Chapter 7

Interacting Systems in Thermodynamic Equilibrium: Phase Transitions

7.1 Thermodynamic considerations and outline

When there are interactions between the particles (or spins) of a many-body system, the eigenstates of the system, and hence its full thermodynamics, cannot, in general, be determined exactly.

However, it can be understood from a very general thermodynamic argument that in an interacting system the states at low and at high temperature can be different in a *qualitative* way and, thus, separated by a phase transition.

These concepts will be made more concrete in this chapter[1]

1. Phase transitions with spontaneous symmetry breaking:

   In a thermodynamic system controlled by temperature, the free energy

$$F = U - TS \tag{7.1}$$

[1] The Bose- Einstein condensation is the only example of a system without interaction which has a phase transition, induced only by statistical correlations.

is always minimized in termodynamic equilibrium.

(a) For $T \to 0$, this means that U is minimized, since $-TS$ gives a vanishing contribution. With interactions, U is often minimal for an *ordered state.*

Example: Spin system with ferromagnetic interactions.
The ground state is a ferromagnetic state with all spins aligned parallel to each other.

This ground state is unique (3rd law of thermodynamics) e.g. the ferromagnetic state with given magnetization direction.

(b) For sufficiently high T, F is always dominated by the term $-TS$, since for $T \to \infty$ the classical limit is approached, where $U \sim T$ and $-TS \sim -T \ln T$ (see chapter 6.4.1, equipartation theorem).

Thus, for $T \to \infty$ the system maximizes the entropy S (instead of minimizing U) and forms a *disordered* state, realized by a large number of microstates.

Example:

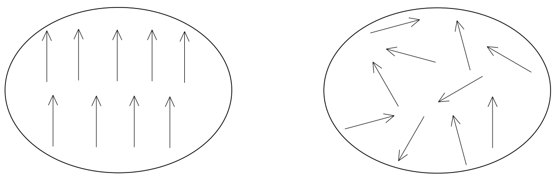

Figure 7.1: Paramagnetic state

The ordered state has often (not always! countercase: liquid- gas) a reduced symmetry compared to the disordered one.

Example: In the ferromagnetic state the rotational symmetry is broken.

Therefore, the thermodynamic states at high and at low T must be separated by a phase transition. The existence of a phase transition as a function of T can be inferred alone from the thermodynamic interplay of energy minimization and entropy maximization, if the interactions tend to form a ground state which has lower symmetry than the Hamiltonian itself.

There are, however, phase transitions which cannot be understood in this way:

2. Phase transitions without spontaneous symmetry breaking or long-range ordering.

   Example: Liquid-gas transition

3. Phase transitions at $T = 0$, controlled by other parameters (coupling strength, particle density, ...)

   $\rightarrow$ "quantum phase transitions".

In this chapter we will first consider examples of real systems using approximate methods, in order to formulate the phenomenon "phase transition" and several concepts connected with it more precisely.

Starting from the approximate treatment, we will then, in particular, formulate the concept of universality near phase transitions, which will lead us to the renormalization group method (RG).

A classification of the different types of phase transitions will be briefly discussed at the end.

As examples we will describe

(a) Ordering transitions in magnetic systems

(b) Liquid-gas transitions in a classical, interacting gas.

7.2 Spin systems with interactions

7.2.1 Origin of the spin-spin interaction and models

The nucleus as well as the electron system of a single atom can carry a finite magnetic moment or spin. The magnetic moment of the electron system of an atom is comprised of the orbital angular momentum and of the spin of the electrons. It is in general non-zero for atoms with incompletely filled shells and must be finite, if the number of electrons is odd.

Types of spin-spin interactions

(a) Direct dipole interaction:

This interaction is very weak because of the smallness of the individual magnetic moments involved and can usually be neglected.

(b) Exchange interaction:

This interaction is induced by the symmetry of the quantum mechanical many-body wave function and by the (large!) Coulomb interaction between the electrons. Therefore, this interaction can be large.

- **Hund's rule ferromagnetic interaction**

  between electrons in orbitals which are *spatially close* (large Coulomb-interaction) but (almost) *orthogonal* to each other (no hybridization mixing)

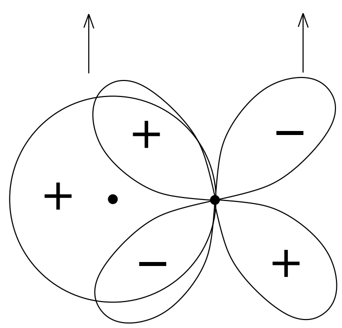

Figure 7.2: Feromagnetic interaction

$$|\psi\rangle = |\phi\rangle_{\text{orbital}} \otimes |\chi\rangle_{\text{spin}} \tag{7.2}$$

Spatial part of 2-particle wave function antisymmetric to minimize strong Coulomb repulsion.

$\Rightarrow$ Spin part of wave function symmetric $\rightarrow$ triplet

$\Rightarrow$ Coulomb induced ferromagnetic interaction.

