Non-Universal Anomalies and $\boldsymbol{R}$ Symmetries in Heterotic String Models

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Chapter 1

Introduction

For more than half a century physicists have tried to unify the four fundamental forces of nature in a single framework. The first successful step in this direction was the emergence of the Salam–Weinberg theory of electroweak interactions. Since then, the Standard Model (SM) of particle physics has been proposed to describe the electromagnetic, weak, and strong interaction in a single gauge theory with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. However, the SM is only tested up to the scale of electroweak symmetry breaking, $M_{\text{ew}} \sim 100$ GeV. In addition, gravity is not included in the SM. To unify the aforementioned interactions with gravity, described by Einstein's theory of general relativity at a scale of $M_{\text{P}} \sim 10^{19}$ GeV, an ultraviolet extension of the SM is needed.

One possible ingredient of such an extension is *supersymmetry*. In its simplest form, it predicts the existence of one superpartner for each SM particle which differs in spin by $\frac{1}{2}$. Among other things, supersymmetry removes quadratic divergences from scalar mass terms, provides a natural WIMP dark matter candidate, and makes gauge coupling unification possible. The idea to unify the three SM gauge groups into a single bigger one is well motivated, since in the minimal supersymmetric extension of the Standard Model (MSSM) the three gauge couplings seem to unify at a scale $M_{\text{GUT}} \sim 10^{16}$ GeV. Phenomenologically appealing candidates for gauge groups of such a *grand unified theory* (GUT) are $SU(5)$ and $SO(10)$ [1].

The probably best-developed theory to describe an ultraviolet completion of the SM coupled to gravity is *string theory*. In this framework, point-like particles are replaced by one-dimensional strings whose typical length is fixed by the so-called *string scale*, which is of a similar order as the GUT and Planck scales. In this thesis, we are particularly interested in heterotic string theory [2, 3], which only contains closed strings. It automatically provides supergravity in ten dimensions with an $E_8 \times E_8$ or $SO(32)$ gauge theory in its low energy limit. In order to obtain four-dimensional theories with $\mathcal{N} = 1$ supersymmetry, e.g. the MSSM, six of these dimensions have to be compactified on a smooth Calabi–Yau manifold or an orbifold.

On top of being free of ultraviolet divergences, string theory has an elegant way of yielding anomaly-free and thus consistent theories. It incorporates anomaly cancellation via the *Green–Schwarz mechanism* [4], which imposes strong constraints on the gauge group, in particular, only $E_8 \times E_8$ and $SO(32)$ are possible.

In this thesis, we deal with four-dimensional theories obtained by compactifying the heterotic string on orbifolds [5–7] and their smooth Calabi–Yau counterparts, obtained via a blow-up procedure (cf. [8–11], and [12, 13] for a gauged linear sigma model approach). The discussion of anomalies differs in these two cases. In orbifold constructions, imposing modular invariance of the string partition function is sufficient to guarantee consistency of the theory via the Green–Schwarz mechanism [14], where the axion dual to the Kalb–Ramond two-form cancels the anomalies. Since there is only one axion but many different possible anomalies, anomaly freedom requires that all anomalies are related such that the different anomalies can be canceled with one universal axion. In smooth Calabi–Yau compactifications, however, the picture is different. Anomalies are again canceled via the Green–Schwarz mechanism [15–17]. But, in contrast to the orbifold case, there can be more than one axion and the axionic couplings are in general non-universal, even if there is only one axion. This is true because in the blow-up phase, the additional axions arise from internal cohomology two-forms and their couplings are determined by the choice of the gauge bundle used for blowing-up the orbifold. This anomaly cancellation mechanism becomes important when considering vector bundles or line bundles over Calabi–Yau manifolds (see e.g. [18–20] for recent applications).

Both of the above mentioned compactification schemes are plagued by a number of phenomenological problems, like too fast proton decay and the so-called μ problem. One of the prevailing methods to circumvent these issues are (discrete) R symmetries. Since R symmetries act differently on fermions and bosons one might conclude that in string theory, R symmetries emerge as remnants of the local Lorentz symmetry. However, the exact origin of R symmetries both in orbifold and smooth Calabi–Yau compactifications is far from understood.

Thesis Outline and Summary

This thesis is organized as follows: In Chapter 2 we briefly review the heterotic string and its compactification on orbifold spaces. As a simple example we discuss the $T^6/\mathbb{Z}_3$ orbifold. The following chapter is devoted to anomalies and their cancellation. Here, we comment on the existence of an anomalous $U(1)$ on orbifolds and give expressions for the relevant anomaly coefficients. In addition, we review the Green–Schwarz mechanism and the concept of anomaly polynomials. In Chapter 4 we discuss the blow-up procedure for heterotic orbifolds. To this end, we introduce the necessary tools from toric geometry and employ them to resolve curvature singularities in compact and non-compact orbifolds. Afterwards, in Chapter 5 we discuss an explicit blow-up model of the $\mathbb{Z}_3$ orbifold in standard embedding. We use three Abelian vector bundles to arrive at a model with gauge group $SO(8) \times SU(3) \times U(1)_{\text{A}} \times U(1)_{\text{B}} \times E_8$. Subsequently, we analyze arising anomalies of both $U(1)$ symmetries to provide an example of non-universal Green–Schwarz anomaly cancellation. This result agrees with our discussion in Chapter 3 in which we argue that non-universal axion couplings are indeed the general case for smooth Calabi–Yau compactifications, thus clarifying some confusion in recent literature. Furthermore, we comment on the consequences of anomaly non-universality for bottom-up GUT models and give an example of a $\mathbb{Z}_6$ (non-R) symmetry which commutes with $SU(5)$ grand unification and satisfies all important phenomenological requirements. In the remainder of the chapter we discuss remnant discrete symmetries of the exemplary string model, using a gauged linear sigma model description. We focus especially on R symmetries and conclude that the $\mathbb{Z}_3$ orbifold exhibits a $\mathbb{Z}_6^R$ symmetry which is completely broken by the blow-up procedure. We also attempt to clarify some notational confusion regarding the order of discrete R symmetries on orbifolds. The results presented in this chapter have been published in [21]. In Chapter 6 we give an outlook on future research directions, especially the investigation of superpotential couplings with regard to their charge under R symmetries. We try to find a connection between R symmetries in orbifold and smooth Calabi–Yau models and attempt to understand how they are broken during blow-up procedures. Afterwards, we give a short conclusion.

Chapter 2

Heterotic Strings on Orbifolds

Among all string theories, the heterotic string [2,3] provides the most interesting phenomenological applications. In ten-dimensional space-time it supports $\mathcal{N} = 1$ super Yang–Mills theory coupled to supergravity with gauge group $E_8 \times E_8$ or $SO(32)$. As mentioned before, to relate this to four-dimensional theories with $\mathcal{N} = 1$ supersymmetry, six dimensions have to be compactified. Heterotic orbifolds [5–7] constitute simple geometrical compactification spaces. Orbifolds are flat everywhere except for isolated curvature singularities at the so-called *fixed points*. It turns out that these singularities do not pose an obstruction to performing exact string computations. Thus, a very large number [22, 23] of viable and phenomenologically interesting string compactifications on orbifolds are known.

2.1 The Heterotic String

We start with a brief review of the heterotic string following the description in [24,25], in order to set the notation and introduce some of the underlying concepts which are relevant to the following chapters.

Heterotic string theory is the only string theory which contains closed strings only. One of the key aspects of this theory is to exploit the fact that left- and right-moving excitations of these closed strings decouple. This results in left-moving bosonic strings living in $d = 26$ dimensions and right-moving superstrings in $d = 10$ dimensions. The string is described by maps $X^\mu(\tau, \sigma)$ which embed the two-dimensional world-sheet (with coordinates τ and σ) into the ten-dimensional target space. Left- and right-movers are then treated separately in the following way,

$$X^\mu(\tau, \sigma) = X^\mu_{\mathrm{L}}(\sigma^+) + X^\mu_{\mathrm{R}}(\sigma^-)\,, \tag{2.1}$$

where we have introduced the so-called *light-cone coordinates* $\sigma^\pm = \tau \pm \sigma$.

World-sheet action and mode expansion

The world-sheet action for the ten-dimensional fields of the heterotic string is of the form

$$S_{10\mathrm{d}} = \frac{1}{\pi}\int \mathrm{d}^2\sigma \left(2\partial_+ X^\mu \partial_- X_\mu + \mathrm{i}\psi^\mu_{\mathrm{R}}\partial_-\psi_{\mathrm{R},\mu}\right) , \tag{2.2}$$

where $\mu = 0\ldots9$ and $\partial_\pm$ denotes the derivative with respect to $\sigma^\pm$. Thus, the right-moving superstring is described by ten-dimensional bosons $X^\mu_{\mathrm{R}}(\sigma^-)$ and Majorana–Weyl fermions $\psi^\mu_{\mathrm{R}}(\sigma^-)$ which are related by world-sheet supersymmetry. The left-moving string is described by $X^\mu_{\mathrm{L}}(\sigma^+)$ and 16 additional bosonic degrees of freedom $X^I_{\mathrm{L}}(\sigma^+)$, where $I = 1\ldots16$.

As a step towards quantization of the theory we perform a mode expansion of the bosonic and fermionic degrees of freedom. Respecting the periodic boundary conditions on the cylindric world-sheet, i.e.

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)\,, \tag{2.3}$$

one finds for the right-moving bosons

$$X^\mu_{\mathrm{R}}(\sigma^-) = x^\mu_{\mathrm{R}} + p^\mu_{\mathrm{R}}\cdot\sigma^- + \frac{\mathrm{i}}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq0}}\frac{1}{n}\alpha^\mu_n e^{-2\mathrm{i}n\sigma^-} . \tag{2.4}$$

The fermions allow for either periodic (Ramond) or anti-periodic (Neveu–Schwarz) boundary conditions: $\psi^\mu_{\mathrm{R}}(\sigma^- + \pi) = \pm\psi^\mu_{\mathrm{R}}(\sigma^-)$. These two sectors have to be treated separately and have the following mode expansions

$$\psi^\mu_{\mathrm{R}}(\sigma^-) = \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2\mathrm{i}n\sigma^-} , \qquad \text{(R)} \tag{2.5a}$$

$$\psi^\mu_{\mathrm{R}}(\sigma^-) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2\mathrm{i}r\sigma^-} . \qquad \text{(NS)} \tag{2.5b}$$

Let us now turn to the left-movers. The ten-dimensional bosonic degrees of freedom have a mode expansion similar to the right-moving ones, i.e.

$$X^\mu_{\mathrm{L}}(\sigma^+) = x^\mu_{\mathrm{L}} + p^\mu_{\mathrm{L}}\cdot\sigma^+ + \frac{\mathrm{i}}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq0}}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2\mathrm{i}n\sigma^+} . \tag{2.6}$$

In order to preserve modular invariance, the 16 additional bosonic degrees of freedom X^I have to be compactified on a 16-dimensional torus T^{16}, described by a lattice

$$\Gamma_{16} = \left\{2\pi\sum n_i e_i | n_i \in \mathbb{Z}\right\}. \tag{2.7}$$

Γ_{16} is spanned by 16 linearly independent vectors $e_i \in \mathbb{R}^{16}$ and is required to be even and self-dual. With 16 dimensions being compactified on $T^{16} = \mathbb{R}^{16}/\Gamma_{16}$, the X^I can be thought of as strings which are closed after encircling the torus a certain number of times:

$$X^I(\sigma^+ + \pi) = X^I(\sigma^+) + 2\pi \sum_{i=1}^{16} w_i e_i^I \,, \tag{2.8}$$

where w_i are called *winding numbers*. It turns out that Γ_{16} can only be the root lattice of $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$. This corresponds to the well-known result from ten-dimensional anomaly cancellation [4] that the only gauge groups which support a consistent heterotic string theory are $E_8 \times E_8$ and $SO(32)$. Since the gauge group descends in this way from the 16 compactified dimensions, the X^I are frequently called *gauge degrees of freedom*. Their mode expansion is given by

$$X_{\mathrm{L}}^I(\sigma^+) = x_{\mathrm{L}}^I + p_{\mathrm{L}}^I \cdot \sigma^+ + \frac{\mathrm{i}}{2} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \tilde{\alpha}_n^I e^{-2\mathrm{i}n\sigma^+} \,, \tag{2.9}$$

where $p_{\mathrm{L}} \in \Gamma_{16}$ is the *internal momentum* of the string.

Quantization and massless spectrum

In order to quantize the theory, the expansion modes encountered in the previous paragraph are taken to be operator-valued. They act on a Hilbert space $\mathcal{H}$ and obey the commutation relations

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0 \,, \tag{2.10a}$$

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = [\alpha_m^\mu, \alpha_n^\nu] = n\delta_{n+m,0}\eta^{\mu\nu} \,, \tag{2.10b}$$

$$[\tilde{\alpha}_m^I, \tilde{\alpha}_n^J] = n\delta_{n+m,0}\delta^{IJ} \,. \tag{2.10c}$$

The fermionic modes are elements of a Grassmann algebra and thus fulfill the anticommutation relations

$$\{b_r^\mu, b_s^\nu\} = \delta_{r+s,0}\eta^{\mu\nu} \,, \tag{2.11a}$$

$$\{d_m^\mu, d_n^\nu\} = \delta_{n+m,0}\eta^{\mu\nu} \,. \tag{2.11b}$$

Demanding reality of X and the Majorana property of ψ then amounts to the following conditions on the modes,

$$\alpha_n = \alpha_{-n}^\dagger \,, \quad \tilde{\alpha}_n = \tilde{\alpha}_{-n}^\dagger \,, \quad d_n = d_{-n}^\dagger \,, \quad b_r = b_{-r}^\dagger \,. \tag{2.12}$$

Hence we observe that some of the modes act as creation operators and some as annihilation operators on $\mathcal{H}$. However, negative- or zero-norm states can appear in the spectrum alongside the physical positive-norm states. In order to remove these unphysical states from the theory, we impose a number of constraints on the world-sheet energy-momentum tensor $T_{\alpha\beta}$ and the world-sheet supercurrent J_α:

$$T_{++} = T_{--} = J_- = 0\,. \tag{2.13}$$

The above quantities can be expressed in terms of the fields as follows

$$T_{++} = -\partial_+ X^\mu_{\mathrm{L}} \partial_+ X_{\mathrm{L},\mu} - \partial_+ X^I_{\mathrm{L}} \partial_+ X_{\mathrm{L},I}\,, \tag{2.14a}$$

$$T_{--} = -\partial_- X^\mu_{\mathrm{R}} \partial_- X_{\mathrm{R},\mu} - \frac{\mathrm{i}}{2} \psi^\mu_{\mathrm{R}} \partial_- \psi_{\mathrm{R},\mu}\,, \tag{2.14b}$$

$$J_- = \psi^\mu_{\mathrm{R}} \partial_- X_{\mathrm{R},\mu}\,. \tag{2.14c}$$

The mode expansions for these operators, i.e.

$$T_{++} = \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-2in\sigma^+}\,, \quad T_{--} = \sum_{n\in\mathbb{Z}} L_n e^{-2in\sigma^-}\,, \tag{2.15a}$$

$$J_-^{(\mathrm{R})} = \sum_{n\in\mathbb{Z}} F_n e^{-2in\sigma^-}\,, \quad J_-^{(\mathrm{NS})} = \sum_{s\in\mathbb{Z}+\frac{1}{2}} G_s e^{-2is\sigma^-}\,, \tag{2.15b}$$

define the generators of the super-Virasoro algebra. Using these we can define a physical state $|\phi\rangle \in \mathcal{H}$ by demanding

$$(L_0 - a_{\mathrm{R}})|\phi\rangle = (\tilde{L}_0 - a_{\mathrm{L}})|\phi\rangle = 0\,, \tag{2.16a}$$

$$L_n|\phi\rangle = \tilde{L}_n|\phi\rangle = 0\,, \qquad \forall n > 0\,, \tag{2.16b}$$

$$F_n|\phi\rangle = 0\,, \qquad \forall n \geq 0\,, \tag{2.16c}$$

$$G_s|\phi\rangle = 0\,, \qquad \forall s \geq 0\,. \tag{2.16d}$$

Note that the zero-mode equations contain shifts $a_{\mathrm{L}} = 1$ and $a_{\mathrm{R}} = 0$ or $\frac{1}{2}$ – depending on the boundary conditions (Ramond or Neveu–Schwarz) – which arise from the normal-ordering of the oscillators.

We are now in a position to discuss mass operators of the theory. In light-cone gauge the mass-squared operator reads

$$M^2 = M^2_{\mathrm{L}} + M^2_{\mathrm{R}}\,, \tag{2.17}$$

with level matching condition $M^2_{\mathrm{L}} = M^2_{\mathrm{R}}$. The operators in eq. (2.17) are given by

$$\frac{M^2_{\mathrm{L}}}{8} = -\frac{p_{\mathrm{L},\mu} p^\mu_{\mathrm{L}}}{8} = \frac{p_I p^I}{2} + \tilde{N} - 1\,, \tag{2.18a}$$

$$\frac{M^2_{\mathrm{R}}}{8} = -\frac{p_{\mathrm{R},\mu} p^\mu_{\mathrm{R}}}{8} = \frac{q^2}{2} + N - a_{\mathrm{R}}\,, \tag{2.18b}$$

where q is the right-moving momentum taken from the weight lattice of $SO(8)$, which is the little group of the Lorentz group $SO(1,9)$. We have implicitly bosonized the theory, hence in this formulation two world-sheet Majorana–Weyl fermions play the role of one holomorphic boson. For the sake of completeness we give the explicit form of the oscillator number operators N and $\tilde{N}$:

$$\tilde{N} := \sum_{n=1}^{\infty} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n,\mu} + \tilde{\alpha}^{I}_{-n}\tilde{\alpha}_{n,I}\,, \tag{2.19a}$$

$$N := \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n,\mu}\,. \tag{2.19b}$$

Let us now proceed to the determination of the massless spectrum. The massive modes are of no particular interest to string phenomenology, since the masses are of the order of the string scale $M_{\mathrm{S}} \sim 10^{17}$ GeV [26]. Thus, in a low energy effective theory massive states are not present.

