that the enthalpy remains constant during the expansion, and that the process is an irreversible one.

b) The JOULE-THOMSON coefficient is defined to be $\delta = (\frac{\partial T}{\partial p})_H$. Show that $\delta = [T(\frac{\partial V}{\partial T}_p) - V]/c_p$, and that, for an ideal gas, $\delta = 0$.

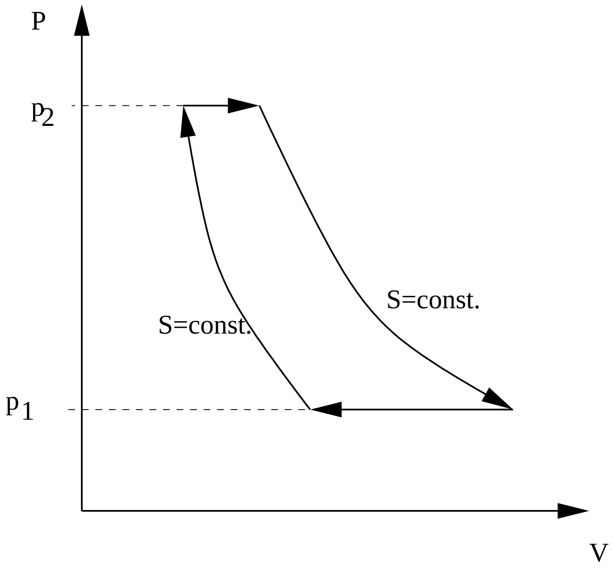

Figure 1: JOULE-THOMSON cycle

c) Calculate the efficiency ratio of the JOULE-THOMSON cycle as a function of p_1 and p_2 (see Fig. 1).

4.3 Elastic Wire

A force K is exerted on an elastic wire of length L. In the HOOKE regime, the equation of state is

$$K = k(L - L_0) + A_1(T - T_0),$$

where T is the temperature; L_0 and T_0 denote the standard length and the temperature for $K = 0$, resp., and k and A_1 are constants.

a) Give the differential of the internal energy, $\mathrm{d}U$, in terms of the thermodynamic differentials $\mathrm{d}S$ and $\mathrm{d}L$, where S is the entropy. Give a formula for the entropy S.

b) Construct a thermodynamic potential depending on the independent variables L and T. Find the MAXWELL relations which can be derived from that potential.

c) Calculate the temperature expansion coefficient, $\frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_K = \alpha_K$ at constant force as well as the elasticity $\frac{1}{L}\left(\frac{\partial L}{\partial K}\right)_T = \kappa_T$.

4.4 Thermodynamic Potentials and the Ideal Gas

In exercise 3.3 we have shown that the entropy of an ideal gas at constant particle number is given by

$$S(T, V) = S_0 + Nk_\mathrm{B}\left[\frac{f}{2}\ln\frac{T}{T_0} + \ln\frac{V}{V_0}\right].$$

a) By means of the two equations of state give the HELMHOLTZ free energy and the GIBBS free enthalpy. Then calculate the chemical potential.

b) From the expression for F, rederive the thermal equation of state.

Enjoy!