- **Hopping-induced antiferromagnetic interaction**

  between electrons in orbitals which are *spatially separated* (small Coulomb interaction), but have large overlap (strong hopping between orbitals)

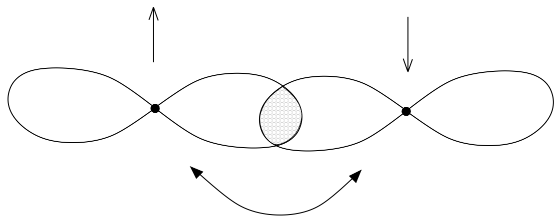

Figure 7.3: Antiferomagnetic interaction

Kinetic energy minimized by virtual hopping of an electron between the two orbitals. This is only possible, if the electrons in the two orbitals are antiferromagnetically oriented, because of the Pauli principle.

$\Rightarrow$ antiferromagnetic coupling.

Using these microscopic couplings one can write down effective spin models for the electrons which do not involve the details of the orbitals any longer but only the spin couplings.

We will consider only spin-$\frac{1}{2}$ systems here.

1. **Heisenberg model**

   Localized interacting spins on a lattice.

$$H_{\text{Heisenb.}} = -\sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j - \mu \vec{B} \cdot \sum_i \vec{S}_i \tag{7.3}$$

most often: nearest neighbor coupling:

$$J_{ij} = \begin{cases} J & , \quad i,j, \text{ n.n.} \\ 0 & , \quad \text{else} \end{cases} \tag{7.4}$$

$$J_{i,j} > 0 \quad : \quad \text{ferromagnetic} \tag{7.5}$$

$$J_{i,j} < 0 \quad : \quad \text{antiferromagnetic} \tag{7.6}$$

$$\vec{B} \quad : \quad \text{external magnetic field} \tag{7.7}$$

$$\vec{S}_i = \frac{1}{2}\begin{pmatrix} \vec{\sigma}_x \\ \vec{\sigma}_y \\ \vec{\sigma}_z \end{pmatrix} \quad : \quad \text{vector of Pauli matrices} \tag{7.8}$$

$$\sigma_x(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y(0) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.9}$$

which have quantum dynamics according to the Heisenberg equation of motion.
$\rightarrow$ In general not exactly solvable.

2. **Ising model**

$$H_{\text{Ising}} = -\sum_{i,j} J_{ij} S_{iz} S_{jz} - \mu B \sum_i S_z \tag{7.10}$$

Since all the x, y components are *set equal to zero*, the spins have no temporal dynamics in this model, but can only take the values $S_{iz} = \pm\frac{1}{2}$ in a statistical manner.
$\rightarrow$ classical spin model

3. **x-y model**
Planar spins on a lattice

$$H_{x-y} = -\sum_{i,j} J_{ij}(S_{ix}S_{jx} + S_{iy}S_{jy}) - \mu B \sum_i S_{ix} \tag{7.11}$$

4. **Stoner model**
Itinerant electrons with short-range (screened) Coulomb repulsion U on a lattice

$$H_{\text{Stoner}} = \sum_{\vec{k},\sigma} \epsilon_{\vec{k}} c^{+}_{\vec{k}\sigma} c_{\vec{k}\sigma} + U \sum_{i,j} \delta_{i,j} \widehat{n}_{i\uparrow} \widehat{n}_{j\downarrow}, \quad U > 0 \tag{7.12}$$

$\widehat{n}_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}, \quad \sigma = \uparrow, \downarrow:$ electron number on site i with spin σ.

7.2.2 Mean-field theory for the Heisenberg model

The ferromagnetic Spin-$\frac{1}{2}$ Heisenberg model with nearest neighbor coupling reads

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - \mu \vec{B} \cdot \sum_i \vec{S}_i \ , \quad J > 0 \tag{7.13}$$

where $\langle \cdot, \cdot \rangle$ denotes nearest neighbor sites.

- Ground state ($T = 0$):
  All spins aligned, ferromagnetic long-range order
- High-T limit ($k_B T \gg J$):
  Independent spins, paramagnetic

Since for the quantum model the excited many-body states and energies cannot be found exactly, one must apply an approximation to describe the thermodynamics.

Mean-field (MF) approximation:
We select an arbitrary single spin $\vec{S}_i$ at lattice site i and assume that the effect of all the surrounding spins on $\vec{S}_i$ can be described in an averaged way by an effective field $\vec{B}_{MF}$ (Weißfield). The resulting mean-field Hamiltonian is then

$$H_{MF} = -\mu(\vec{B} + \vec{B}_{MF}) \cdot \sum_i \vec{S}_i \tag{7.14}$$

Since H_{MF} is non-interacting, it can be solved exactly. However, $\vec{B}_{MF}$ is now a variational parameter which must be determined in an optimal way so as to minimize the free energy.