In the right-moving sector we deduce from demanding (2.18b) to be zero that massless modes have $N = 0$ (since N has integral eigenvalues, $N > 0$ leads to massive states) and $q^2 = 1$, which is satisfied by

- $q_{\mathrm{V}} = (\underline{\pm 1, 0, 0, 0})$, which are the weights of the $\mathbf{8}_{\mathrm{V}}$ representation of $SO(8)$. The underline denotes permutations of the entries.
- $q_{\mathrm{S}} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an even number of plus signs, which correspond to the weights of the $\mathbf{8}_{\mathrm{S}}$ representation of $SO(8)$.

In the left-moving sector the picture is slightly different. From eq. (2.18a) with $M_{\mathrm{L}}^2 = 0$ we conclude that massless left-moving states must have either $p^2 = 2$ and $\tilde{N} = 0$, or $p^2 = 0$ and $\tilde{N} = 1$. In the first case, the internal momenta p are the 480 roots of $E_8 \times E_8$ or $SO(32)$. In the second case, the massless states are found to be the oscillator states $\tilde{\alpha}^{\mu}_{-1}|0\rangle_{\mathrm{L}}$ and $\tilde{\alpha}^{I}_{-1}|0\rangle_{\mathrm{L}}$. The full spectrum of the heterotic string is then obtained by computing the tensor product of the previously discussed states:

$$|q\rangle_{\mathrm{R}} \otimes \tilde{\alpha}^{\mu}_{-1}|0\rangle_{\mathrm{L}} \quad \mathcal{N} = 1,\ d = 10 \text{ SUGRA multiplet}\,, \tag{2.20a}$$

$$|q\rangle_{\mathrm{R}} \otimes \tilde{\alpha}^{I}_{-1}|0\rangle_{\mathrm{L}} \quad 16 \text{ Cartan generators of } E_8 \times E_8 \text{ or } SO(32)\,, \tag{2.20b}$$

$$|q\rangle_{\mathrm{R}} \otimes |p\rangle_{\mathrm{L}} \quad 480 \text{ generators of } E_8 \times E_8 \text{ or } SO(32)\,. \tag{2.20c}$$

The 496 states in (2.20b) and (2.20c) transform in the adjoint representation $\mathbf{248} + \mathbf{248}$ of $E_8 \times E_8$ or the $\mathbf{496}$ of $SO(32)$. It is also straightforward to verify that the gauge singlet states in (2.20a) indeed constitute an $\mathcal{N} = 1$ SUGRA multiplet:

While the oscillators α^{I}_{-1} transform trivially under $SO(8)$, the $\tilde{\alpha}^{\mu}_{-1}$ transform in the $\mathbf{8}_{\mathrm{V}}$ vector representation of $SO(8)$. Thus for the case of bosonic q, (2.20a) yields the tensor product

$$\mathbf{8}_{\mathrm{V}} \times \mathbf{8}_{\mathrm{V}} = \mathbf{1} + \mathbf{28} + \mathbf{35}_{\mathrm{V}}\,, \tag{2.21}$$

which can, by merely counting degrees of freedom, be identified with the dilaton Φ, the antisymmetric two-form $B_{\mu\nu}$, and the graviton $g_{\mu\nu}$, respectively. A similar procedure for the fermionic part of q leaves us with

$$\mathbf{8}_{\mathrm{S}} \times \mathbf{8}_{\mathrm{V}} = \mathbf{8}_{\mathrm{C}} + \mathbf{56}_{\mathrm{C}}\,, \tag{2.22}$$

which corresponds to the dilatino ψ and the gravitino ψ_μ, respectively.

2.2 Orbifold Compactifications

As mentioned before, in order to reproduce the MSSM or the Standard Model of particle physics from string theory, six of the ten stringy dimensions have to be compactified. This is done by choosing the ten-dimensional target space M_{10} to be of the form

$$M_{10} = M_{1,3} \times M_6\,, \tag{2.23}$$

where $M_{1,3}$ is four-dimensional Minkowski space and M_6 denotes the six-dimensional internal space. Perhaps the most important restriction on the compactification space M_6 is the requirement of $\mathcal{N} = 1$ supersymmetry in four dimensions. Demanding one remnant covariantly constant spinor after breakdown of the ten-dimensional Lorentz group restricts the holonomy group of M_6 to be $SU(3)$. This constraint is satisfied by Calabi–Yau manifolds and orbifolds.

This section is devoted to orbifold geometry and topology, as well as the propagation of heterotic strings on orbifolds. We discuss the most important concepts by means of a simple example, the $T^6/\mathbb{Z}_3$ orbifold. Compactifications on smooth Calabi–Yau manifolds, especially blow-ups of heterotic orbifolds, will be discussed in Chapter 4.

2.2.1 Orbifold Geometry

By definition, an orbifold is a manifold with a discrete symmetry modded out. To construct a toroidal orbifold $\mathcal{O}$ suitable for string compactification, we start with a six-dimensional torus T^6 and divide out a symmetry of the underlying torus lattice Γ, also known as *point group*:

$$\mathcal{O} = T^6/P\,. \tag{2.24}$$

Note that P has to act crystallographically on Γ, hence the number of possible point groups is finite. P acts on the three complex coordinates z_i of the torus according to

$$\theta : z_i \longrightarrow \theta z_i = e^{2\pi \mathrm{i} v_i} z_i\,, \tag{2.25}$$

where $v = (v_1, v_2, v_3)$ is called *twist vector*. Note that in this thesis we only consider orbifolds with $P = \mathbb{Z}_N$ whose torus lattices can be factorized into three two-dimensional tori. The condition of $\mathcal{N} = 1$ supersymmetry in four dimensions amounts to requiring the point group, which is identified with the orbifold holonomy group, to be a subgroup of $SU(3)$. This implies

$$\sum_{i=1}^{3} v_i = 0 \bmod 1\,. \tag{2.26}$$

Furthermore, since the point group is of order N the twist vector has to satisfy $Nv_i = 0 \bmod 1$.

Before we start discussing the propagation of strings on orbifolds, let us briefly comment on a different way to construct orbifolds, namely via the so-called *space group*. The space group S is defined by an equivalence relation for the complex torus coordinates,

$$z \sim (\theta^k, n)z := \theta^k z + n\,, \qquad k \in \{1, 2, \ldots, N-1\}\,, \tag{2.27}$$

where $n = n_i e_i$ are lattice translations and θ is the generating element of the point group introduced in (2.25). Two elements $g_1, g_2 \in S$ obey the multiplication law

$$g_1 \circ g_2 = (\theta^k, n) \circ (\theta^l, m) = (\theta^{k+l}, \theta^k m + n)\,. \tag{2.28}$$

The orbifold $\mathcal{O}$ is then constructed as the quotient space

$$\mathcal{O} = \mathbb{C}^3/S\,. \tag{2.29}$$

Note that P and S do not act freely on $\mathbb{C}^3$: There is a finite number of fixed points f which satisfy $f = (\theta^k, n)f$ and are thus invariant under S. Due to the non-trivial holonomy these fixed points constitute curvature singularities on $\mathcal{O}$. If one of the complex coordinates is invariant under a twist θ^k, i.e. $\theta^k z_i = z_i$, there exists a fixed subspace of complex dimension two, which for orbifolds on factorizable tori is always a fixed torus.

2.2.2 Strings on Orbifolds

Our next step is to consider the compactification of six bosonic coordinates X^μ, $\mu = 4 \ldots 9$ on the orbifold $\mathcal{O}$. It is convenient to express these real coordinates in terms of three complex coordinates

$$Z^a = X^{2a+2} + \mathrm{i}X^{2a+3}\,, \qquad a = 1 \ldots 3\,. \tag{2.30}$$

Due to the orbifold geometry, i.e. the space group action, the boundary conditions (2.3) for the string coordinates change:

$$Z(\tau, \sigma + 2\pi) \overset{!}{=} \theta^k Z(\tau, \sigma) + n_i e_i\,, \tag{2.31a}$$

$$\psi(\tau, \sigma + 2\pi) \overset{!}{=} \pm\theta^k \psi\,. \tag{2.31b}$$

To give a similar expression for the coordinates X^I, we have to specify how the space group is embedded into the gauge degrees of freedom. A convenient choice is to assign to every twist θ_i

a *shift vector* V_i^I and to every lattice translation by e_i a *Wilson line* W_i. The new boundary condition for the gauge degrees of freedom then reads

$$X^I(\sigma^+ + 2\pi) \stackrel{!}{=} X^I(\sigma^+) + V_g^I \,, \tag{2.32}$$

where

$$V_g^I = kV^I + \sum_{i=1}^{6} n_i W_i^I \tag{2.33}$$

denotes the so-called *local shift vector*, associated with a space group element $g \in S$ at a particular fixed point. Note that from now on we will drop the index $I = 1 \ldots 16$ whenever the notation is clear without ambiguity. The full group, which acts on both the three complex-dimensional space and the gauge degrees of freedom, we call *orbifold group* O.

An important distinction in terms of physical states is made between the untwisted sector ($k = 0$) and the twisted sectors ($k \neq 0$). In the following, we discuss both cases separately.

Twisted sector

Strings in the twisted sector are subject to the modified boundary conditions (2.31). Thus, massless twisted strings are localized at fixed points corresponding to some *constructing element* $g = (\theta^k, n_i e_i)$ of S. Similar to the way we introduced the local shift V_g it is now useful to define the local twist

$$v_g = kv \,, \tag{2.34}$$

which denotes the twist acting on any string localized at the fixed point with constructing element g. As a consequence of the twisted boundary conditions the right- and left-moving momenta are shifted by v_g and V_g, respectively. The new masslessness conditions can then be written as

$$\frac{M_{\mathrm{R}}^2}{8} \stackrel{!}{=} 0 = \frac{q_{\mathrm{sh}}^2}{2} + \delta c - \frac{1}{2} \,, \tag{2.35a}$$

$$\frac{M_{\mathrm{L}}^2}{8} \stackrel{!}{=} 0 = \frac{p_{\mathrm{sh}}^2}{2} + \tilde{N} + \delta c - 1 \,, \tag{2.35b}$$

where we have defined the shifted momenta as

$$q_{\mathrm{sh}} := q + v_g \,, \qquad p_{\mathrm{sh}} := p + V_g \,. \tag{2.36}$$

Note another important difference to the untwisted mass operators (2.18): The presence of twisted oscillators in the mode expansions of twisted strings leads to a shift in the zero-point energies of both right- and left-moving sector, given by

$$\delta c = \sum_{i=1}^{3} v_g^i \left(1 - v_g^i\right) \,, \tag{2.37}$$

for $0 \leq v_g^i < 1$.

In order to determine the Hilbert space of physical states we have to consider how a closed string state $|q_{\rm sh}\rangle_{\rm R} \otimes \tilde{\alpha}|p_{\rm sh}\rangle_{\rm L}$ with constructing element $g \in S$ transforms under some element $h \in S$. It turns out that in case of commuting space group elements, which is given throughout this thesis, the state transforms with a phase, i.e.

$$|q_{\rm sh}\rangle_{\rm R} \otimes \tilde{\alpha}|p_{\rm sh}\rangle_{\rm L} \xrightarrow{h} e^{2\pi{\rm i}\left(p_{\rm sh}^I V_h^I - R^i v_h^i\right)}|q_{\rm sh}\rangle_{\rm R} \otimes \tilde{\alpha}|p_{\rm sh}\rangle_{\rm L}\,, \tag{2.38}$$

where we have introduced the so-called R charge

$$R^i = q_{\rm sh}^i - \widetilde{N}^i + \widetilde{N}^{*\,i}\,. \tag{2.39}$$

The combination of oscillators and shifted momenta is chosen in such a way that R^i is invariant under picture changing [27]. Since physical states have to transform trivially under h, it follows from (2.38) the projection condition

$$p_{\rm sh}^I V_h^I - R^i v_h^i \stackrel{!}{=} 0 \bmod 1\,. \tag{2.40}$$

One can show that the twisted sector cannot contribute gauge group factors to the compactified spectrum. In fact, the only states which survive (2.40) form matter representations in the form of four-dimensional $\mathcal{N} = 1$ chiral multiplets.

Untwisted sector

Strings in the untwisted sector fulfill the trivial boundary conditions (2.3). Hence the solutions to the masslessness conditions (2.18) are simply given by the ten-dimensional spectrum of the heterotic string. However, to obtain the correct massless physical spectrum on the orbifold we still have to project out those states which are not invariant under the orbifold action.

From the transformation behavior of the operators and states under the twist θ, given by

$$|q\rangle \longrightarrow e^{-2\pi{\rm i}q^i v_i}|q\rangle\,, \tag{2.41a}$$

$$\tilde{\alpha}_n^i \longrightarrow e^{2\pi{\rm i}v_i}\tilde{\alpha}_n^i, \tag{2.41b}$$

$$|p\rangle \longrightarrow e^{2\pi{\rm i}p^I V_I}|p\rangle\,, \tag{2.41c}$$

and knowing that physical states must transform trivially, we draw a number of conclusions. First, from the ten-dimensional supergravity multiplet (2.20a), only an $\mathcal{N} = 1$, $d = 4$ supergravity multiplet and untwisted moduli like the Kähler and complex structure moduli survive. Second, from the 16 Cartan generators (2.20b) we get 16 four-dimensional vector multiplets. Thus, the rank of the gauge group has not been reduced. Last but not least, the 480 generators of $E_8 \times E_8$ from (2.20c) transform under $g \in S$ as follows,

$$|q\rangle_{\rm R} \otimes |p\rangle_{\rm L} \xrightarrow{g} e^{2\pi{\rm i}\left(p_I V_g^I - q_i v_g^i\right)}|q\rangle_{\rm R} \otimes |p\rangle_{\rm L}\,. \tag{2.42}$$

Thus, if $q_i v_g^i = 0$, the left-moving momentum must satisfy

$$p_I V_g^I \stackrel{!}{=} 0 \bmod 1\,. \tag{2.43}$$

The surviving p are the roots of the four-dimensional gauge group. As stated in our previous discussion of the twisted sector, these are in fact the only states contributing to the gauge group. In the case $q_i v_g^i \neq 0$, the condition reads

$$p_I V_g^I - q_i v_g^i \stackrel{!}{=} 0 \bmod 1\,, \tag{2.44}$$

in analogy to (2.40). The set of invariant states arising from (2.44) give rise to untwisted matter transforming in representations specified by their weights p.

Modular invariance and other constraints

The gauge embedding of the twist, i.e. the shift V and Wilson lines W_i, has to satisfy a number of consistency conditions.

For a $\mathbb{Z}_N$ orbifold, acting N times with an orbifold group element on the gauge degrees of freedom and demanding that $\theta^N = \mathbb{1}$ leads to a constraint on the shift vector,

$$NV \in \Gamma_{16}\,. \tag{2.45}$$

Thus, a twist of order N is translated into a shift of the same order in the gauge sector. A similar rule can be derived for Wilson lines. If N_i denotes the order of the orbifold action in the i-th subtorus, acting N_i times on the coordinates Z^a and X_{L}^I with $g = (\theta, e_i; V, W_i) \in O$ results in the condition that W_i has to be of order N_i, i.e.

$$N_i W_i \in \Gamma_{16}\,. \tag{2.46}$$

A number of additional constraints are imposed by modular invariance of one-loop string amplitudes. One-loop diagrams for closed strings are simply tori, parameterized by a modulus τ. The torus modulus is invariant under $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ transformations of the form

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{2.47}$$

generated by $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$. Therefore, the string amplitude itself has to be invariant under (2.47), which for $\mathbb{Z}_N$ orbifolds leads to the consistency conditions

$$N(V^2 - v^2) = 0 \bmod 2\,, \tag{2.48a}$$

$$N_i(W_i \cdot V) = 0 \bmod 2\,, \tag{2.48b}$$

$$N_i(W_i \cdot W_i) = 0 \bmod 2\,, \tag{2.48c}$$

where as before N_i denotes the order of W_i.

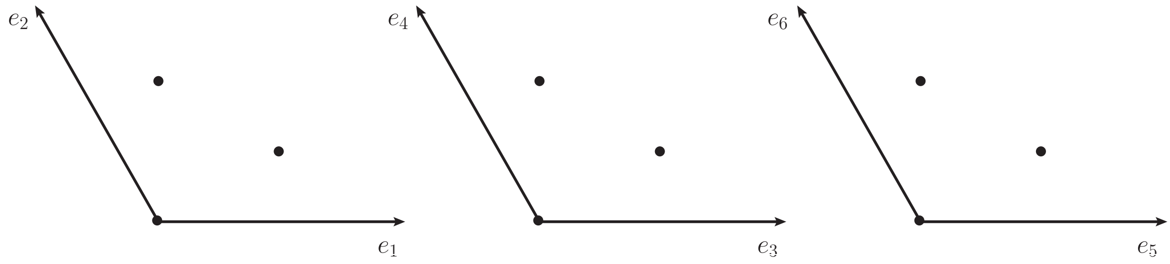


Figure 2.1: *The factorized $SU(3)^3$ torus lattice of the $\mathbb{Z}_3$ orbifold, spanned by the basis vectors $e_1 = e_3 = e_5 = (1, 0)$ and $e_2 = e_4 = e_6 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The black dots mark the $3^3 = 27$ fixed points.*

2.2.3 Example: The $T^6/\mathbb{Z}_3$ Orbifold

Equipped with the previously discussed techniques we can now proceed to study the simplest example of an orbifold which admits $\mathcal{N} = 1$ supersymmetry in four dimensions, $T^6/\mathbb{Z}_3$. The geometry of the $\mathbb{Z}_3$ orbifold has been discussed in great detail in various works [5–7,28][1]. Hence, we merely summarize the most important results for the simple case of *standard embedding* in absence of Wilson lines. We will use exactly this orbifold compactification as starting point for a blow-up model with non-universal anomalies in Chapter 5.

Let us first specify what is meant by *standard embedding*. We choose the six-dimensional torus lattice to be factorized as

$$\Gamma = \Gamma_{SU(3)} \times \Gamma_{SU(3)} \times \Gamma_{SU(3)} , \tag{2.49}$$

where $\Gamma_{SU(3)}$ denotes the root lattice of $SU(3)$. Each sublattice lies in one of the three complex planes of the tori. This is schematically depicted in Figure 2.1. In this case the orbifold twist θ is realized by the twist vector

$$v = \frac{1}{3}(1, 1, -2) . \tag{2.50}$$

The term standard embedding in this context means that the twist is embedded into the gauge group in the simplest non-trivial way,

$$V = \frac{1}{3}\left(1, 1, -2, 0^5\right)\left(0^8\right) . \tag{2.51}$$

Following the procedure discussed in the previous sections one can now verify that the shift breaks the gauge group as

$$E_8 \times E_8 \longrightarrow E_6 \times SU(3) \times E_8. \tag{2.52}$$

We remark that in $E_8 \times E_8$ heterotic string theory there are in fact five inequivalent shift vectors which satisfy the modular invariance conditions (2.48) [22]. Thus, in the absence of Wilson lines there are five different $\mathbb{Z}_3$ orbifolds whose gauge groups are listed in Table 2.1.

[1]For a detailed account of the $SO(32)$ heterotic string on $\mathbb{Z}_3$ orbifolds, see [29].

Shift vector	Gauge group
$\left(0^8\right)\left(0^8\right)$	$E_8 \times E_8$
$\frac{1}{3}\left(1,1,-2,0^5\right)\left(0^8\right)$	$E_6 \times SU(3) \times E_8$
$\frac{1}{3}\left(1,1,-2,0^5\right)\left(1,1,-2,0^5\right)$	$E_6 \times SU(3) \times E_6 \times SU(3)$
$\frac{1}{3}\left(1^2,0^6\right)\left(-2,0^7\right)$	$E_7 \times SO(14) \times U(1)^2$
$\frac{1}{3}\left(-2,1^4,0^3\right)\left(-2,0^7\right)$	$SU(9) \times SO(14) \times U(1)$

Table 2.1: *All modular invariant inequivalent shift vectors for the $\mathbb{Z}_3$ orbifold with resulting gauge group.*

Returning to the simple case of the standard embedding, we perform the analysis outlined in Section 2.2.2 to determine the massless chiral spectrum in the remaining four-dimensional theory. It turns out that in the untwisted sector, the only matter representations which survive the projection conditions are three copies of $\left(\mathbf{27},\overline{\mathbf{3}},\mathbf{1}\right)$. A similar analysis in the twisted sector reveals that at each of the 27 fixed points there are localized matter representations $(\mathbf{27},\mathbf{1},\mathbf{1})$ and $3\cdot(\mathbf{1},\mathbf{3},\mathbf{1})$. Thus, the full massless chiral spectrum reads

$$3\left(\mathbf{27},\overline{\mathbf{3}},\mathbf{1}\right)+27\left[\,(\mathbf{27},\mathbf{1},\mathbf{1})+3\,(\mathbf{1},\mathbf{3},\mathbf{1})\,\right]. \tag{2.53}$$

This completes our discussion of the $\mathbb{Z}_3$ orbifold for now. However, as already mentioned we will come back to this particular example in Chapter 5.

Chapter 3

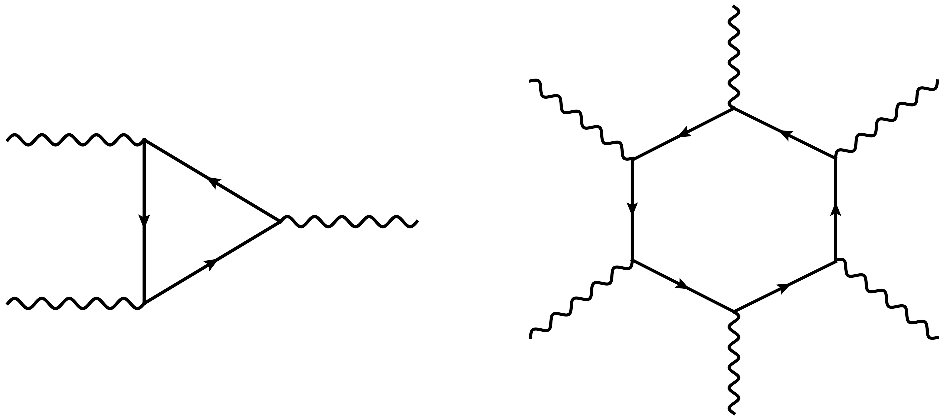
Anomaly Cancellation

Anomalies arise when a classical symmetry is not preserved in the process of quantization. If the anomalous symmetry is a gauge symmetry, the theory is rendered inconsistent. Therefore, anomaly cancellation is of crucial importance in the construction of consistent string models. The conditions for omalous[1] theories in ten dimensions have been studied by Green and Schwarz [4] for the first time. Criteria for discrete symmetries have been investigated in [30,31]. The computation of anomaly coefficients can be performed in various ways: Using a simple Feynman diagram approach, calculating the anomalous variation of the path integral measure via a method developed in [32, 33], or via the anomaly polynomial [34, 35]. In this chapter we comment on these different methods and discuss the concept of anomaly polynomials in detail. Afterwards, we explore the presence and consequences of an anomalous $U(1)$ symmetry, commonly denoted by $U(1)_{\text{anom}}$, on heterotic orbifolds.

3.1 Field Theory Picture

In a d-dimensional field theory, anomalies are represented by non-vanishing $(\frac{d}{2}+1)$-sided polygon graphs, see Figure 3.1 for examples in four and ten dimensions. Since only diagrams with chiral fermions in the loop give non-vanishing contributions, anomalies only exist in even dimensions. Concretely, let us consider a theory with a gauge group which is a product of a non-Abelian part, denoted by G, and a number of $U(1)$ symmetries. We also include a number of chiral fermions f which transform in the representation $\boldsymbol{r}_f$ of G and carry $U(1)_i$ charge q_f^i. Now one can simply try to calculate the triangle diagram in Figure 3.1a. Considering only $U(1)$ gauge fields coupled to two other $U(1)$ gauge fields, two gravitons or two non-Abelian

[1] We use this term coined by Donagi for anomaly-free theories. We also remark that *anomalous*, stemming from the greek word *homalos* (= even, smooth) is in no way related to *abnormal* (= against the rule).



(a) (b)

Figure 3.1: *Anomaly graphs in four (a) and ten (b) dimensions.*

gauge fields, we obtain for the $U(1)$ current J_i

$$\partial_\mu J_i^\mu \sim A_{G^2-U(1)_i} \operatorname{tr} F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{1}{s_{ijk}} A_{U(1)^3_{ijk}} F_{j,\mu\nu}\widetilde{F}_k^{\mu\nu} + \frac{1}{24} A_{\mathrm{grav}^2-U(1)_i} \operatorname{tr} R_{\mu\nu}\widetilde{R}^{\mu\nu}\,. \tag{3.1}$$

Here, $F_{\mu\nu}$, $F_{i,\mu\nu}$, and $R_{\mu\nu}$ denote the field strengths of G, $U(1)_i$, and the Lorentz group, respectively. The symmetry factor s_{ijk} satisfies

$$s_{iii} = 3!\,, \quad s_{iij} = 2!\,, \quad s_{ijk} = 1!\,, \tag{3.2}$$

and takes into account permutations of the legs in the triangle diagram. The anomaly coefficients A are found to be

$$A_{G^2-U(1)} = \sum_f q_f \ell(\boldsymbol{r}_f)\,, \qquad A_{\mathrm{grav}^2-U(1)} = \sum_m q_m\,, \qquad A_{U(1)^3_{ijk}} = \sum_m q_m^i q_m^j q_m^k\,, \tag{3.3}$$

where the first sum runs over all fermions f in $\boldsymbol{r}_f$ and $\ell(\boldsymbol{r}_f)$ denotes the Dynkin index of $\boldsymbol{r}_f$. All these anomaly coefficients have to vanish for the theory to be omalous.

For discrete symmetries, especially $\mathbb{Z}_N$ symmetries, instead of $U(1)$'s the anomaly coefficients are of the same form as (3.3) but only have to vanish mod N (or mod $\frac{N}{2}$ if N is even) [30, 31]. Furthermore, it has been argued in various places [31, 36] that quadratic and cubic discrete anomalies are not relevant for the discussion.

3.2 Anomaly Polynomial and Green–Schwarz Mechanism

A more convenient way to discuss anomalies and their cancellation is via the so-called *anomaly polynomial*. Whenever a current like (3.1) is not conserved, the path integral measure transforms non-trivially, i.e.

$$\int \mathcal{D}\Psi e^{\mathrm{i}S} \longrightarrow \int \mathcal{D}\Psi e^{\mathrm{i}\mathcal{A}} e^{\mathrm{i}S}\,. \tag{3.4}$$

Thus, the effective action Γ_S appears non-invariant under the considered gauge or local Lorentz transformation with parameter λ or Θ, respectively:

$$\Gamma_S \longrightarrow \Gamma_S + \mathcal{A}(\lambda, \Theta)\ . \tag{3.5}$$

As discussed before, the anomaly $\mathcal{A}$ only arises in the quantum theory and could be calculated using Feynman diagrams or a regularization method developed by Fujikawa [32, 33]. In a d-dimensional theory the anomaly can be expressed as an integral of a d-form,

$$\mathcal{A} = \int I_d^{(1)}\ . \tag{3.6}$$

It has been argued [37,38] that the non-trivial information about the anomaly can be succinctly given in terms of a (formal) closed and gauge-invariant $(d+2)$-form I_{d+2}, which is related to I_d via the Stora–Zumino descent equations

$$I_{d+2} = \mathrm{d}I_{d+1}^{(0)}\,, \qquad \delta_\lambda I_{d+1}^{(0)} = \mathrm{d}I_d^{(1)}\ . \tag{3.7}$$

I_{d+2} is called *anomaly polynomial* and $I_{d+1}^{(0)}$ is the Chern–Simons form.

Let us now specialize to the case of interest, which is ten-dimensional $\mathcal{N} = 1$ supergravity coupled to super Yang–Mills theory, as a low energy effective theory of the $E_8 \times E_8$ heterotic string. Green and Schwarz discovered [4, 39] that only reducible anomalies can be canceled, i.e. I_{12} has to factorize into an eight-form and a four-form according to

$$I_{12} = Y_8 X_4\ . \tag{3.8}$$

Taking anomalous contributions from the gravitino, the dilatino, and the gauginos into account, one obtains for Y_8 and X_4,

$$Y_8 = \frac{1}{8}\operatorname{tr}\mathfrak{R}^4 + \frac{1}{32}\left(\operatorname{tr}\mathfrak{R}^2\right)^2 - \frac{1}{8}\operatorname{tr}\mathfrak{R}^2\operatorname{tr}\mathfrak{F}^2 + \frac{1}{24}\operatorname{tr}\mathfrak{F}^4 - \frac{1}{8}\left(\operatorname{tr}\mathfrak{F}^2\right)^2\,, \tag{3.9}$$

$$X_4 = \operatorname{tr}\mathfrak{R}^2 - \operatorname{tr}\mathfrak{F}^2\,, \tag{3.10}$$

where $\mathfrak{R}$ and $\mathfrak{F}$ denote the ten-dimensional Riemann and field strength tensors, respectively[2]. The field strength tensors are constrained to be the ones of $E_8 \times E_8$ or $SO(32)$. In the case at hand, the anomaly is canceled by the non-trivial variation of the antisymmetric two-form B_2 from the supergravity multiplet (2.21), given by

$$\delta B_2 = \operatorname{tr}\Theta\mathrm{d}\mathfrak{W} - \operatorname{tr}\lambda\mathrm{d}\mathfrak{A}\,, \tag{3.11}$$

where $\mathfrak{W}$ and $\mathfrak{A}$ denote spin and gauge connection, respectively. In particular, the anomalous variation (3.5) of the supergravity action is now canceled by the Green–Schwarz action for B_2, which is of the form

$$S_{\mathrm{GS}} = \int \frac{1}{2}H_3 \wedge *H_3 + cB_2X_4\,, \tag{3.12}$$

[2]Note that we usually omit wedge products between forms for convenience.

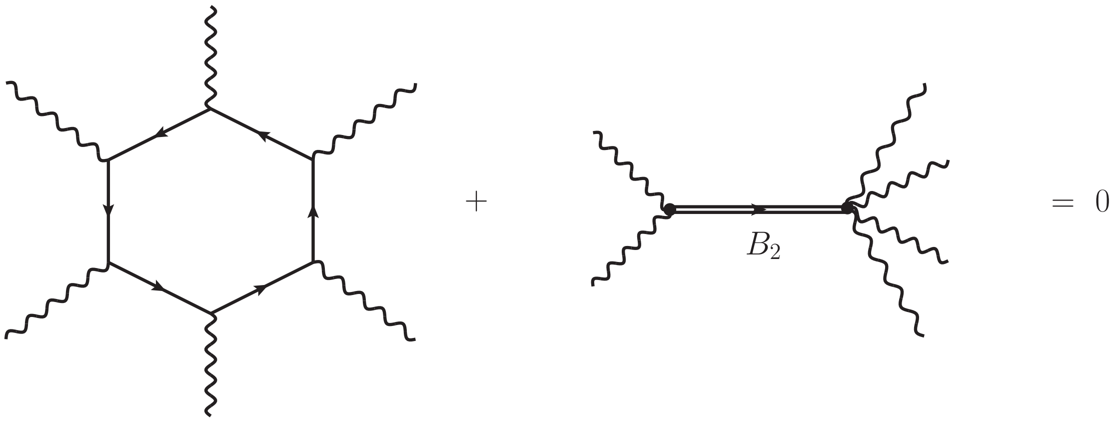


Figure 3.2: *Anomalous hexagon diagram and tree level diagram with the exchange of B_2. The anomalous variation of B_2 exactly cancels the ten-dimensional gauge anomaly.*

where the field strength of B_2 is given by

$$H_3 = \mathrm{d}B_2 + \frac{1}{c}(\omega_3^{\mathrm{YM}} - \omega_3^{\mathrm{L}}) \,. \tag{3.13}$$

Here c is a free parameter and ω_3^{YM} and ω_3^{L} denote the Chern–Simons three-forms

$$\omega_3^{\mathrm{YM}} = \mathrm{tr}\left[\mathfrak{A}\mathfrak{F} - \frac{2\mathrm{i}}{3}\mathfrak{A}^3\right] , \tag{3.14a}$$

$$\omega_3^{\mathrm{L}} = \mathrm{tr}\left[\mathfrak{W}\mathfrak{R} - \frac{2\mathrm{i}}{3}\mathfrak{W}^3\right] . \tag{3.14b}$$

This cancellation process is schematically depicted in Figure 3.2. Note that I_{12} can also contain more than one anomalous contribution, i.e.

$$I_{12} = \sum_a Y_8^a X_4^a \,, \tag{3.15}$$

in general. In this case, each of the contributions to (3.15) has to be canceled separately by variations of appropriate two-forms C_2^a.

Four-dimensional anomaly cancellation

Let us now focus on four-dimensional theories. Here, reducible anomalies factorize as $I_6 = \sum_a Y_2^a X_4^a$, and can thus again be canceled by two-forms. However, in four dimensions two-forms are dual to zero-forms (scalar fields, in particular axions) in the sense that their field strengths satisfy $*H_3 = H_1$. Thus, we can phrase the following discussion in terms of scalars only, which is much more convenient. The two-form Y_2 can only be a field strength of a $U(1)$ gauge group factor, $Y_2 = \mathrm{d}A_1$, and the $U(1)$ is anomalous if $X_4 \neq 0$. Then X_4 contains either two more $U(1)$'s or a square of a non-Abelian group (again including gravity), and by an abuse of notation we denote these as Abelian, non-Abelian and gravitational anomalies, respectively.

For heterotic string compactifications we have to distinguish between two fairly different cases: Compactifications on orbifolds and on smooth Calabi–Yau manifolds with vector bundles, including orbifold blow-ups.