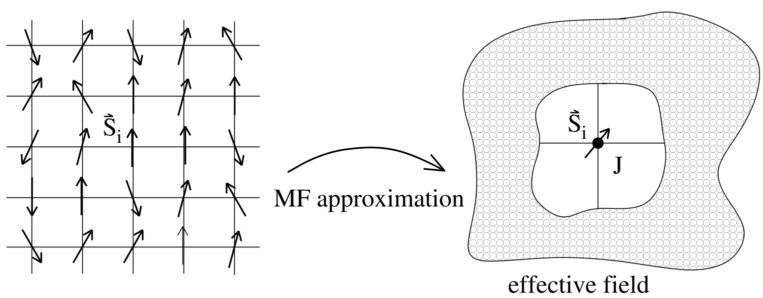

Figure 7.4: Mean-field approximation

In mean-field approximation the equilibrium density operator is

$$\widehat{W}_{MF} = \frac{1}{Z_{MF}} e^{-\frac{\widehat{H}_{MF}}{k_B T}} \tag{7.15}$$

with the MF partition sum (system of N free spins $\frac{1}{2}$)

$$Z_{MF} = tr\{e^{-\frac{\widehat{H}}{k_B T}}\} = (2\cosh x)^N \tag{7.16}$$

with the abbreviations

$$x = \frac{\mu(B + B_{MF})}{2k_B T}, \qquad h = \frac{\mu B}{2k_B T} \tag{7.17}$$

($\vec{B}_{MF} \parallel \vec{B} \parallel \widehat{z}$ without loss of generality).

The MF thermal average of any quantity is the average taken with respect to $\widehat{W}_{MF}$, e.g. the average spin and the spin-spin correlation function (see free spin $\frac{1}{2}$, chapter 3):

$$\begin{aligned}
\langle \vec{S}_i \rangle_{MF} &= \frac{1}{2}\widehat{e}_z \tanh x && (7.18)\\
\langle \vec{S}_i \cdot \vec{S}_j \rangle_{MF} &= \langle \vec{S}_i \rangle_{MF} \cdot \langle \vec{S}_j \rangle_{MF} && (7.19)\\
&= \frac{1}{4}\tanh^2 x\ , && (7.20)
\end{aligned}$$

and the MF free energy

$$\begin{aligned}
F\{\widehat{W}_{MF}\} &= U - TS = tr\{\widehat{H}\widehat{W}_{MF} + k_B T \widehat{W}_{MF} \ln(\widehat{W}_{MF})\} && (7.21)\\
&= -k_B T \ln Z_{MF} + \langle \widehat{H} - \widehat{H}_{MF} \rangle_{MF} && (7.22)\\
&= -k_B T \ln Z_{MF} - J \sum_{\langle i,j \rangle} \langle \vec{S}_i \cdot \vec{S}_j \rangle_{MF} + \mu \vec{B}_{MF} \cdot \sum_i \langle \vec{S}_i \rangle_{MF} && (7.23)
\end{aligned}$$

$$\begin{aligned}
F &= F(B_{MF}) && (7.24)\\
&= N\left[-k_B T \ln(2\cosh x) - J\frac{Z}{2} \cdot \frac{1}{4}\tanh^2 x + k_B T (x - h)\tanh x\right] && (7.25)
\end{aligned}$$

Z = number of nearest neighbors.

To find the optional field B_{MF} we find the minimal (stationary) point of F wrt. B_{MF}:

$$0 = \frac{\partial F}{\partial x} = \left[\frac{1}{4} JZ \tanh x + k_B T(x-h)\right] \frac{1}{\cosh^2 x} \tag{7.26}$$

or

$$\boxed{\mu B_{MF} = \frac{1}{2} JZ \tanh \frac{\mu(B_{MF}+B)}{2k_B T}} \qquad \text{variational condition for } B_{MF}. \tag{7.27}$$

This result can be understood pictorially in that B_{MF} is the effective field acting on $\vec{S}_i$, created by the surrounding spins, i.e. comparing equations (7.13), (7.14):

$$\mu B_{MF} = J \sum_{j,n,n,i} \langle \vec{S}_j \rangle_{MF} = JZ \cdot \frac{1}{2} \tanh x \tag{7.28}$$

in agreement with the variational result.

Since $\langle \vec{S}_j \rangle_{MF}$ of the surrounding spins is the same as for the central spin $\langle \vec{S}_i \rangle_{MF}$ (because of translational invariance), B_{MF} is also called "selfconsistent field" (similar to Hartree-Fock approximation).

Graphical solution of the selfconsistency equation

1. $B = 0$ (no external field)

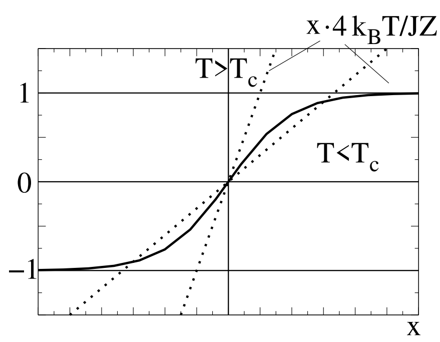

Figure 7.5: Graphical solution for $B = 0$ (no external field)

(a) For $T > T_c$,

$$\boxed{T_c = \frac{JZ}{4k_B}} \quad \text{f.m. transition temperature (Curie temperature)} \tag{7.29}$$

only the solution $x = 0$, $B_{MF} = \langle \vec{S}_i \rangle_{MF} = 0$ exists.
$\rightarrow$ Paramagntic high-T phase.