In the orbifold case, the only field taking part in the cancellation process is the antisymmetric two-form b_2 which survives the dimensional reduction of B_2. Its dual, denoted by a, is known as the *model independent axion*. Since there is only one possible field we can write

$$I_6 = Y_2 X_4 = Y_2 \left(\operatorname{tr} R^2 - \sum_a c_a \operatorname{tr} F_a^2 \right) . \tag{3.16}$$

Notice that we have chosen a notation where the ten-dimensional field strengths split into internal background flux and four-dimensional field strength as $\mathfrak{F} = \mathcal{F} + F$ and $\mathfrak{R} = \mathcal{R} + R$, assuming that four-dimensional backgrounds and massless internal fluctuations are absent. The sum in (3.16) runs over all unbroken gauge group factors. The anomaly is then canceled by the four-dimensional Green–Schwarz action (3.12),

$$S_{\mathrm{GS}}^{4d} \sim \int a\, X_4 \, . \tag{3.17}$$

We observe that a couples universally to all four-dimensional gauge group factors, apart from the somewhat trivial coefficients c_a which can be determined by group theory arguments only. In Section 3.3 we will discuss how this universal coupling is related to the existence of an anomalous $U(1)$ in certain orbifold constructions. In particular, the anomalous $U(1)$ manifests itself as a shift of a.

Let us now proceed to the second case, compactification on smooth Calabi–Yau manifolds as blow-ups of heterotic orbifolds. The most important difference is given by the fact that a number of additional axions β_r enter the theory during the blow-up procedure. Specifically, we resolve the orbifold singularities by gluing in a number of *exceptional divisors* E_r. The details of this procedure are discussed in Chapter 4 and an explicit example is given in Chapter 5. For now, it suffices to argue that the Kalb–Ramond two-form B_2 can be expanded as

$$B_2 = b_2 + \alpha_i R_i - \beta_r E_r \, , \tag{3.18}$$

where R_i denote so-called *inherited divisors* which will not play a role in our example model. The transformation of the β_r under the anomalous symmetry follows from dimensional reduction of (3.11) after expansion of the internal flux $\mathcal{F}$. We will work this out explicitly in Section 5.1 during the discussion of gauge boson masses of anomalous $U(1)$'s. From the above discussion we conclude that these axions will in general not couple universally to all gauge group factors.

Let us now work out a way to determine the four-dimensional anomaly coefficients via the anomaly polynomial. Keeping in mind the Bianchi identity for the six-dimensional backgrounds (see also Section 4.3),

$$\mathrm{d}H = \operatorname{tr}\mathcal{R}^2 - \operatorname{tr}\mathcal{F}^2 = 0\,, \tag{3.19}$$

where the second equality holds only in cohomology, we can insert the decomposition of $\mathfrak{F}$ and $\mathfrak{R}$ into the ten-dimensional polynomial given by (3.8) and (3.9) and keep the relevant terms, i.e. the ones cubic in $\mathcal{F}$ and $\mathcal{R}$. The result reads

$$\begin{aligned} I_6 = \frac{1}{(2\pi)^6}\int\limits_X \Bigg[&\frac{1}{6}\operatorname{tr}\big(\mathcal{F}'F'\big)^2 + \frac{1}{4}\left(\operatorname{tr}\mathcal{F}'^2 - \frac{1}{2}\operatorname{tr}\mathcal{R}^2\right)\operatorname{tr}F'^2 \\ &-\frac{1}{8}\left(\operatorname{tr}\mathcal{F}'^2 - \frac{5}{12}\operatorname{tr}\mathcal{R}^2\right)\operatorname{tr}R^2\Bigg]\operatorname{tr}\big(\mathcal{F}'F'\big) + \big(F',\mathcal{F}' \leftrightarrow F'',\mathcal{F}''\big)\,. \end{aligned} \tag{3.20}$$

The field strengths of the first and the second E_8 are denoted by $\mathcal{F}'$ and $\mathcal{F}''$, respectively. Apparently, each E_8 factor contributes three different terms:

- $\int_X \operatorname{tr}(\mathcal{F}'F')^2 \cdot \operatorname{tr}(\mathcal{F}'F')$ gives rise to Abelian anomalies only. This is also true for bundles with non-Abelian structure group H because the generators of H, for which the trace gives a nonvanishing contribution, are broken by the bundle.
- $\int_X \big(\operatorname{tr}\mathcal{F}'^2 - \frac{1}{2}\operatorname{tr}\mathcal{R}^2\big)\operatorname{tr}F'^2 \cdot \operatorname{tr}(\mathcal{F}'F')$ gives Abelian and non-Abelian anomalies, and
- $\int_X \big(\operatorname{tr}\mathcal{F}'^2 - \frac{5}{12}\operatorname{tr}\mathcal{R}^2\big)\operatorname{tr}R^2 \cdot \operatorname{tr}(\mathcal{F}'F')$ contributes purely gravitational anomalies.

We observe some partial anomaly universality: The non-Abelian anomalies arising from the first E_8 are captured by one anomalous $U(1)$ factor with universal coefficients, and similar for the second E_8. Furthermore, if one E_8 is unbroken, i.e. $\mathcal{F}'' = 0$, the Bianchi identity (3.19) implies that the non-Abelian and gravitational anomalies are captured by the same $U(1)$, and their coefficients are proportional to each other.

This completes our discussion of anomaly polynomials. However, we make excessive use of the described methods in order to determine the anomaly coefficients of a $T^6/\mathbb{Z}_3$ orbifold blow-up model in Section 5.1.

3.3 Anomalous $U(1)$ on Orbifolds

In generic orbifold compactifications the four-dimensional gauge group can contain many $U(1)$ factors. In this section we briefly discuss what has been established in the literature [40–42] for some time: Although many of the $U(1)$'s may appear anomalous, one can always perform a basis change to rotate the generators in a way that only one $U(1)$ is truly anomalous, which is then denoted by $U(1)_{\text{anom}}$. We investigate how exactly this can be realized, before we discuss the four-dimensional version of the Green–Schwarz mechanism which cancels these anomalies. At the end of this section, we comment on the conditions for an anomalous $U(1)$ to be present in the first place.

Uniqueness of $U(1)_{\text{anom}}$

As mentioned above, for a set of apparently anomalous $U(1)$ factors we can always rotate the charges in such a way that only one anomalous $U(1)$ remains. Starting from a number of gauge group factors $U(1)_i$ and a state $|p_{\text{sh}}\rangle_{\text{L}}$ with charges q_i given by

$$q_i|p_{\text{sh}}\rangle_{\text{L}} = t_i^I H^I |p_{\text{sh}}\rangle_{\text{L}} = t_i^I p_{\text{sh}}^I |p_{\text{sh}}\rangle_{\text{L}} \,, \tag{3.21}$$

where H^I denote the 16 Cartan generators of $E_8 \times E_8$, we have from (3.3) potentially non-vanishing anomaly coefficients

$$A_i = \sum_f q_i^f = \sum_f t_i^I p_{\text{sh}}^{I,f} \,. \tag{3.22}$$

In [43] it has been worked out that a unique anomalous generator in a proper normalization can be constructed by

$$t_{\text{anom}} = \sum_i \frac{A_i}{t_i^I t_{i,I}} t_i = \frac{1}{12} \sum_f p_{\text{sh}}^f \,, \tag{3.23}$$

where we have used (3.21) as well as the fact that the generators t_i are chosen to be orthogonal. Using (3.22) one can prove that $U(1)_{\text{anom}}$ is indeed anomalous and all other $U(1)$'s, whose generators satisfy $\tilde{t}_i^I t_{\text{anom}}^I = 0$, are not.

Furthermore, the authors of [43] have shown that there is an anomalous space group element corresponding to t_{anom}, given by $g_{\text{anom}} = (\theta^{k_{\text{anom}}}, n_{\text{anom}}^i e^i)$. This is in fact related to discrete anomalies arising from stringy symmetries like the space group selection rule or H-momentum conservation. In particular, t_{anom} can be expressed as a superposition of the shift vector V and the Wilson lines W^i of the orbifold model,

$$t_{\text{anom}} = k_{\text{anom}} V + \sum_i n_{\text{anom}}^i W^i \,, \tag{3.24}$$

which holds up to lattice vectors of $\Gamma_{E_8 \times E_8}$.

Anomaly cancellation and universality

Since $U(1)_{\text{anom}}$ is a gauge symmetry, all anomalies have to be absent or canceled by a Green–Schwarz mechanism for the theory to be consistent. As we have seen in Section 3.2, ten-dimensional anomalies are canceled by the variation of a counterterm $\int B_2 \wedge X_8$. In four dimensions the cancellation proceeds via a shift of an axion a descending from B_2 [43]:

Consider a chiral superfield Ψ and a vector superfield V transforming under $U(1)_{\text{anom}}$ as

$$\Psi \longrightarrow e^{\mathrm{i}q_{\text{anom}}\Lambda}\Psi\,, \qquad V \longrightarrow V + \mathrm{i}(\Lambda - \Lambda^{\dagger})\,, \tag{3.25}$$

where Λ is a chiral superfield. This transformation is canceled by the variation of a complex dilaton field $S = s + \mathrm{i}a$ of the form

$$S \longrightarrow S + \mathrm{i}\delta_{\text{GS}}\Lambda\,. \tag{3.26}$$

δ_{GS} can be shown to be proportional to $A_{H^2-U(1)_{\text{anom}}}$, where H denotes Abelian, non-Abelian and gravitational gauge group factors. The exact relation between those three types of anomalies has been worked out in [42, 44] and reads

$$\delta_{\text{GS}} \sim A_{G^2-U(1)_{\text{anom}}} = \frac{1}{24}A_{\text{grav}^2-U(1)_{\text{anom}}} = \frac{1}{6|t_{\text{anom}}|^2}A_{U(1)^3_{\text{anom}}} = \frac{1}{2|t_i|^2}A_{U(1)^2_i-U(1)_{\text{anom}}}\,. \tag{3.27}$$

This *universality condition* follows from the fact that several kinds of anomalies have to be canceled by only one axion a, frequently called the model independent axion. Therefore, anomaly universality only holds for orbifold constructions. As we will see, for example in Chapter 5, on smooth Calabi–Yau manifolds there can generically be many axions, so that anomaly universality is neither required nor fulfilled in general.

Another important point which follows from (3.27) is that the Kähler potential for the dilaton S, which is generically of the form $K_S = -\ln(S + S^{\dagger})$, has to be modified to be invariant under $U(1)_{\text{anom}}$:

$$K_S \longrightarrow K'_S = -\ln\left(S + S^{\dagger} - \frac{1}{2}\delta_{\text{GS}}V\right)\,, \tag{3.28}$$

where V denotes the vector multiplet of $U(1)_{\text{anom}}$. The modified Kähler potential (3.28) contains a D-term of the schematic form

$$[K_S]_D \sim \operatorname{tr} q_{\text{anom}} \sim A_{\text{grav}^2-U(1)_{\text{anom}}}\,, \tag{3.29}$$

where $\operatorname{tr} q_{\text{anom}} = \sum_f q^f_{\text{anom}}$. With (3.27) it follows that the presence of an anomalous $U(1)$ symmetry automatically induces a non-vanishing Fayet–Iliopoulos (FI) D-term. The interpretation of this FI term and its connection to supersymmetry breaking have been discussed in various places in the literature [41, 43].

Presence of $U(1)_{\text{anom}}$

In the literature, many models with an anomalous $U(1)$ have been known for some time. However, until the authors of [40] were able to derive general conditions on the appearance of $U(1)_{\text{anom}}$ the detailed spectrum and $U(1)$ charges of each model had to be analyzed to determine whether an anomalous $U(1)$ was present or not. These conditions can be summarized as follows.

- Any orbifold compactification with an unbroken E_8 gauge group factor has no anomalous $U(1)$. This is true for the following reason: From the universality condition (3.27) it follows that if a $U(1)$ symmetry has no mixed $U(1) - G^2$ anomaly for a certain G, it has no anomaly of any kind. Thus, it suffices to check for those anomalies which are easiest to calculate, namely the mixed anomaly with the largest gauge group. Since the masslessness conditions (2.35) forbid E_8 matter fields, mixed $U(1) - E_8^2$ anomalies always vanish.

- Even for models with two broken E_8 factors – labeled E_8' and E_8'' – which contain $U(1)$ symmetries, the presence of $U(1)_{\text{anom}}$ is not guaranteed. In particular, if there is no mixing between E_8' and E_8'' for massless twisted matter there is no anomalous $U(1)$. In this context, absence of mixing means that no massless field transforming under an unbroken subgroup of E_8'' is charged under any $U(1)$ descending from E_8' and vice versa.

- Another condition is related to the presence of Wilson lines and is therefore of less importance to the remainder of this thesis: An omalous $U(1)$ which survives the breaking of E_8 by a shift V will not become anomalous when switching on Wilson lines orthogonal to V. This is due to the fact that in case of orthogonal shift and Wilson lines there exists a certain discrete symmetry which cancels potential anomalies of the $U(1)$.

Apparently, these rules impose strong restrictions on the presence of $U(1)_{\text{anom}}$. Again, we stress the fact that these rules only hold for orbifold compactifications. On smooth Calabi–Yau manifolds some of the contributing assumptions – such as anomaly universality – have to be dropped so that other methods have to be invoked to classify models with anomalous U(1) symmetries.

Chapter 4

Blow-Up of Heterotic Orbifolds

In Chapter 2 it was argued that apart from orbifolds also smooth Calabi–Yau manifolds provide viable compactification spaces. We also realized that orbifolds, while providing a useful background for explicit string computations, exhibit curvature singularities. The blow-up procedure described in this chapter resolves those singularities, and hence provides a conceptual bridge between orbifolds and smooth Calabi–Yau spaces. This idea is schematically depicted in Figure 4.1.

We start this chapter by reviewing the most important properties of generic Calabi–Yau manifolds. To that end, we introduce important quantities like the Kähler form and concepts like intersection numbers of codimension-one submanifolds. We then very briefly discuss toric geometry[1] as a useful tool to investigate blow-ups of heterotic orbifolds, and subsequently describe the resolution procedure itself. Since in this thesis we only use the blow-up procedure as a means to investigate anomalies on smooth Calabi–Yau manifolds, we do not cover the full depth of this mechanism. The interested reader is referred to [8–11, 47–49] which discuss this topic in some detail.

There is another point worth stressing: While toric geometry provides helpful mathematical tools to study blow-ups, there is a perhaps more intuitive picture in field theory: From Chapter 3 we know that the existence of an anomalous $U(1)$ on the orbifold implies the appearance of certain Fayet–Iliopoulos terms. In order to preserve $\mathcal{N} = 1$ supersymmetry these terms have to be canceled. Thus, certain twisted fields charged under $U(1)_{\text{anom}}$ have to attain a vev. When giving a vev to these so-called *blow-up modes* the corresponding fixed points are smoothed out and the curvature singularities are resolved. It turns out that the bundle vectors which characterize the gauge flux are the weight vectors of the blow-up modes. In Chapter 5 we will use both toric geometry and the above field theory picture to describe a blow-up model of the $\mathbb{Z}_3$ orbifold.

[1]For a thorough treatment, see [45, 46].

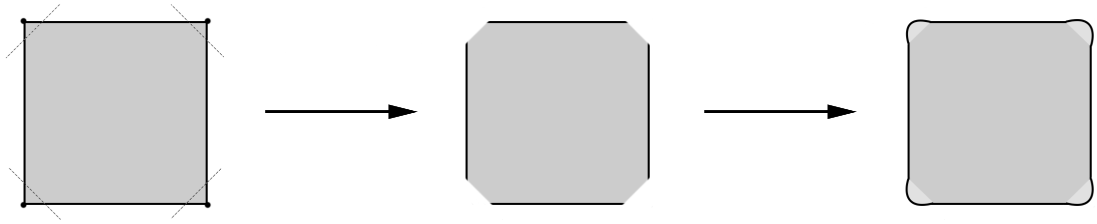


Figure 4.1: *To obtain a blown-up orbifold, we cut out the neighborhoods around curvature singularities and replace them with smooth hypersurfaces.*

4.1 Smooth Calabi–Yau Manifolds

As mentioned earlier, in order to arrive at a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry via compactification on a compact, six-dimensional smooth manifold X, there must be exactly one covariantly constant spinor on X. One can show that this leads to the condition that X must admit a metric of $SU(3)$ holonomy. It was conjectured by Calabi [50] and later proved by Yau [51] that any Kähler manifold[2] with vanishing first Chern class $c_1 := \frac{1}{2\pi}\,\mathrm{tr}\,\mathcal{R}$, where

$$\mathcal{R} = R_{i\overline{\jmath}}\mathrm{d}z^i\mathrm{d}\overline{z}^{\overline{\jmath}} \tag{4.1}$$

is the curvature two-form, admits such a metric. Compact Kähler manifolds with $c_1(X) = 0$ are thus called *Calabi–Yau manifolds.* Note that the requirement of vanishing first Chern class is equivalent to the following statements:

1. The Ricci tensor on X vanishes.
2. The metric on X has holonomy group $SU(3)$.
3. X admits a globally defined and nowhere vanishing holomorphic three-form.
4. X has a trivial canonical bundle.

Let us briefly comment on some of the equivalences of these statements. The equivalence of $c_1(X) = 0$ to the first statement is highly non-trivial and was covered in Yau's proof. The equivalence of the first two statements can be proved by arguing that for any Kähler metric the holonomy group must be contained in $U(3)$ (see, for example [52]), and that the condition of vanishing Ricci-form eliminates a $U(1)$ factor within this $U(3)$. Thus, the holonomy group must be contained in $SU(3)$. The converse can be proved in a similar fashion. Furthermore, the

[2] A Kähler manifold is a complex Hermitian manifold whose Kähler form J is closed, i.e. $\mathrm{d}J = 0$.

equivalence of the last two statements can be shown by interpreting forms on complex manifolds as sections of certain vector bundles: The canonical bundle K_M of a complex manifold M of real dimension $2n$ is a complex vector bundle, sections of which are $(n, 0)$-forms. Since K_M is a holomorphic line bundle, triviality implies $K_M = M \times \mathbb{C}$. Thus, there is a nowhere vanishing $(m, 0)$-form corresponding to the unit section $M \times \{1\}$, namely the unique constant function 1. Again, the converse of this statement can be proved analogously.

The investigation of Calabi–Yau manifolds and their application as string compactification spaces has been a highly discussed subject for many years. However, a thorough treatment of this incredibly rich topic is beyond the scope of this thesis. We restrict ourselves to reviewing the most important properties and quantities describing Calabi–Yau manifolds. For details, we recommend [52–55].

Hodge diamond and Euler characteristic

Let us denote the covariantly constant spinor on X by η. Being covariantly constant, it satisfies $D_i\eta = 0$, which implies that

$$R_{ijkl}\Gamma^{kl}\eta = 0 \quad \Rightarrow \quad \Gamma^k R_{ik}\eta = 0\,, \tag{4.2}$$

where R_{ijkl} and R_{ik} denote Riemann and Ricci tensor, respectively[3]. From (4.2) it follows that $R_{ik} = 0$, and hence X is Ricci-flat. Using the fact that any tensor field constructed from products of η with itself is again covariantly constant, we can now introduce a number of basic forms on X: The Kähler form $J_{ij} = \overline{\eta}\Gamma_{ij}\eta$, which is closed since any Calabi–Yau manifold is also a Kähler manifold, the complex structure $I^i_j = g^{ik}J_{kj}$, and finally the aforementioned holomorphic three-form

$$\Omega_{ijk} = \eta^T\Gamma_{ijk}\eta\,. \tag{4.3}$$

As mentioned before, it can be shown that a Kähler manifold has $c_1 = 0$ if and only if there is a unique and nowhere vanishing holomorphic (3,0)-form Ω, thus $h^{(3,0)} = 1$. In addition, it is easy to verify that for any Calabi–Yau manifold $h^{p,0} = h^{3-p,0}$ and thus $h^{0,0} = 1$. Using this and a number of other properties the Hodge numbers of any Calabi–Yau manifold can be

[3] This result can be obtained using curvature and gamma matrix identities.

written as a Hodge diamond,

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{4.4}$$

We observe that the Dolbeault cohomology of X is completely determined by $h^{1,1}$ and $h^{2,1}$, which give the number of Kähler moduli and complex structure moduli, respectively. Using this result, we can express the Euler characteristic of X as

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q} = 2(h^{1,1} - h^{2,1}) \,. \tag{4.5}$$

Note that for compactifications with standard embedding, i.e. in cases where the spin connection is identified with the gauge connection, the number of chiral families in four dimensions is given by $\frac{\chi}{2}$.

Divisors and intersections

In the following chapters we will frequently make use of special complex codimension-one submanifolds of X called *divisors*, denoted by S_i. These divisors are homology four-cycles, which by Poincaré duality correspond to cohomolgy (1,1)-forms. The number of independent S_i is therefore given by $h^{1,1}$. By abuse of notation we denote the form and cycle by the same symbol S_i. Divisors and their intersections encode important topological information of the Calabi–Yau manifold, which can be accessed by rather simple means. The intersection number of three divisors is defined as

$$\mathrm{Int}(S_i S_j S_k) = \int_X S_i \wedge S_j \wedge S_k = \int_{S_i S_j S_k} 1 \,. \tag{4.6}$$

Notice that Poincaré duality was used in the second equality. As will become clear in Chapter 5, eq. (4.6) will be of great importance when investigating blow-ups of orbifolds. We will use intersection numbers to calculate the volumes of X, divisors S and cycles C, given by

$$\mathrm{vol}(X) = \frac{1}{3!} \int_X J \wedge J \wedge J \,, \quad \mathrm{vol}(S) = \frac{1}{2} \int_S J \wedge J \,, \quad \mathrm{vol}(C) = \int_C J \,. \tag{4.7}$$

Since $\mathrm{d}J = 0$ as discussed before, the Kähler form defines a non-trivial equivalence class in $H^{1,1}$. We can therefore choose a basis of divisors and expand J as follows,

$$J = \sum_i^{h^{1,1}} t_i S_i \,, \tag{4.8}$$

where t_i denote the Kähler moduli. They correspond to scalar fields in the four-dimensional low energy effective theory.

4.2 Toric Geometry and Resolution of Orbifold Singularities

Toric geometry provides useful concepts to study blow-ups of orbifolds. The principle idea is to employ a new set of coordinates on so-called *toric varieties* which allow us to describe a singularity of the type $\mathbb{C}^n/\mathbb{Z}_N$ and replace it with smooth hypersurfaces. These smooth spaces turn out to be the divisors introduced in Section 4.1. Hence, it is necessary to look at each orbifold fixed point separately and then glue the resulting spaces together in order to obtain blow-ups of compact orbifolds like $T^{2n}/\mathbb{Z}_N$. Note that we do not wish to explore the mathematical details in great depth. We follow the description in [11, 46, 47].

Toric varieties and toric diagrams

A toric variety can be thought of as a generalization of a complex projective space. It consists of a set of homogeneous coordinates $z_1, \ldots, z_n$ as well as a set of k projective relations of the form

$$(z_1, \ldots, z_n) \sim \left(\lambda_j^{\alpha_1} z_1, \ldots, \lambda_j^{\alpha_n} z_n\right) , \qquad \text{for } j = 1 \ldots k \,, \tag{4.9}$$

where $\lambda_j \in \mathbb{C}^*$. Since the aim is to mod out a discrete symmetry we need to specify collections of coordinates which are not allowed to vanish simultaneously, specified by the exclusion set Z. One then defines a toric variety $\mathcal{T}$ of dimension $d = n - k$ as

$$\mathcal{T} = \frac{\mathbb{C}^n - Z}{(\mathbb{C}^*)^k} \,. \tag{4.10}$$

Notice the similarity to the definition of the complex projective space $\mathbb{CP}^n$, in which case the exclusion set is simply given by the origin of $\mathbb{C}^{n+1}$:

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}}{\mathbb{C}^*} \,. \tag{4.11}$$

A very useful geometrical tool in toric geometry is the *toric diagram*. It is given by an n-dimensional lattice isomorphic to $\mathbb{Z}_n$ which is triangulated into a set of cones. Each of

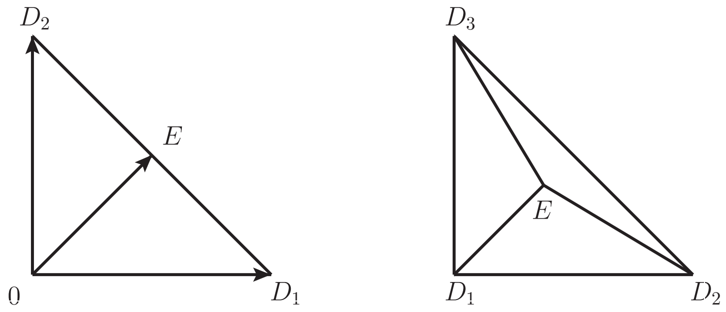


Figure 4.2: *Toric diagram of Res($\mathbb{C}^2/\mathbb{Z}_2$) and the two-dimensional projection of the toric diagram of Res($\mathbb{C}^3/\mathbb{Z}_3$). There is one ordinary divisor for each complex plane and one exceptional divisor for each fixed point. Note that the diagram of Res($\mathbb{C}^3/\mathbb{Z}_3$) depicts the only possible triangulation.*

the lattice vectors is associated with one of the homogeneous coordinates z_i. The projective relations (4.9) then translate to a set of k linear relations

$$\sum_i \alpha_i v_i = 0\,, \tag{4.12}$$

where v_i denote the lattice vectors and $\alpha_i \in \mathbb{R}^k$ are linearly independent vectors. As an example, the toric diagrams for the resolved singularities Res($\mathbb{C}^2/\mathbb{Z}_2$) and Res($\mathbb{C}^3/\mathbb{Z}_3$) are depicted in Figure 4.2. A very instructive derivation can be found in [11, 47].

Divisors and intersections: A second glance

We have already encountered divisors as complex codimension-one submanifolds of generic Calabi–Yau manifolds. Let us now investigate their connection to toric geometry. We start by defining *ordinary divisors*, labeled D_i, as the zero loci of the homogeneous coordinates of a toric variety,

$$D_i = \{z_i = 0\}\,. \tag{4.13}$$

As explained before, they are $n-2$ cycles which, by Poincaré duality, correspond to (1,1)-forms.

Each ordinary divisor D_j is associated with one vector v_j in the toric diagram. However, in many cases the set of ordinary divisors is not sufficient to resolve a given singularity. One therefore needs another class of divisors, called *exceptional divisors* E_r, which are associated with new homogeneous coordinates x_r. Exceptional and ordinary divisors can then be brought together in the toric diagram, which can in turn be used to determine the intersection numbers (4.6) after choosing a particular triangulation. Note that the case of $\mathbb{C}^3/\mathbb{Z}_3$ is in this sense particularly simple, since there exists only one triangulation. Details are given in Appendix A.

Resolutions of compact orbifolds are more complicated to describe than the previously discussed non-compact case. The non-compact resolutions have to be glued together in an appropriate way to form the correct compact resolution space. For example, we need 27 copies of the variety $\text{Res}(\mathbb{C}^3/\mathbb{Z}_3)$ to describe a blow-up of the $T^6/\mathbb{Z}_3$ orbifold. In this process another class of divisors is introduced, called *inherited divisors* R_k. They extend the toric diagram to so-called *auxiliary polyhedra*. The exact form of the gluing is encoded in certain equivalence relations between ordinary, exceptional, and inherited divisors. The most important aspects of this gluing procedure are summarized in Appendix A.2. However, for details we refer to the literature mentioned above. Furthermore, one can show that on the resolved orbifold the inherited and exceptional divisors form a basis of (1,1)-forms. Thus, the expansion of the Kähler form (4.8) becomes

$$J = \sum_k a_k R_k - \sum_r b_r E_r \,. \tag{4.14}$$

From the previous discussion follows that a_k and b_r are the Kähler moduli describing the size of the tori and blow-up cycles, respectively. Therefore, the relevant volumes on a blow-up space (4.7) can be determined via intersection numbers of divisors, deduced from toric diagrams or auxiliary polyhedra.

Another important fact which will be exploited during the analysis in Chapter 5 is the following. The total Chern class of $\mathcal{T}$, which is usually defined via the curvature two-form $\mathcal{R}$,

$$c(\mathcal{T}) = \det\left(1 + \frac{\mathcal{R}}{2\pi \mathrm{i}}\right) , \tag{4.15}$$

can be expressed in terms of divisors and products of divisors. From this follows, for instance, that the second Chern class $c_2(\mathcal{T})$ is given by

$$c_2(\mathcal{T}) = \frac{1}{2}\sum_S (c_1(\mathcal{T}) - S_i)S_i = -\frac{1}{2}\sum_S S_i^2 \,, \tag{4.16}$$

where S stands for all classes of divisors and the sum runs over all divisors. Note that the second equality only holds if $\mathcal{T}$ is Calabi–Yau.

4.3 Model Building on Resolved Orbifolds

In order to construct explicit models on blow-ups of heterotic orbifolds and to match their spectra with the ones from the underlying orbifold theory, we have to consider the low energy effective theory called *heterotic supergravity*. Since it is unknown so far how to construct the explicit metric on resolution spaces, we can not perform exact CFT calculations as in the orbifold case. During the dimensional reduction of the effective theory the gauge group breaking is achieved by wrapping gauge bundles around the resolutions. For a gauge bundle $\mathcal{F}$

with structure group H, the gauge group $G = E_8 \times E_8$ is then broken to the commutant of H in G. Since the description of gauge fluxes with non-Abelian structure group is fairly complicated, we restrict ourselves to sums of $U(1)$ line bundles. These can be expanded in terms of the Cartan subalgebra elements H_I of $E_8 \times E_8$ and exceptional divisors as

$$\mathcal{F} = E_r V_r^I H_I \,, \tag{4.17}$$

where V_r^I are called *bundle vectors*. Thus, for this choice of expansion the background flux is supported at the exceptional divisors only, which, as explained in Section 4.2 reside at the orbifold singularities. From our discussion of gauge group breaking on heterotic orbifolds in Section 2.2.2 it is quite clear that the bundle vectors V_r have to be related in some way to the local shift vectors V_g at each fixed point. In fact it turns out that [11]

$$V_{g,r}^I H_I = V_r^I H_I + \lambda \,, \qquad \lambda \in \Gamma_{E_8 \times E_8} \,. \tag{4.18}$$

This relation can be proved using Stokes' theorem over an appropriate curve.

However, in order to render a consistent theory with $\mathcal{N} = 1$ supersymmetry in four dimensions, the gauge bundle and thus the bundle vectors have to satisfy a number of additional consistency conditions. For instance, the gauge flux has to be quantized properly, which means it has to lie in the $E_8 \times E_8$ root lattice when integrated over any compact curve. Furthermore, it has to satisfy the *Bianchi identities* and the *Donaldson–Uhlenbeck–Yau* (DUY) equations, both of which will be discussed in the following.

Bianchi identities

We have briefly mentioned the Bianchi identities as consistency conditions in the low energy effective theory in Chapter 3. The field strength H_3 of the Kalb–Ramond two-form and its exterior derivative

$$\mathrm{d}H_3 = \mathrm{tr}\,\mathcal{R}^2 - \mathrm{tr}\,\mathcal{F}^2 \,, \tag{4.19}$$

have to be globally defined as they appear in the low energy effective action, e.g. in (3.12). Since $\mathrm{d}H_3$ is exact, it follows from Stokes' theorem that its integral over any closed four-cycle C must vanish

$$0 = \int_C \mathrm{d}H_3 = \int_C (\mathrm{tr}\,\mathcal{R}^2 - \mathrm{tr}\,\mathcal{F}^2) \,. \tag{4.20}$$

This result is commonly known as *integrated Bianchi identity.* As explained before, one particular basis of four-cycles (which are dual to the two-forms) is spanned by the exceptional and inherited divisors. Thus, using (4.20) and knowledge about the relevant intersection numbers one can find conditions on the bundle vectors V_r^I. For example, in the case of a blow-up of $T^6/\mathbb{Z}_3$ one finds that the Bianchi identities read

$$V_r^2 = \sum_I V_r^I V_r^I = \frac{4}{3}\,, \tag{4.21}$$

for all bundle vectors.

Donaldson–Uhlenbeck–Yau equations

In order to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions, the gauge flux on a Calabi–Yau X has to be F-flat and D-flat. It has been proved[4] that this is equivalent to demanding

$$\mathcal{F}_{ab} = \mathcal{F}_{\bar{a}\bar{b}} = 0\,, \tag{4.22a}$$

$$\mathcal{F}_{a\bar{b}} g^{a\bar{b}} = 0\,, \tag{4.22b}$$

where g denotes the metric on X. These conditions are known as *Hermitian Yang–Mills equations*. Notice that (4.22a) indeed restricts $\mathcal{F}$ to be a (1,1)-form, and thus the ansatz in terms of exceptional divisors we made in (4.17) is justified. The DUY equations are an integrated version of the D-flatness condition (4.22b) [56]. At tree level they read [15]

$$\frac{1}{2}\int_X J \wedge J \wedge \mathcal{F} = 0\,. \tag{4.23}$$

Inserting the ansatz (4.17) for the gauge flux we obtain constraints on the bundle vectors of the form

$$\frac{1}{2}\int_X J \wedge J \wedge E_r V_r^I H_I = \sum_r \mathrm{vol}(E_r) V_r \; = 0\,, \tag{4.24}$$

where we have used (4.14) for J and assumed that $\mathrm{vol}(E_r) > 0$ in the Kähler cone $a_k \gg b_r > 0$.

Chiral spectrum

In the beginning of this chapter we mentioned that the gauge flux wrapped around the resolved singularities is responsible for the gauge group breaking from $E_8 \times E_8$ to some subgroup. Analogous to the case of orbifold compactifications, only those roots of $E_8 \times E_8$ which are orthogonal to the bundle vectors V_r survive the compactification. However, the others can still appear in the massless chiral spectrum of the four-dimensional theory as weights of matter representations. The multiplicity of these chiral multiplets can then be deduced from the gaugino

[4] See, for example, [55] for a very well explained account.

contribution to the ten-dimensional anomaly polynomial. Integrating over the resolution space yields the multiplicity operator [10],

$$N = \frac{1}{(2\pi)^3} \int_X \left(\frac{1}{6} \mathcal{F}^3 - \frac{1}{24} \operatorname{tr} \mathcal{R}^2 \cdot \mathcal{F} \right) , \tag{4.25}$$

which corresponds to a special version of the Atiyah–Singer index theorem. Acting with (4.25) on the 496 gaugino states of $E_8 \times E_8$ gives the number of each of these states in the four-dimensional chiral spectrum. Note that the integral depends on the resolution (and therefore on the particular choice of triangulation), and so does the spectrum. Anomaly freedom of the chiral spectrum is on the one hand ensured by the Bianchi identities, as they forbid the presence of non-Abelian anomalies. On the other hand, all Abelian anomalies are canceled by an appropriate Green–Schwarz mechanism as explained in Chapter 3.