(b) For $T < T_c$ two solutions $x \neq 0$ exist with the non-zero magnetization

$$M = -\left.\frac{\partial F}{\partial B}\right|_{B=0} = \mu N \langle |\vec{S}_j| \rangle_{MF} = M_0 \tanh x|_{B=0} \ , \tag{7.30}$$

$M_0 = \mu N s$ ($s = \frac{1}{2}$) saturation magnetization with $x =$ solution of selfconsistency equation.

(c) For $T \lesssim T_c$ one obtains by expanding $\tanh(x)$ for small x

$$\frac{\mu B_{MF}}{2k_B T} = x = \pm\sqrt{3}\sqrt{1 - \frac{4k_B T}{JZ}} = \pm\sqrt{3}\sqrt{1 - \frac{T}{T_c}} \tag{7.31}$$

and a finite, spontaneous magnetization

$$M(T) = M_0 \tanh\sqrt{3\left(1 - \frac{4k_B T}{JZ}\right)} \approx M_0\sqrt{3\left(1 - \frac{4k_B T}{JZ}\right)} + O(T_c - T) \tag{7.32}$$

(ferromagnetism).

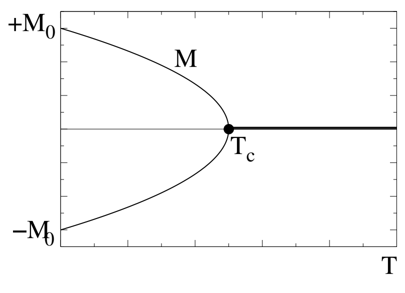

Figure 7.6:

2. $B \neq 0$:
   In finite external field, the magnetization curve is shifted.

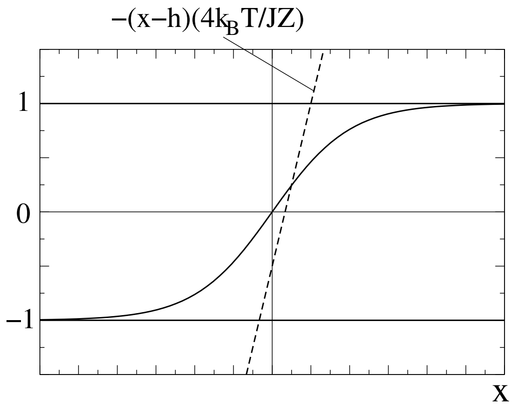

Figure 7.7:

Magnetic susceptibility:
For small external field B and $T \approx T_c$: $x \ll 1$, the selfconsistency equation reads

$$
\begin{aligned}
(x-h)\frac{4k_BT}{JZ} &= \tanh x \approx x - \frac{1}{3}x^3 + O(x^5) && (7.33)\\
\frac{\partial x}{\partial B} &= 2k_BT\frac{\partial x}{\partial h} && (7.34)\\
\left(\left.\frac{\partial x}{\partial h}\right|_{B=0} - 1\right)\underbrace{\frac{4k_BT}{JZ}}_{\frac{T}{T_c}} &= \left.\frac{\partial x}{\partial h}\right|_{B=0} - \left.x^2\right|_{B=0}\left.\frac{\partial x}{\partial h}\right|_{B=0} && (7.35)\\
\left.\frac{\partial x}{\partial h}\right|_{B=0} &= \frac{\frac{T}{T_c}}{\frac{T}{T_c} - 1 + \left.x^2\right|_{B=0}} && (7.36)\\
&= \frac{\frac{T}{T_c}}{\frac{T}{T_c} - 1 + 3\left|1 - \frac{T}{T_c}\right|} && (7.37)
\end{aligned}
$$

$$= \begin{cases} \frac{1}{2}\frac{\frac{T}{T_c}}{1-\frac{T}{T_c}} & , \quad T < T_c \\ \frac{1}{4}\frac{\frac{T}{T_c}}{\frac{T}{T_c}-1} & , \quad T > T_c \end{cases} \tag{7.38}$$

Hence,

$$\chi(T) = \left.\frac{\partial M}{\partial B}\right|_{B=0} \tag{7.39}$$

$$= -\left.\left(\frac{\partial^2 F}{\partial B^2}\right)\right|_{B=0} = M_0 \frac{\partial}{\partial B} \tan x(B,T)|_{B=0} \tag{7.40}$$

$$= M_0 \frac{1}{\cosh^2 x(0,T)} \cdot \frac{1}{2k_B T}\frac{\partial x}{\partial h} \tag{7.41}$$