Chapter 5

A $T^6/\mathbb{Z}_3$ Blow-Up Model

We are now in a position to fit the previously introduced concepts together and study non-universal anomalies on smooth Calabi–Yau manifolds by means of a blow-up model of the $\mathbb{Z}_3$ orbifold. In this chapter we construct a very simple family of models via three Abelian vector bundles which resolve the $\mathbb{Z}_3$ orbifold to a smooth CY manifold with gauge group $SO(8)\times U(1)^2\times SU(3)\times E_8$. Furthermore, we investigate potential anomalies of both $U(1)$ factors in detail and work out the spectra and anomaly coefficients of three specific configurations. We stress the fact that the axions which cancel the arising anomalies couple non-universally to the gauge group factors and the Lorentz group. Afterwards, we investigate remnant discrete symmetries, in particular R symmetries, from the orbifold and the blow-up perspective. At the end of this chapter we discuss implications of anomaly non-universality for bottom-up models with phenomenological restrictions. An explicit example of a $\mathbb{Z}_6$ non-R symmetry is given which commutes with $SU(5)$ grand unification and fulfills all phenomenological requirements while having non-universal anomaly coefficients.

5.1 Blow-Up Procedure

Let us now construct an explicit string-derived realization illustrating the arguments made in Chapter 3. We consider a model on the $T^6/\mathbb{Z}_3$ orbifold in $E_8 \times E_8$ heterotic string theory. In analogy to the example in Section 2.2.3 we choose the case of standard embedding with shift vector given by (2.51) and without Wilson lines. Thus, the unbroken gauge group is $E_6 \times SU(3) \times E_8$ and the massless chiral spectrum is given by (2.53). In particular, there is no anomalous $U(1)$ in this setup. In order to resolve the orbifold singularities and construct a smooth Calabi–Yau manifold we follow the procedure outlined in Chapter 4: The orbifold fixed points are replaced by 27 exceptional divisors E_r, where $r = 1 \dots 27$. We choose an Abelian background flux $\mathcal{F}$ which can be expanded in terms of the $E_8 \times E_8$ Cartan subalgebra and exceptional divisors according to (4.17).

The bundle vectors V_r^I are chosen to coincide with shifted momenta of twisted orbifold states. Since our primary goal is to break the E_6 and leave the $SU(3)$ factor unbroken, we take

$$V_1 = \frac{1}{3}\left(\ \ 2,\ \ 2,\ \ 2,\ \ 0, 0^4\right)\left(0^8\right)\,, \tag{5.1a}$$

$$V_2 = \frac{1}{3}\left(-1,-1,-1,\ \ 3, 0^4\right)\left(0^8\right)\,, \tag{5.1b}$$

$$V_3 = \frac{1}{3}\left(-1,-1,-1,-3, 0^4\right)\left(0^8\right)\,, \tag{5.1c}$$

from the twisted states $(\mathbf{27},\mathbf{1},\mathbf{1})$. We assign V_1 to the first k fixed points, V_2 to the next p fixed points and V_3 to the remaining $q = 27-k-p$ ones.

However, as explained in Chapter 4 the bundle vectors have to fulfill flux quantization, the Bianchi identities, and the DUY equations. Since the bundle vectors are given by shifted momenta of the twisted states $(\mathbf{27},\mathbf{1},\mathbf{1})$, the flux quantization condition $3V_r^I \in \Gamma_{E_8\times E_8}$ is automatically fulfilled. It is also easy to check that the Bianchi identity (4.21) is satisfied. The relevant tree level DUY equations are given by (4.24). They can be fulfilled for all configurations (k,p,q) with arbitrarily large exceptional divisor volumes. In particular, (4.24) can be rewritten as

$$\sum_{r=1}^{k}\mathrm{vol}(E_r)\cdot V_1 + \sum_{r=k+1}^{k+p}\mathrm{vol}(E_r)\cdot V_2 + \sum_{r=k+p+1}^{k+p+q}\mathrm{vol}(E_r)\cdot V_3 = 0\,. \tag{5.2}$$

Since in our model the bundle vectors add up to zero, this simplifies to the condition

$$\sum_{r=1}^{k}\mathrm{vol}(E_r) = \sum_{r=k+1}^{k+p}\mathrm{vol}(E_r) = \sum_{r=k+p+1}^{k+p+q}\mathrm{vol}(E_r)\,. \tag{5.3}$$

Each of the bundle vectors in Eq. (5.1) breaks E_6 to a differently embedded $SO(10)\times U(1)$, but only two breakings are independent. We are thus left with the gauge group

$$G = SO(8)\times U(1)_A \times U(1)_B \times SU(3)\times E_8\,. \tag{5.4}$$

The two $U(1)$ factors are generated by

$$t_A = \left(2,2,2,0,0^4\right)\left(0^8\right)\,, \qquad t_B = \left(0,0,0,2,0^4\right)\left(0^8\right)\,. \tag{5.5}$$

In this normalization all charges are integer. The $U(1)$ generators are related to the bundle vectors (5.1) via

$$V_1 = \frac{1}{3}t_A\,, \quad V_2 = -\frac{1}{6}t_A + \frac{1}{2}t_B\,, \quad V_3 = -\frac{1}{6}t_A - \frac{1}{2}t_B\,. \tag{5.6}$$

The induced decomposition of the **27** of E_6 (via $SO(10)\times U(1)$) is

$$\begin{aligned}\mathbf{27} &\longrightarrow \mathbf{16}_1+\mathbf{10}_{-2}+\mathbf{1}_4\\ &\longrightarrow (\mathbf{8}_{\text{S}})_{1,-1}+(\mathbf{8}_{\text{C}})_{1,1}+(\mathbf{8}_{\text{V}})_{-2,0}+\mathbf{1}_{-2,-2}+\mathbf{1}_{-2,2}+\mathbf{1}_{4,0}\,.\end{aligned} \tag{5.7}$$

As outlined in Chapter 4, this blow-up procedure in the field theory picture corresponds to giving a vev to the twisted states with shifted momenta (5.1). In particular, the three bundle vectors (5.1) on the Calabi–Yau side correspond on the orbifold side to a vev of the three singlets in (5.7). The massless chiral spectrum on the blow-up then depends on the distribution (k,p,q) of the bundle vectors over the 27 fixed points. The untwisted sector and the twisted $(\mathbf{1},\mathbf{3})$'s in the orbifold spectrum (2.53) always contribute $72\cdot(\mathbf{1},\mathbf{3})$ and $9\cdot(\mathbf{8},\mathbf{3})$, while some of the twisted **8**'s get massive or vector-like. As a result, we get

$$(p-q)\cdot(\mathbf{8}_{\text{V}},\mathbf{1})+(k-q)\cdot(\mathbf{8}_{\text{S}},\mathbf{1})+(k-p)\cdot(\mathbf{8}_{\text{C}},\mathbf{1})\ . \tag{5.8}$$

Furthermore, the $U(1)$ charges of the states from the twisted sector will be shifted by the charges of the blow-up modes at the respective fixed point. The blow-up mode at the r-th fixed point is redefined as [16, 49]

$$\Phi_r^{\text{BU-Mode}}=e^{b_r+i\beta_r}\,, \tag{5.9}$$

where b_r are the Kähler moduli of the r-th blow-up cycle dual to E_r, and β_r are the localized axions. In addition, the twisted orbifold states Φ^{Orb} are redefined as

$$\Phi_r^{\text{BU}}=\Phi^{\text{Orb}}\cdot e^{-(b_r+i\beta_r)}\,. \tag{5.10}$$

With this, the $U(1)$ charges of the blow-up states are given by

$$q(\Phi^{\text{BU}})=q(\Phi^{\text{Orb}})-q(\Phi^{\text{BU-Mode}})\,. \tag{5.11}$$

For a detailed discussion of this redefinition procedure and a complete match of the orbifold and blow-up spectra, see [16, 17].

5.2 Anomaly Coefficients

Once the spectrum on the blow-up is determined, we can calculate the anomaly coefficients of $U(1)_A$ and $U(1)_B$ via the corresponding triangle graphs using (3.3). As an important consistency check, we compare the result with the coefficients appearing in the four-dimensional anomaly polynomial (3.20).

From the structure of the anomaly polynomial we draw a number of conclusions: First we take (3.20) for the first E_8 (and hence drop the primes on all field strengths) and express it in terms of the second Chern class c_2 as follows

$$I_6 = \frac{1}{(2\pi)^6} \int_X \left[\frac{1}{6} \operatorname{tr}(\mathcal{F}F)^2 - \frac{1}{4} c_2 \operatorname{tr} F^2 + \frac{7}{48} c_2 \operatorname{tr} R^2 \right] \operatorname{tr}(\mathcal{F}F) \ . \tag{5.12}$$

Here we have used that $\operatorname{tr} \mathcal{R}^2 = -2c_2$ and $\operatorname{tr} \mathcal{F}^2 = \operatorname{tr} \mathcal{R}^2$ in cohomology, as long as the second E_8 remains unbroken. We expand the flux and the four-dimensional field strength as $\mathcal{F} = E_r V_r^I H_I$ and $F = F^I H_I$ and insert k times the contribution from V_1, p times the contribution from V_2, and q times the contribution of V_3 to obtain

$$\begin{aligned} I_6 \sim\ & F_A^3 \cdot \left(\frac{k-6}{12} \right) + F_A F_B^2 \cdot \left(\frac{k-18}{4} \right) \\ & + F_A \left[\operatorname{tr} F_{SU(3)}^2 + \operatorname{tr} F_{SO(8)}^2 - \frac{7}{12} \operatorname{tr} R^2 \right] \cdot \left(\frac{k-9}{2} \right) \\ & + F_B \left[\frac{1}{8} F_B^2 + \frac{1}{48} F_A^2 + \operatorname{tr} F_{SU(3)}^2 + \operatorname{tr} F_{SO(8)}^2 - \frac{7}{12} \operatorname{tr} R^2 \right] \cdot \left(\frac{p-q}{2} \right) \ . \end{aligned} \tag{5.13}$$

We have also used relations (5.6) and $\int_X c_2 \, E_r V_r^I F_I = -6 \sum_r V_r^I F_I$. From (5.13) it is evident that $U(1)_B$ is omalous if $p = q$, that the cubic $U(1)_A$ anomaly vanishes for $k = 6$, and that the non-Abelian anomalies of $U(1)_A$ vanish for $k = 9$. In particular, there is no configuration with omalous $U(1)_A$.

5.2.1 Gauge Boson Masses

Although $U(1)_B$ is omalous in some bundle configurations, both $U(1)$'s are always massive. The axion associated with $U(1)_B$ always feels a shift proportional to λ due to the transformation (3.11) and hence gets a mass via the Stückelberg mechanism. From the expansion of B_2 given by (3.18) and the transformation (3.11) we deduce that β_r transforms as

$$\delta\beta_r = \operatorname{tr} \lambda V_r \ . \tag{5.14}$$

Notice that the inherited divisors R_i and thus the axions α_i do not play a role in our case, since we have chosen to expand the gauge flux in exceptional divisors only. Thus, the anomalies in our model are canceled by the non-universal axions β_r. There can be no contribution from the universal b_2 because of the absence of an anomalous $U(1)$ on the orbifold. In particular, (5.6) indicates that the first k of the β_r cancel the anomalies of $U(1)_A$ and the rest cancel a mixture of both $U(1)$'s. Hence the axions can be redefined such that only two of the $1 + 27$ possible axions transform with a shift, and the others are invariant under (3.5).

Specifically, to see that both $U(1)$'s are massive we consider the mass term for the four-dimensional gauge bosons arising from the action (3.12),

$$\int_X H_3 \wedge *H_3 = A_\mu^I A^{\mu\, J} \left(M^2 \right)^{IJ} + \cdots \ , \tag{5.15}$$

with the mass matrix

$$\left(M^2\right)^{IJ} = V_r^I V_s^J \cdot \int\limits_X E_r \wedge *_6\, E_s\,. \tag{5.16}$$

Here, $*_6$ denotes the six-dimensional Hodge star. Using the result from [57],

$$*_6\, E_s = \frac{3}{4}\frac{\mathrm{vol}(E_s)}{\mathrm{vol}(X)} J \wedge J - \frac{1}{2} E_s \wedge J \tag{5.17}$$

and the DUY equations (4.24) we find for the mass matrix

$$\left(M^2\right)^{IJ} = \frac{9}{2} V_r^I V_s^J \delta_{rst} b_t = \frac{1}{2}\begin{pmatrix} m_1 & m_1 & m_1 & m_2 \\ m_1 & m_1 & m_1 & m_2 \\ m_1 & m_1 & m_1 & m_2 \\ m_2 & m_2 & m_2 & m_3 \end{pmatrix}, \tag{5.18}$$

and $\left(M^2\right)^{IJ} = 0$ for $I, J > 4$. The entries m_i are given by

$$\begin{aligned} m_1 &= 4\sum_{r=1}^{k} b_r + \sum_{r=k+1}^{k+p} b_r + \sum_{r=k+p+1}^{27} b_r\,, \\ m_2 &= -3\sum_{r=k+1}^{k+p} b_r + 3\sum_{r=k+p+1}^{27} b_r\,, \\ m_3 &= 9\sum_{r=k+1}^{k+p} b_r + 9\sum_{r=k+p+1}^{27} b_r\,, \end{aligned} \tag{5.19}$$

which means that $(M^2)^{IJ}$ always has rank two in the blow-up phase.

5.2.2 Examples

As an example of the above statements we discuss three simple models with different (k, p, q). Their spectra and anomaly coefficients are summarized in Table 5.1(a) and 5.1(b), respectively. Model 1 has the configuration $(k, p, q) = (9, 9, 9)$. As explained before, $U(1)_B$ is omalous and $U(1)_A$ only has Abelian anomalies. Model 2 has $(k, p, q) = (25, 1, 1)$. Here, $U(1)_B$ is still omalous and $U(1)_A$ has Abelian and non-Abelian anomalies. Model 3 with configuration $(k, p, q) = (13, 13, 1)$ is an example of the most general case, where all anomaly coefficients are nonzero.

In all models the anomaly coefficients from the triangle diagrams and from the anomaly polynomials match. All models exhibit Green–Schwarz anomaly cancellation with non-universal axions.

(k,p,q)	Massless chiral spectrum
$(9,9,9)$	$24\,(\mathbf{1},\mathbf{3})_{4,0} + 24\,(\mathbf{1},\mathbf{3})_{-2,-2} + 24\,(\mathbf{1},\mathbf{3})_{-2,2}$ $+\,3\,(\mathbf{8}_\mathrm{V},\mathbf{3})_{-1,1} + 3\,(\mathbf{8}_\mathrm{S},\mathbf{3})_{-1,-1} + 3\,(\mathbf{8}_\mathrm{C},\mathbf{3})_{2,0}$
$(25,1,1)$	$72\,(\mathbf{1},\mathbf{3})_{4,0}$ $+\,24\,(\mathbf{8}_\mathrm{S},\mathbf{1})_{3,1} + 24\,(\mathbf{8}_\mathrm{C},\mathbf{1})_{3,-1}$ $+\,3\,(\mathbf{8}_\mathrm{V},\mathbf{3})_{-1,1} + 3\,(\mathbf{8}_\mathrm{S},\mathbf{3})_{-1,-1} + 3\,(\mathbf{8}_\mathrm{C},\mathbf{3})_{2,0}$
$(13,13,1)$	$36\,(\mathbf{1},\mathbf{3})_{4,0} + 36\,(\mathbf{1},\mathbf{3})_{-2,2}$ $+\,12\,(\mathbf{8}_\mathrm{V},\mathbf{1})_{0,2} + 12\,(\mathbf{8}_\mathrm{S},\mathbf{1})_{3,1}$ $+\,3\,(\mathbf{8}_\mathrm{V},\mathbf{3})_{-1,1} + 3\,(\mathbf{8}_\mathrm{S},\mathbf{3})_{-1,-1} + 3\,(\mathbf{8}_\mathrm{C},\mathbf{3})_{2,0}$

(a) *Chiral massless spectra of the three example models.*

(k,p,q)	$U(1)$ factor	$A_{SO(8)^2}$	$A_{SU(3)^2}$	$A_{U(1)^2_A}$	$A_{U(1)^2_B}$	A_{grav^2}
$(9,9,9)$	$U(1)_A$	0	0	54	-18	0
	$U(1)_B$	0	0	0	0	0
$(25,1,1)$	$U(1)_A$	4	4	342	14	28
	$U(1)_B$	0	0	0	0	0
$(13,13,1)$	$U(1)_A$	1	1	126	-10	7
	$U(1)_B$	1	1	24	24	7

(b) *Anomaly coefficients.*

Table 5.1: *Details of the three example models in an obvious notation. The gauge group is always $SO(8) \times U(1)_A \times U(1)_B \times SU(3)$, with the unbroken E_8 factor omitted.*

5.3 Remnant Discrete Symmetries

Discrete symmetries find a large number of applications in string phenomenology. They are, for example, very useful for explaining the absence of a perturbative μ term and of dimension-four and -five proton decay operators. While discrete non-R symmetries often stem from broken $U(1)$ symmetries, discrete R symmetries arise as remnants of the internal Lorentz symmetry after compactification. Although the discussed family of models does not exhibit a phenomenologically interesting gauge group the detailed investigation of discrete symmetries is still worthwhile, since the procedure outlined in this section can be performed in a similar way in other models. After a short comment on non-R symmetries we analyze remnant R symmetries. A full introduction into the theory of R symmetries, especially on orbifolds, is, however, beyond the scope of this thesis. Therefore, we recommend [44, 58] for further reading. Another instrument we use is the GLSM description [59] of complete intersection Calabi–Yaus (CICYs). For a thorough introduction, see [12, 13].