$B = 0$, $T \approx T_c$: $x \approx 0$

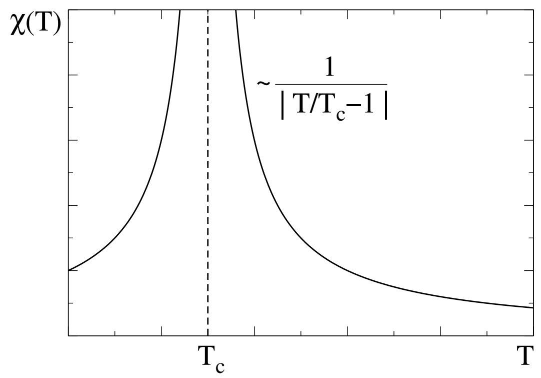

Figure 7.8:

$$\chi(T) = \frac{M_0}{2k_B T_c}\frac{\alpha}{\left|\frac{T}{T_c}-1\right|}, \quad \alpha = \begin{cases} \frac{1}{2} & , \quad T < T_c \\ \frac{1}{4} & , \quad T > T_c \end{cases} \tag{7.42}$$

The magnetic susceptibility diverges st the transition with a power law $\left|\frac{T}{T_c}-1\right|^{-1}$.

From the mean-field solution of the ferromagnetic transition one can extract several important, general concepts of phase transitions:

1. **Classification of phase transitions**
   For $T < T_c$ the thermodynamic state is qualitatively different than for $T > T_c$, e.g. finite magnetization broken rotational symmetry below T_c.
   The transition between the two types of states is called *phase transition*.

   Remark: Definition of qualitative charge non-trivial

   For the f.m. transition the susceptibility $\chi = -\frac{\partial^2 F}{\partial B^2}$ diverges at the transition, while $M = -\frac{\partial F}{\partial B}$ is continuous.

   Definition:
   A *phase transition is of nth order*, if the n-th derivatives of F wrt. to intensive variables diverges at the transition and all lower-order derivatives are continuous.

   We first consider 2nd order phase transitions.

2. **Long-range order and order parameter**
   In all the thermodynamic states at $T < T_c$ the spins have a finite component $\langle \vec{S}_i \rangle_{MF}$ aligned parallel to each other over an infinite distance, i.e. there is *long-range order*.
   The states for $T < T_c$ have a finite spontaneous (i.e. even for $\vec{B} = 0$) magnetization $M(T < T_c) \neq 0$. This region of states is called *ordered phase*.
   For $T > T_c$ the magnetization vanishes, $\vec{M}(T \geq T_c) = 0$. This region is called *disordered phase*.
   A quantity (like $\vec{M}$) which vanishes in the disordered phase and is finite in the ordered phase and which can, hence, be used to characterize the phases, is called *ordered parameter* (OP).

3. **Spontaneous symmetry breaking**
   The states with finite magnetization $|\vec{M}|$ are infinitely degenerate, since the direction of $\vec{M}$ is arbitrary. For a given state, however, the direction of $\vec{M}$ is fixed: The ordered state does not have the full $3D$ rotational symmetry of the Hamiltonian. This fact is called spontaneous *symmetry breaking*.
   The spontaneous breaking of a *continuous* symmetry has important consequences. They result from the fact, that the (symmetry-breaking) ground

state is infinitely degenerate:

(a) Existence of gapless collective excitations which "mix" in the degenerate ground states in a position-dependent way: *Goldstone modes*

Example: Magnons (spin waves) in a ferromagnet

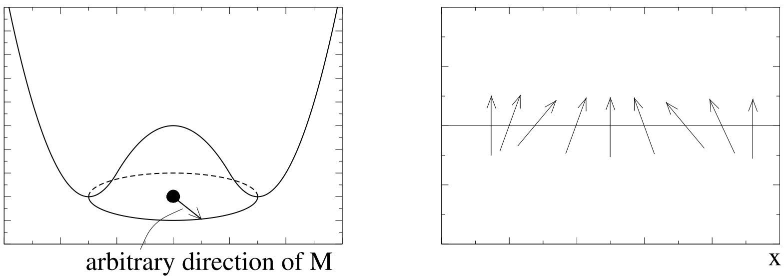

Figure 7.9:

(b) Existence of *domains* with different orientation of $\vec{M}$.

Since Goldstone modes are gapless, they dominate the low-energy excitation spectrum. The Goldstone modes are not included in the MF theory since no spatial dependence is considered. Therefore, the MF theory does not give T-dependencies correctly, like specific heat.

4. **Critical exponents**

At a 2nd order transition the order parameter vanishes with a power law

$$\vec{M}(T) \sim \left(1 - \frac{T}{T_c}\right)^{\beta} \tag{7.43}$$

The susceptibilities (2nd derivatives of F, response functions) diverge with a power law

$$\chi(T) \sim \left|1 - \frac{T}{T_c}\right|^{-\gamma} \tag{7.44}$$

$$c_V(T) \sim \left|1 - \frac{T}{T_c}\right|^{-\alpha} \tag{7.45}$$

The exponents $\alpha, \beta, \gamma, \dots$ are called critical exponents; "critical behavior" of physical quantities at the transition.
In mean field theory:

$$\beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \alpha = 0 \quad \text{(step)} \tag{7.46}$$

This result is universal for any 2nd order transition in MF approximation (will be shown in G.-L.-theory). The exact exponents are in general different from the MF result. The reason is that thermal fluctuations (caused by Goldstone modes) are not taken into account in MF approximation.

7.2.3 Ginzburg-Landau-Theory for 2nd order phase transitions

The free energy density is a function of the mean field $\frac{\mu B_{MF}}{2k_BT} = x$, i.e. of the order parameter field (see 6.2.2):

$$\frac{M}{M_0} = m = \tanh \frac{\mu(B + B_{MF})}{2k_BT} = \begin{cases} 0 & , \quad \text{paramagnet} \\ \neq 0 & , \quad \text{ferromagnet} \end{cases} \tag{7.47}$$

In equilibrium, the value of B_{MF} (and thus m) is determined by the stationarity condition on F (selfconsistency equation). However, in a general state F has the form

$$F = \int d^3x\, f \tag{7.48}$$

$$f(T, m, h) = \frac{N}{V}\left[-k_BT\ln(2\cosh x) - J\frac{Z}{2}\cdot\frac{1}{4}\tanh^2 x + k_BT(x - h)\tanh x\right] \tag{7.49}$$

In order to learn general properties of a 2nd-order transition, one can exploit that the order parameter m vanishes continuously at the transition and, hence, that f can be expanded in powers of m:

$$f = \overbrace{-nk_BT\ln 2}^{f_0} + \frac{1}{2}A\underbrace{\frac{T - T_c}{T_c}}_{\tau} m^2 + \frac{1}{4}Bm^4 - mh + O(m^6) \tag{7.50}$$

The coefficients A, B can be calculated from a microscopic theory for a given system.
However, we are here interested in the general behavior near the transition without calculating the coefficients explicitly. From very general principles they have the properties

1. Only even powers of m appear in the expansion (spatial inversion symmetry) if there is no external field h.

2. The minima of $f(m)$ are at the equilibrium values of m. This implies that the coefficient of the m^2 term is (see figure)

$$\begin{cases} > 0 & , \quad T \geq T_c \\ < 0 & , \quad T < T_c \end{cases} \tag{7.51}$$

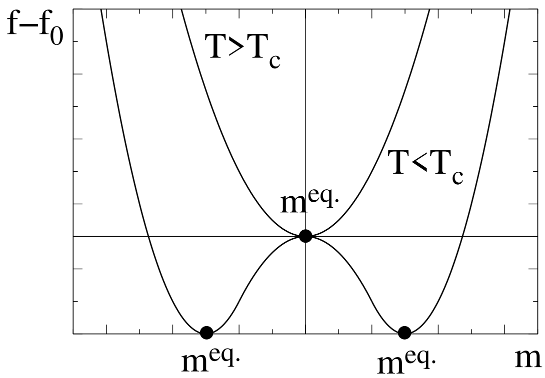

Figure 7.10:

This has been taken into account in the factor

$$\tau = \frac{T - T_c}{T_c} \qquad \text{reduced temperature} \tag{7.52}$$

and $A > 0$.

3. The coefficient of the m^4 term and of all higher-order terms must be $R > 0$ in order to stabilize the system.

In order to include spatial fluctuations, the free energy density can be considered position dependent, which introduces a gradient term ("rigidity" of the OP field)

$$\Delta f = f - f_0 = \overline{f}\left\{\frac{1}{2}\xi_0^2(\vec{\nabla}\vec{m})^2 + \frac{1}{2}\tau\vec{m}^2 + \frac{1}{4}b(\vec{m}^2)^2 - \vec{m}\cdot\vec{h}\right\} \tag{7.53}$$

$\vec{m} = \vec{m}(\vec{x})$, $\quad \xi_0$: OP coherence length.

Although the G.-L. theory has been developed here for a ferromagnetic transition, it can be written for any 2nd order transition and allows to explore general properties of a transition in MF approximation, eg. spec. heat, critical exponents etc.

7.2.4 Principle of the Higgs mechanism

One of the important problems of the standard model of elementary particle physics is the question, how the gauge bosons obtain their non-zero rest mass. Gauge bosons are the particles which mediate the elementary interactions. They are the field quanta of the gauge fields, e.g. $W^{\pm}$ and Z^0 bosons for the weak interaction (SU(2) symmetry), photons for the electromagnetic interaction (U(1) symmetry) and gluons for the strong interaction (SU(3) symmetry). The gauge symmetry prohibits a rest mass of the gauge bosons. For the example of the electromagnetic interaction we show in this chapter first, how the U(1) gauge symmetry implies that photons are massless, and second, as an application of a Ginzburg-Landau theory, how spontaneous symmetry breaking can lead to a finite rest mass of the gauge bosons. The latter is called the Anderson-Higgs mechanism.

U(1) gauge symmetry

We consider a scalar boson matter field $\psi(\underline{x})$ with rest mass m, the Higgs boson field, obeying the Klein-Gordon equation ($\hbar = 1, c = 1$)

$$\left(-\partial_\mu\partial^\mu + m^2\right)\psi = 0. \tag{7.54}$$

The free energy density of the field $\psi(\underline{x})$ is

$$f_\psi = \psi^*\left(-\partial_\mu\partial^\mu + m^2\right)\psi - Ts, \tag{7.55}$$

where s is the entropy density considered later. Note that in the Klein-Gordon theory $n = i\psi^* (\partial_t \psi)$ is interpreted as a particle density.