5.3.1 Non-*R* Symmetries

Since the blow-up is generated from twisted orbifold fields that get a vev, discrete symmetries can arise from remnants of the $U(1)$ gauge groups under which the blow-up fields were charged.

In the family of models at hand, the discrete non-R symmetries are the ones left over from the two broken $U(1)$'s. One finds from the branching of the **27** of E_6 in (5.7) that the bundle vectors (5.1) correspond to the blow-up modes

$$(\mathbf{1},\mathbf{1})_{4,0}\,,\quad (\mathbf{1},\mathbf{1})_{-2,-2}\,,\quad (\mathbf{1},\mathbf{1})_{-2,2}\,. \tag{5.20}$$

When these get a vev, a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup survives[1]. It is easy to check that both $\mathbb{Z}_2$ factors are omalous.

5.3.2 *R* Symmetries

The discussion of remnant R symmetries is more involved. R symmetries are those transformations that do not commute with supersymmetry, which in superspace language means that the Grassmann coordinate θ transforms non-trivially. Since there is only one such coordinate in four-dimensional $\mathcal{N} = 1$ supersymmetry, this can at most be a single $U(1)$ or $\mathbb{Z}_N$: If there are several such symmetries, they can be redefined such that only one of them transforms θ, while the others act as usual non-R symmetries. This convention only fixes the charges of the fields up to an admixture of non-R symmetries that leave θ invariant. The normalization is commonly chosen such that θ transforms with charge 1, which implies that the superpotential W has charge 2. Furthermore, a $\mathbb{Z}_2$ R symmetry can be turned into a non-R symmetry by a combination with a sign reversal on the fermions, so $\mathbb{Z}_2$ symmetries do not lead to true R symmetries.

In the following, we begin by reviewing R symmetries from the orbifold point of view. After that, we discuss them from the Calabi–Yau perspective.

R symmetries on the orbifold

The $T^6/\mathbb{Z}_3$ orbifold possesses a discrete $(\mathbb{Z}_3)^3$ rotational symmetry stemming from rotating each torus independently by $\frac{2\pi i}{3}$ (note that this is a symmetry of the compactification space but not an orbifold space group element). In the literature [44], one often finds the R charge of the

[1] Generically, this is a $\mathbb{Z}_4 \times \mathbb{Z}_4$. In our family of models, however, all states have charge 0 or 2 under both $\mathbb{Z}_4$'s.

orbifold state defined as in (2.39). Notice that this only holds for space-time bosons. The ones of space-time fermions are obtained via $q_{\text{sh}}^{\text{f}} = q_{\text{sh}} - \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, and hence $R_{\text{f}}^i = R^i - \frac{1}{2}$. R charge conservation requires that for a superpotential coupling involving L chiral superfields Φ_α the charges satisfy

$$\sum_{\alpha=1}^{L} R_\alpha^i \equiv 1 \bmod N_i\,, \quad i = 1, 2, 3\,. \tag{5.21}$$

Here N_i is the order of the orbifold twist in the i-th torus. Note that in this convention the superpotential W has R charge 1 and thus θ has R charge $\frac{1}{2}$.

In cases where rotating a sub-torus independently by $\frac{2\pi \mathrm{i}}{N_i}$ is a symmetry, an orbifold state Φ transforms as

$$\mathcal{R}\,:\,\Phi \longrightarrow e^{2\pi \mathrm{i} v\cdot R}\,\Phi \tag{5.22}$$

with $v = \left(0, \frac{1}{N_1}, 0, 0\right)$ and similarly for the other sublattice rotations. Explicitly, this transformation acts in the following way:

- The bosonic R charge from (2.39) is quantized in units of $\frac{1}{N_i}$, so under (5.22) bosons get a phase $e^{\frac{2\pi \mathrm{i}}{N_i^2}}$.
- The R charges of the fermions are shifted by $-\frac{1}{2}$, so θ transforms with a phase $e^{\frac{2\pi \mathrm{i}}{2N_i}}$, i.e. sublattice rotations act as a $\mathbb{Z}_{2N_i}$ R symmetry.
- Finally, the order of (5.22) acting on the fermions is given by the least common multiple of N_i^2 and $2N_i$.

To summarize, the R transformations form a $\mathbb{Z}_{2N_i^2}$ symmetry, under which the charges of bosons, fermions and θ are of the form $2k$, $2k - N$ and N, respectively, where k is an integer. If N is even, so are all charges, and consequently, only a $\mathbb{Z}_{N_i^2}$ is realized on the fields. Due to this slightly confusing symmetry pattern, one finds at least three different R charge normalizations in the literature:

1. W has charge 1, and the smallest charge quantization is in units of $\frac{1}{2N_i}$. This is inspired by the orbifold R rule (5.21).
2. W has charge 2, and the smallest charge quantization is in units of $\frac{1}{N_i}$, which fits with the usual four-dimensional R symmetry conventions.
3. W has charge $2N_i$, and the smallest charge quantization is in units of 1.

In the case at hand, each two-torus can be rotated independently (with $N_i = 3$ for $i = 1, 2, 3$) and we use the second normalization, such that we speak of a $\mathbb{Z}_6^R$ symmetry where fermion charges are quantized in multiples of $\frac{1}{3}$, bosonic ones in multiples of $\frac{2}{3}$, and θ has charge 1. Note that in particular the twisted states Φ corresponding to the **27** of E_6 have $R = \frac{1}{3}(0,1,1,1)$ and thus transform with charge $\frac{1}{9}$ under each $\mathbb{Z}_3$ sublattice rotation. Clearly, this is not a bona fide $\mathbb{Z}_6$ symmetry because applying it six times does not give the identity of the fields, but it fits with the standard R symmetry normalization from four-dimensional supersymmetry, and the orbifold R charge conservation (5.21) becomes a mod 6 condition.

R symmetries from the blow-up perspective

To find unbroken R symmetries after switching on vevs to generate the blow-up, we seek combinations of the three sublattice rotations $\mathcal{R}_i$ and the two $U(1)$ generators $t_{A,B}$ which leave the blow-up modes invariant,

$$\mathbf{1}_{q_A,q_B} \longrightarrow (\mathcal{R}_1)^r\,(\mathcal{R}_2)^s\,(\mathcal{R}_3)^t\,(t_A)^{q^A}\,(t_B)^{q^B}\,\mathbf{1}_{q_A,q_B} = \mathbf{1}_{q_A,q_B}\,, \tag{5.23}$$

for $(q_A,q_B) = (4,0)$, $(-2,2)$ and $(-2,-2)$. This implies that $r+s+t \equiv 0 \bmod 3$, i.e. only a trivial $\mathbb{Z}_2$ R symmetry remains. Note that by combining with discrete non-R $\mathbb{Z}_N$ symmetries, higher $\mathbb{Z}_N$ R symmetries (with $N > 3$) can be obtained. For the examples presented here, the discrete non-R symmetries are $\mathbb{Z}_2 \times \mathbb{Z}_2$, such that in this case no R symmetry enhancement by mixing with other symmetries is possible. Hence for the models at hand it is expected that no non-trivial R symmetry will be left after blowing up. However, as discussed in Section 5.4, both R and non-R symmetries can forbid the unwanted superpotential terms once anomaly universality is not required.

GLSM description

We now investigate how to reproduce this from the perspective of the resolution space. One way to uncover discrete R symmetries on the resolution Calabi–Yau manifolds is to describe the model via a *gauged linear sigma model* (GLSM). In our case, this GLSM is a (0,2) supersymmetric two-dimensional field theory with $U(1)$ gaugings, which in the infrared limit flows to the superconformal world-sheet description. In this framework, changes in topology like blowing-up an orbifold to a smooth CY manifold can be described by phase transitions of the underlying GLSM [59]. For the sake of clarity, we focus on the $(k,p,q) = (9,9,9)$ model, where the blow-up can be described with just three exceptional divisors. However, using the results from [13], the following analysis can be repeated for more general configurations and for other orbifolds in the same fashion.

The starting point is the observation that the geometry of our model can be realized as the blow-up of a complete intersection in $\left(\mathbb{P}^2[3]\right)^3/\mathbb{Z}_3$. The coordinates and charges for the GLSM realization are given in Table 5.2. The notation is chosen as follows: The $z_{i\rho}$ correspond to the inherited divisors, where $i = 1,2,3$ labels the torus and $\rho = 1,2,3$ labels the fixed point.

Charges	z_{11}	z_{12}	z_{13}	z_{21}	z_{22}	z_{23}	z_{31}	z_{32}	z_{33}	x_{111}	x_{211}	x_{311}	c_1	c_2	c_3
R_1	1	1	1	0	0	0	0	0	0	0	0	0	-3	0	0
R_2	0	0	0	1	1	1	0	0	0	0	0	0	0	-3	0
R_3	0	0	0	0	0	0	1	1	1	0	0	0	0	0	-3
E_{111}	1	0	0	1	0	0	1	0	0	-3	0	0	0	0	0
E_{211}	0	1	0	1	0	0	1	0	0	0	-3	0	0	0	0
E_{311}	0	0	1	1	0	0	1	0	0	0	0	-3	0	0	0

Table 5.2: *Charge assignment of the GLSM superfields describing the geometry in the notation of [13].*

The three coordinates $x_{\alpha 11}$ label the three exceptional divisors where each resolves nine of the 27 orbifold fixed points. The c_i correspond to charges of extra chiral superfields which are inserted to ensure invariance of the superpotential.

The orbifold twist acts with a phase $e^{\frac{2\pi i}{3}}$ on z_{11}, z_{21} and z_{31}. From this and Table 5.2, one derives the F-term equations for the c_i

$$0 = z_{11}^3 x_{111} + z_{12}^3 x_{211} + z_{13}^3 x_{311}\,, \tag{5.24a}$$

$$0 = z_{21}^3 x_{111} x_{211} x_{311} + z_{22}^3 + z_{23}^3\,, \tag{5.24b}$$

$$0 = z_{31}^3 x_{111} x_{211} x_{311} + z_{32}^3 + z_{33}^3\,, \tag{5.24c}$$

specifying the complete intersection, and the D-term equations

$$|z_{1\alpha}|^2 + |z_{21}|^2 + |z_{31}|^2 - 3|x_{\alpha 11}|^2 = b_{\alpha 11}\,, \qquad \alpha \in \{1,2,3\}\,, \tag{5.25a}$$

$$|z_{i1}|^2 + |z_{i2}|^2 + |z_{i3}|^2 = a_i\,, \qquad i \in \{1,2,3\}\,, \tag{5.25b}$$

specifying the geometric phase (the vevs of the c_i have already been set to zero). Note that the Fayet-Iliopoulos parameters a_i and $b_{\alpha 11}$ are related to the sizes of the tori and the exceptional divisors, respectively. We have chosen a phase where $a_i \gg 0$ and $a_i \gg b_{\alpha 11}$. For $b_{\alpha 11} \gg 0$, one uncovers the blow-up regime, while $b_{\alpha 11} \to -\infty$ corresponds to the orbifold regime.

To find R symmetries in this picture, we have to find holomorphic automorphisms of the ambient space which leave (5.24) and (5.25) invariant. In addition, the automorphisms must transform the holomorphic $(3,0)$–form Ω non-trivially [58]. The latter can be seen from the definition of Ω in (4.3). Ω can acquire at most a phase $\gamma = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R}$, i.e. $\Omega \to \gamma\Omega$. This means that $\eta \to \pm\gamma^{\frac{1}{2}}$ and thus the superpotential W transforms as $W \to \gamma W$, i.e. like Ω. On the orbifold, the twisted $\mathbf{27}^3$ coupling is allowed, so the $\mathbf{27}$ of E_6 has to transform with a phase $\gamma^{\frac{1}{3}}$.

In our case, we find that the F- and D-term constraints are invariant under the $\mathbb{Z}_3$ transformations[2]

$$z_{i\alpha} \longrightarrow e^{\frac{2\pi \mathrm{i}}{3}\cdot k_{i\alpha}} z_{i\alpha} \qquad \forall\ i,\ \alpha\,. \tag{5.26}$$

Furthermore, there is the $\mathbb{Z}_3$ symmetry

$$(x_{111}, x_{211}, x_{311}) \longrightarrow e^{\frac{2\pi \mathrm{i}}{3}\cdot k}(x_{111}, x_{211}, x_{311})\,. \tag{5.27}$$

Note that the presence of these symmetries is inherited from the symmetries of the orbifold. In other words, the polynomials in (5.24) are not the most general ones in $(\mathbb{P}^2[3])^3$ but have been chosen to be compatible with the orbifold action. In particular, the complex structure of the elliptic curves has been frozen at $\tau = e^{\frac{2\pi \mathrm{i}}{3}}$, so that we are already at a special sublocus of the whole moduli space which exhibits enhanced symmetries. At even more special points in moduli space, there appear certain symmetries under coordinate exchange: When $a_2 = a_3$, there is a symmetry

$$z_{2\alpha} \longleftrightarrow z_{3\alpha}\,, \quad \alpha = 1, 2, 3\,. \tag{5.28}$$

When $b_1 = b_2 = b_3$, we find an S_3 permutation symmetry acting on

$$\{(z_{11}, x_{111})\,, (z_{12}, x_{211})\,, (z_{13}, x_{311})\}\,. \tag{5.29}$$

We can interpret these as exchanges of exceptional or inherited divisors, which are symmetries whenever the corresponding volumes, given by the Kähler parameters a_i and b_α, are equal. Focusing on the $\mathbb{Z}_3$ symmetries, we find combinations such that $\Omega \to e^{\frac{2\pi \mathrm{i}}{3}}\Omega$. Thus $\gamma = e^{\frac{2\pi \mathrm{i}}{3}}$ and the **27** of E_6 transforms with $e^{\frac{2\pi \mathrm{i}}{9}}$, which reproduces the quantization in multiples of $\frac{1}{9}$ from the orbifold, and hence the same $\mathbb{Z}_6^R$ symmetry.

So far, we have used the GLSM merely as a book-keeping device to realize the geometry of the blow-up space. It does, however, contain more information. Specifically, from the preceding discussion it seems that the $\mathbb{Z}_3$ symmetries (5.27) cannot be broken in the GLSM, since the $z_{i\alpha}$ appear only cubed or as absolute values. This seems puzzling, since we previously found that all R symmetries are generically broken on the blow-up. On the other hand, from the GLSM point of view the $\mathbb{Z}_3$ symmetries are merely accidental symmetries, and we would expect them to be broken by quantum effects. However, notice that up to now we have not incorporated the gauge bundle into the GLSM description, which, as in the field theory description, is expected to be responsible for the breaking of the R symmetry.

To see this explicitly, in analogy to the blow-up modes (5.1) we consider the line bundle $\mathcal{L} = \mathcal{O}(0, 0, 0, 2, -1, -1)^3 \oplus \mathcal{O}(0, 0, 0, 0, 3, -3)$. The chiral spectrum is then given by various line bundle cohomology groups (see Table 5.3). Using `cohomCalg` [60, 61] we can reproduce

[2] Note that not all of these symmetries are independent, since some can be related using the GLSM $U(1)$ charges.

Representation	Bundles
$(\mathbf{1},\mathbf{3})$	$\mathcal{O}(0,0,0,4,-2,-2) \oplus \mathcal{O}(0,0,0,-2,4,-2) \oplus \mathcal{O}(0,0,0,-2,-2,4)$
$(\mathbf{8},\mathbf{3})_{\text{V,S,C}}$	$\mathcal{O}(0,0,0,2,-1,-1)$, $\mathcal{O}(0,0,0,-1,2,-1)$, $\mathcal{O}(0,0,0,-1,-1,2)$
$(\mathbf{8},\mathbf{1})_{\text{V,S,C}}$	$\mathcal{O}(0,0,0,0,-3,3)$, $\mathcal{O}(0,0,0,-3,0,3)$, $\mathcal{O}(0,0,0,-3,3,0)$

Table 5.3: *The bundles whose cohomology groups determine the chiral spectrum. The number of left-chiral representations in each case is given by $h^1(V) - h^2(V)$.*

the chiral spectrum of the $(9,9,9)$ model in Table 5.1(a), which is in turn consistent with the orbifold picture.

The transformation of the states under the discrete symmetries can also be calculated via `cohomCalg`. Starting from the symmetries (5.26) and (5.27), which are given in terms of their actions on the GLSM coordinates, we have to determine how they act on the respective cohomologies of our bundle restricted to the Calabi–Yau hypersurface. A priori, it is not clear that the restriction of the symmetry to the Calabi–Yau can be lifted to the gauge bundle. A lift of the discrete symmetry to the gauge bundle which is consistent with the bundle projection and which preserves the group action is known as an *equivariant structure* [20, 62]. It can be shown that this exists for all line bundles in the case of $\mathbb{Z}_M$ symmetries. Given an equivariant structure, we have to check how the relevant bundle cohomologies transform. As in the case of the chiral spectrum, this is done by relating the gauge bundle on the Calabi–Yau to the gauge bundle of the ambient space via the so-called *Koszul resolution*. The transformation of the matter states is then given in terms of the action of the symmetry on the global sections[3] of the gauge bundle, which are given by polynomials in the homogeneous coordinates of the ambient space. Unfortunately, it is beyond this thesis to explore this rich mathematical description.