From equation (7.55) we see that the prefactor of the modulus squared of ψ in the free energy density, $m^2|\psi|^2$, indicates the square of the rest mass. This is, hence, the general place where the rest mass of a particle appears in a field theory.

The electromagnetic 4-vector potential is

$$A^\mu = (\phi, \vec{A}), \qquad\qquad A_\mu = g_{\mu\nu} A^\nu \tag{7.56}$$

and the field tensor in covariant form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{7.57}$$

The electromagnetic free energy density is (neglecting the energy contribution $-Ts_{\text{e.m.}}$ at low T)

$$f_{e.m.} = \frac{1}{8} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) . \tag{7.58}$$

The U(1) gauge symmetry of the total free energy density with respect to local phase transformations of ψ,

$$\psi(\underline{x}) \longmapsto e^{i\Theta(\underline{x})} \psi(\underline{x}) \tag{7.59}$$

implies that the derivatives of ψ are replaced according to

$$i\partial_\mu \longmapsto i\partial_\mu - eA_\mu(\underline{x}) \tag{7.60}$$

(“minimal coupling”), where a gauge field $A_\mu(\underline{x})$ has been introduced which must transform as

$$A_\mu(\underline{x}) \longmapsto A_\mu(\underline{x}) - \frac{1}{e} \partial_\mu \Theta(\underline{x}) \tag{7.61}$$

in order to obey the local U(1) gauge symmetry.

The total free energy density of the interacting system is, hence,

$$\begin{aligned} f\{\psi, \psi^*, A^\mu\} \;&=\; \psi^* \left[(-i\partial_\mu - eA_\mu)(-i\partial^\mu - eA^\mu) + m^2 \right] \psi \\ &\quad -Ts + \frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{7.62}$$

The free electromagnetic field in vacuum (no charges) is a transverse field, $\partial_\mu A^\mu = 0$. This additional constraint means that the 4-vector A^μ consists of 3 independent fields, i.e. is a 3-dim. representation of the rotation group. Hence, A^μ carries an integer spin $S = 1$, and its field quanta are bosons. A^μ is identified with the electromagnetic field and its field quanta are the photons.

It is seen that the energy density u is invariant under the local U(1) gauge transformations (7.59), (7.61), and that a mass term for the gauge field,

$$M_{\mathrm{ph}}^2 A_\mu A^\mu \tag{7.63}$$

would break the gauge symmetry (see (7.61)).

Spontaneous symmetry breaking

We will now describe a condensation transition of the bosonic matter field ψ. To that end we will derive a Ginzburg-Landau form of the free energy.

a) In order to have a condensation the number of bosons (ψ- field) must be fixed by a chemical potential $\mu(T)$.

   Therefore, we transform to the grand canonical potential density w with the natural variable μ:

$$w\{\psi, \psi^*, A^\mu\} = f\{\psi, \psi^*, A^\mu\} - \mu n\{\psi, \psi^*\}. \tag{7.64}$$

   We will be interested only in the condensate part $\psi_0(\underline{x})$ of the matter field, which has momentum $\vec{k}^{(0)} = 0$ and rest energy m,

$$\begin{aligned} \psi_0(\underline{x}) &= |\psi_0| e^{-ik_\mu^{(0)} x^\mu} = |\psi_0| e^{-imt} && (7.65) \\ n^{(0)} &= i\psi^* \partial_t \psi = m|\psi_0|^2 && (7.66) \end{aligned}$$

b) The matter field has a repulsive interaction with itself. For our U(1) system this is the electromagnetic interaction (Coulomb repulsion), since the particles of the matter field ψ carry charge e due to the coupling to A^μ. This interaction would be obtained in second order perturbation theory in the coupling to the electromagnetic field A^μ. For simplicity, we here assume a zero-range repulsive contact interaction, which gives the contribution

$$\frac{1}{4}b|\psi(\underline{x})|^2|\psi(\underline{x})|^2, \qquad b > 0 \tag{7.67}$$

to the free energy density. It will stabilize the system against infinite condensate amplitude $|\psi_0|$.

The total grand potential density is then,

$$\begin{aligned} w\{\psi_0, \psi_0^*, A^\mu\} &= w_0\{\psi_0, \psi_0^*, A^\mu\} + e^2 A_\mu A^\mu |\psi_0|^2 \\ &\quad -\mu(T) m |\psi_0|^2 + \frac{1}{4} b \left(|\psi_0|^2\right)^2 \end{aligned} \tag{7.68}$$

There is no derivative of $\psi_0(\underline{x})$ in w, since the condensate wave function $\psi_0(\underline{x})$ has no spatial dependence and the time derivative cancels in f. The entropy of the condensate $\psi_0(\underline{x})$ vanishes, since it is a single state. All non-condensate contributions, including $-Ts$ and $\frac{1}{8\pi}F_{\mu\nu}F^{\mu\nu}$ are absorbed in w_0. They behave continuously at the condensation transition (which will be considered below).

The quadratic term in w has a sign change at a critical temperature T_c, since

$$\mu(T=0) = m \quad = \quad \text{minimum energy of a matter particle} > 0 \tag{7.69}$$

$$\mu \xrightarrow{T\to\infty} -\infty \qquad \text{(see bosonic systems), and} \tag{7.70}$$

$$e^2\langle A_\mu A^\mu\rangle > 0 \qquad \text{is small,} \qquad e^2\langle A_\mu A^\mu\rangle_{T=0} = 0. \tag{7.71}$$

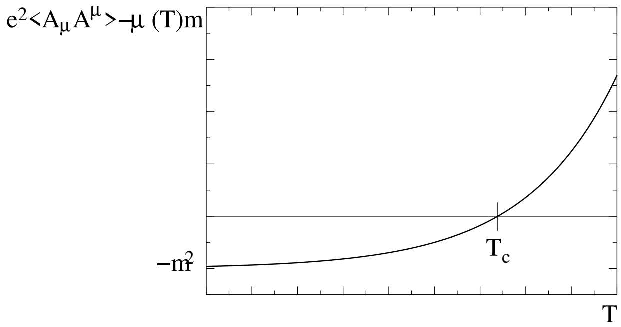

Figure 7.11:

Hence, the grand potential can be written as

$$w = w_0 + \frac{1}{2}a\,\frac{\overbrace{T - T_c}^{\tau}}{T_c}\,|\bar{\psi}_0|^2 + \frac{1}{4}b\left(|\bar{\psi}_0|^2\right)^2 + \dots \quad , \qquad a, b > 0. \tag{7.72}$$

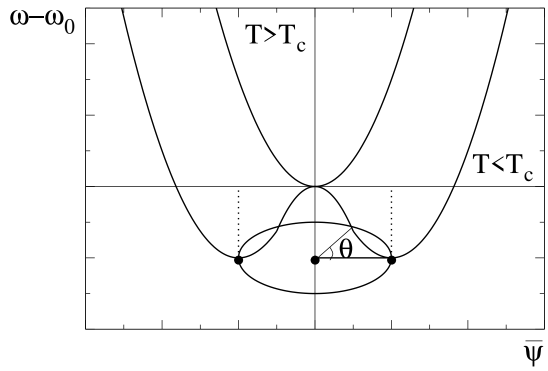

Figure 7.12:

For $T > T_c$, w has minima for nonzero condensate amplitudes $|\bar{\psi}_0|$

$$0 \overset{!}{=} \frac{\partial w}{\partial \bar{\psi}_0^*} a\tau\bar{\psi}_0 + b|\bar{\psi}_0|^3 \tag{7.73}$$

$$\bar{\psi}_0(T) = \sqrt{-\frac{a}{b}\tau} = \sqrt{\frac{a}{b}} \cdot \sqrt{1 - \frac{T}{T_c}}. \tag{7.74}$$

The symmetry-broken ground state has a continuous degeneracy due to an arbitrary global phase $\Theta \in [0, 2\pi[$:

$$\psi_0 = \bar{\psi}_0 e^{i\Theta} e^{-imt}. \tag{7.75}$$

In the condensate phase, $T < T_c$, the free energy of the photon field is

$$f_{A_\mu} = e^2 |\bar{\psi}_0|^2 A_\mu A^\mu + \frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu}, \qquad |\bar{\psi}_0| > 0. \tag{7.76}$$

This means that by the spontaneous symmetry breaking A_μ has dynamically acquired a non-zero mass term with

$$M_{\mathrm{ph}}^2 = e^2 |\bar{\psi}_0|^2 = e^2 \sqrt{\frac{a}{b}} \cdot \sqrt{1 - \frac{T}{T_c}}. \tag{7.77}$$

Physical interpretation:

The physical interpretation can be found by considering the wave equation for A^μ in the symmetry broken state

$$0 = \left(-\partial_t^2 + \partial_{\vec{x}}^2 + M_{\mathrm{ph}}^2\right) A^\mu \qquad (c = 1) \tag{7.78}$$

$$A^\mu = \bar{A}^\mu e^{-i(\omega t - \vec{k}\vec{x})} \tag{7.79}$$

with

$$\omega = \sqrt{k^2 - M_{\mathrm{ph}}^2} = \begin{cases} \sqrt{k^2 - M_{\mathrm{ph}}^2}, & k \geq M_{\mathrm{ph}} \\ -i\sqrt{M_{\mathrm{ph}}^2 - k^2}, & k < M_{\mathrm{ph}}. \end{cases} \tag{7.80}$$

For $c\hbar k < mc^2$ the wave equation for A^μ has only decaying solutions, i.e. the field A^μ cannot statically exist inside the condensate.

Physically, this situation is realized in a superconductor. $\bar{\psi}_0$ is the superconducting condensate amplitude. The fact that photons (i.e. an electromagnetic field) with energy $\hbar\omega < M_{\mathrm{ph}} c^2$ cannot exist inside a superconductor is the Meißner effect.