Instead, we want to resort to the non-compact $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, where a consistent connection between the orbifold and the GLSM bundle description is known [12]. In this case, the bundle is described by chiral-Fermi multiplets $\Lambda^{\hat{I}}$, $\hat{I} = 1, \ldots, 16$, which correspond to the Cartan subalgebra of $E_8 \times E_8$. The $\Lambda^{\hat{I}}$ are charged under the exceptional symmetries $E_{\alpha 11}$, with charges given by the line bundle vectors (5.1) corresponding to the orbifold shifted momenta. Now the coordinates z and x enter the action when determining the charged spectrum [12]: The massless target space modes $\phi_{4\text{d}}(x^\mu)$ appear as deformations of the GLSM kinetic terms for the $\Lambda^{\hat{I}}$ as

$$\int \mathrm{d}^2\theta^+ \phi_{4\text{d}} N_{\hat{I}}^{\hat{J}}(z_{i\alpha}, x_{\alpha 11})\, \Lambda^{\hat{I}} \overline{\Lambda}_{\hat{J}} + \phi'_{4\text{d}} N_{\hat{I}\hat{J}}(z_{i\alpha}, x_{\alpha 11})\, \Lambda^{\hat{I}} \Lambda^{\hat{J}} + \text{h.c.} \tag{5.30}$$

[3]If the bundle is not globally generated, one can twist it by an equivariant ample line bundle and check the transformation for the twisted bundle.

Charges	Λ^a	Λ^4	Λ^I	N_{ab}	N_4^a	N^{a4}	N_{aI}	N_a^I	N_{4I}	N_4^I
E_{111}	2	0	0	-4	2	2	-2	-2	0	0
E_{211}	-1	3	0	2	-4	2	1	1	-3	-3
E_{311}	-1	-3	0	2	2	-4	1	1	3	3

(a) *Charges of the chiral-Fermi fields and of the polynomials arising as kinetic deformations.*

Polynomial	Some contributing monomials
N_{ab}	$x_1^2\left(z_{21}\bar{z}_{22}\right)^2, \left(\bar{z}_{11}^2 z_{12} z_{13}\right)^2, x_1\bar{x}_2\bar{x}_3\bar{z}_{21}z_{22}$
$N_{a,I}$, N_a^I	$x_2x_3z_{21}^2\bar{z}_{22}^2, z_{11}^2\bar{z}_{12}\bar{z}_{13}, \bar{x}_1\bar{z}_{21}\bar{z}_{22}$
N_{4I}, N_4^I	$\bar{x}_2x_3, z_{12}^3\bar{z}_{13}^3, x_1\bar{x}_3^2z_{21}^3\bar{z}_{22}^3$

(b) *Some monomials contributing to the chiral massless spectrum.*

Table 5.4: *Charges of the chiral-Fermi multiplets Λ and the deformation coefficients N and some of the contributing monomials. The monomials for N_4^a and N^{a4} can be obtained from N_{ab} by permutations of indices.*

Here the $N^{\hat{I}\hat{J}}$ and $N_{\hat{I}}^{\hat{J}}$ denote polynomials in the coordinate fields which are chosen such that the expression is gauge invariant. Note that this is a Kähler potential term, so the N's need not be holomorphic.

While locally at each fixed point the gauge group is $SU(3)\times SO(10)\times U(1)\times E_8$, the global model in the end has gauge group $SU(3)\times SO(8)\times U(1)^2\times E_8$. With regard to this, we split the index $\hat{I}$ into $\hat{I}=\left(a,4,I,\tilde{J}\right)$ with $a=1,2,3$, $I=5,\ldots,8$. Furthermore, $\tilde{J}$ corresponds to the second E_8 which is unbroken and hence omitted in the following discussion. The gauge fields are determined by the neutral deformations N_a^b and N_4^4 for $SU(3)\times U(1)^2$ and N_I^J and N_{IJ} for $SO(8)$. We can also read off the charged spectrum from the coefficients: N_{ab}, N_4^a, and N^{a4} correspond to $\left(\mathbf{1},\overline{\mathbf{3}}\right)$ and $(\mathbf{1},\mathbf{3})$, N_{aI} and N_a^I correspond to $\left(\mathbf{8},\overline{\mathbf{3}}\right)$, and N_{4I} and N_4^I correspond to $(\mathbf{8},\mathbf{1})$. The relevant charges of the bundle and the resulting polynomial charges are summarized in Table 5.4(a). Some of the contributing monomials are given in Table 5.4(b). Note that the charges of the N's reproduce some of the line bundle charges of Table 5.3, but not all of them: The missing ones correspond to spinorial roots of E_8 which are not captured in the outlined procedure.

The important fact we deduce from this discussion is the following: Generically, the presence of the N's in (5.30) breaks at least some of the discussed $\mathbb{Z}_3$ symmetries. However, a more thorough understanding of these deformations is needed, e.g. as to which monomials actually

Operator	Charge	Yukawa coupl.	Weinberg op.	$SO(10)$ GUT
$H_u H_d$	$q_{H_u} + q_{H_d}$	$4R - (3q_{\mathbf{10}} + q_{\mathbf{5}})$	$5R - 5q_{\mathbf{10}}$	0
$(LH_u)^2$	$2q_{\mathbf{5}} + 2q_{H_u}$	$4R - (2q_{\mathbf{10}} - q_{\mathbf{5}})$	$2R$	$2R$
$\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$	$q_{\mathbf{10}} + 2q_{\bar{\mathbf{5}}}$	$q_{\mathbf{10}} + 2q_{\bar{\mathbf{5}}}$	$-2R + 5q_{\mathbf{10}}$	$3R$
$\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\bar{\mathbf{5}}$	$3q_{\mathbf{10}} + q_{\bar{\mathbf{5}}}$	$3q_{\mathbf{10}} + q_{\bar{\mathbf{5}}}$	$-R + 5q_{\mathbf{10}}$	$4R$
$\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\bar{\mathbf{5}}_{\boldsymbol{H_d}}$	$3q_{\mathbf{10}} + q_{H_d}$	$2R + 2q_{\mathbf{10}} - q_{\bar{\mathbf{5}}}$	$3R$	$3R$
$\bar{\mathbf{5}}\,\mathbf{5}_{\boldsymbol{H_u}}\bar{\mathbf{5}}_{\boldsymbol{H_d}}\mathbf{5}_{\boldsymbol{H_u}}$	$q_{\bar{\mathbf{5}}} + q_{H_d} + 2q_{H_u}$	$6R - 5q_{\mathbf{10}}$	$6R - 5q_{\mathbf{10}}$	R

Table 5.5: *Charges of relevant operators in MSSM models with $U(1)_X$. The first two operators are the μ-term and the Weinberg operator, respectively. All others violate baryon number conservation. The last three columns are to be read as additional constraints, i.e. allowing the usual Yukawa couplings, allowing the Weinberg operator, and requiring $U(1)_X$ to commute with $SO(10)$ grand unification. Remember that for a coupling to be allowed the charge has to be $2R$, where $R = 1$ and $R = 0$ distinguishes between R symmetries and non-R symmetries.*

contribute in a given phase: Depending on the Kähler parameters, certain coordinates may or may not vanish. This will play a role in determining the appearing operators and hence the symmetry breaking. In particular, we should expect R symmetries to reappear in the orbifold limit $b_{\alpha 11} \to -\infty$. However, in agreement with the field theoretic description it seems that the gauge bundle is responsible for the breaking of the discussed R symmetries.

5.4 Bottom-up Models with Non-Universal Anomalies

Now that we have completed our discussion of the $\mathbb{Z}_3$ blow-up model with non-universal anomalies, we want to comment on the consequences of non-universality for bottom-up models. As a simple example of a model which does not require anomaly universality, consider an extension of the MSSM by an additional $U(1)_X$ symmetry. Here, the anomaly coefficients depend on the $U(1)_X$ charges of the MSSM superfields, i.e. on the imposed phenomenological requirements. We have summarized the $U(1)_X$ charges of all relevant operators and the constraints that follow from phenomenological requirements in Table 5.5.

From this, we draw a number of conclusions. First, anomaly universality is a strong additional constraint which, as was discussed in Chapter 3, is not required for consistency of the theory. This comes about as follows: Assuming the $U(1)_X$ allows all MSSM Yukawa couplings and the Weinberg operator, is flavor-blind, and commutes with $SU(5)$ [but not necessarily with $SO(10)$] results in the anomaly coefficients

$$A_{SU(3)^2-U(1)_X} = \frac{3}{2}\,(3q_{10} + q_{\bar{5}}) - 3R\,, \tag{5.31a}$$

N	$q_{\mathbf{10}}$	$q_{\bar{\mathbf{5}}}$	q_{H_u}	q_{H_d}
6	1	5	4	0

Table 5.6: *Charge assignments for a $\mathbb{Z}_6$ symmetry with $R = 0$ which fulfills all phenomenological requirements.*

$$A_{SU(2)^2-U(1)_X} = (3q_{10} + q_{\bar{5}}) - 3R\,, \tag{5.31b}$$

$$A_{U(1)_Y^2-U(1)_X} = \frac{3}{5}\,(3q_{10} + q_{\bar{5}}) - \frac{9}{2}R\,, \tag{5.31c}$$

which generically do not fulfill the universality condition (3.27). Notice that we have also assumed absence of light Higgs triplets. Our conventions are chosen such that $\ell(\mathbf{fund}_f) = \frac{1}{2}$ and $\ell\big(\mathbf{adj}_f\big) = N$ for $SU(N)$. Furthermore, while generically $U(1)$ normalizations in a bottom-up approach are not fixed, we use the GUT normalization for the hypercharge $U(1)_Y$. We can also go a step further and argue that anomaly universality does not automatically follow from grand unification: Assuming an $SU(5) \times U(1)_X$ theory at some high scale, after the breaking to the MSSM we might expect that $A_{SU(3)^2-U(1)_X} = A_{SU(2)^2-U(1)_X}$. However, this only holds

- for non-R symmetries, since for R symmetries, the gauginos contribute a non-universal factor $\ell\big(\mathbf{adj}_f\big)$,
- and before doublet–triplet splitting, as removing the triplets will change $A_{SU(3)^2-U(1)_X}$ but not $A_{SU(2)^2-U(1)_X}$ (unless $q_{H_u} + q_{H_d} - 2R = 0$, in which case the μ term is not forbidden).

From this also follows that if the anomaly was universal before doublet–triplet splitting, it will not be afterwards and vice versa. However, this of course depends on the type of symmetry breaking mechanism. In a continuous breaking, e.g. a Higgs mechanism, only vector-like pairs of chiral states are removed from the massless spectrum and thus universality is maintained. In contrast, a discrete breaking mechanism, as for example in orbifold compactifications, will generically violate this universality and the above arguments are valid. The above discussion works analogous for the case of a $\mathbb{Z}_N$ instead of $U(1)_X$. In that case the anomaly coefficients are only defined mod N or mod $\frac{N}{2}$, depending on whether N is odd or even.

A second conclusion we draw from Table 5.5 concerns the μ term: Only when $SO(10)$ grand unification is imposed, an R symmetry is necessary to forbid the μ term. For $SU(5)$ grand unification we have

$$2\,(q_{H_u} + q_{H_d}) = 10(R - q_{10}) \bmod N\,, \tag{5.32}$$

and thus the μ term can be forbidden by both R and non-R symmetries once anomaly universality is not imposed. As an explicit example, we give a $\mathbb{Z}_6$ non-R symmetry with corresponding charge assignments in Table 5.6. It fulfills all constraints from Table 5.5 except for $SO(10)$ grand unification.

Chapter 6

Outlook and Conclusion

In the preceding chapter we have seen that the discrete R symmetry on the $\mathbb{Z}_3$ orbifold blow-up, both in field theory and GLSM description, can be understood in terms of the underlying orbifold R symmetry. However, it should be interesting to investigate this connection for more general cases. In particular, for a generic smooth manifold of $SU(3)$ holonomy one expects a $U(1)_R$ symmetry to be the unbroken remnant of the local Lorentz group. One might ask what causes this $U(1)_R$ to be broken to a discrete subgroup in the case of orbifold blow-up manifolds.

In this chapter, we discuss certain orbifold superpotential terms as origins of such a breaking. As an example, we discuss a blow-up of the $\mathbb{Z}_3$ orbifold in standard embedding, i.e. we resolve the singularities using a vector bundle with structure group $SU(3)$. As this is work in progress, the results and arguments made in this chapter still require critical evaluation. In the very last section of this chapter, we give a short conclusion of this thesis.

6.1 R Symmetry Breaking via Superpotential Couplings

Before we consider the superpotential for chiral fields and moduli to investigate couplings which break the R symmetry during the resolution procedure, let us fix the notation. The spectrum of the $T^6/\mathbb{Z}_3$ orbifold is given in (2.53). We adopt the following notation for the untwisted and twisted sector fields:

$$U_i^a \mathrel{\widehat{=}} (\mathbf{27},\overline{\mathbf{3}})_{(\underline{-1,0,0})}\,, \qquad i,a=1\ldots 3\,, \tag{6.1a}$$

$$S_r \mathrel{\widehat{=}} (\mathbf{27},\mathbf{1})_{(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})}\,, \qquad r=1\ldots 27\,, \tag{6.1b}$$

$$T_{r,i}^a \mathrel{\widehat{=}} (\mathbf{1},\mathbf{3})_{(\underline{\frac{2}{3},-\frac{1}{3},-\frac{1}{3}})}\,, \qquad r=1\ldots 27\,;\quad i,a=1\ldots 3\,, \tag{6.1c}$$

where a is the $SU(3)$ index and the subscript on the irreducible representations denotes the orbifold R charges R^i, with $i=1\ldots 3$, in the three complex planes. Using this sign convention, the R charge conservation rule for superpotential couplings of L superfields reads

$$\sum_{\alpha=1}^{L} R^i_\alpha = -1 \text{ mod } N^i = -1 \text{ mod } 3\,. \tag{6.2}$$

Orbifold cubic superpotential

The superpotential for this model contains 496 terms of cubic order (as computed by the `Orbifolder` [63]). We observe the following structure:

- The untwisted fields U_i only couple as $U_1U_2U_3$ since otherwise (6.2) is violated.
- There are no couplings between twisted and untwisted sector.
- The twisted fields S_r couple as S_r^3 or as $S_rS_sS_t$ with $r \neq s \neq t \neq r$, according to a number of non-trivial selection rules for r, s, and t (see, for example [27, 64, 65]).
- The twisted E_6 singlets $T_{r,i}$ only couple as $T_{r,i}T_{s,i}T_{t,i}$ with r, s, and t as above. There are no T^3 couplings at a single fixed point since the antisymmetric $SU(3)$ contraction $T_{i,a}T_{i,b}T_{i,c}\epsilon^{abc}$ vanishes.

Higher order couplings

Order	Couplings	No. of couplings
4	US^2T	1134
5	U^2ST^2	3321
6	U^3S^3, U^3T^3, U^3U^3, S^3T^3, T^3T^3	$> 2 \cdot 10^6$

Table 6.1: *Superpotential couplings allowed by gauge invariance under $E_6 \times SU(3)$. The last column gives the number of couplings computed by the Orbifolder, taking into account R charge conservation and other selection rules.*

The schematic form of all couplings allowed by gauge invariance up to sixth order is summarized in Table 6.1. As will become clear in the following two sections, not all of these cause possible R symmetry violation in the blow-up phase. Of particular interest are the couplings T^3T^3, U^3U^3, and U^3S^3. Furthermore, we investigate couplings of the form U^6T^3.

Blow-up spectrum and superpotential

In the blow-up phase the $SU(3)$ piece of the gauge group is completely broken by the vector bundle. In the field theory picture, this corresponds to giving a vev to the twisted $T^a_{r,i}$ fields of the form [66]

$$\langle T^a_{r,i} \rangle = \lambda_r \delta^a_i \,. \tag{6.3}$$

Thus, only three of the nine T's situated at each fixed point gets a vev. λ_r corresponds to the freedom in choice of the size of the blow-up cycle, and the remaining eight fields become massive. The blow-up vev (6.3) also breaks the orbifold R symmetry in a way that only a $\mathbb{Z}_2^R$ symmetry survives (cf. the discussion in Section 5.3).

The resulting massless spectrum under E_6 contains 36 fields in the **27** as well as $h^{(1,1)} =$ 36 singlets corresponding to the volumes of the three tori and the blow-up cycles around the singularities, respectively [54, 66]. Expanding the Kähler form J in terms of exceptional divisors E_r and inherited divisors R_i as

$$J = a_i R_i - b_r E_r \,, \tag{6.4}$$

these singlets correspond to the 30 Kähler moduli a_i and b_r. The six additional moduli are related to a number of additional non-intersecting divisors S_a, with $a = 1 \ldots 6$. Together with the antisymmetric two-form B_2, expanded as

$$B_2 = b_2 + \alpha_i R_i - \beta_r E_r \,, \tag{6.5}$$

one can construct the so-called complexified Kähler form $J + \mathrm{i}B_2$ and define the complexified Kähler moduli as follows,

$$t_r \equiv b_r + \mathrm{i}\beta_r \,, \qquad \tilde{t}_i \equiv a_i + \mathrm{i}\alpha_i \,. \tag{6.6}$$

For convenience, we choose a normalization of R charge such that $R(\theta) = 1$ and $R(W) = 2$. To determine the R charge of the moduli, we compute the Kähler potential $K(t, \tilde{t})$ according to the description in [54]:

$$\begin{aligned} K(t,\tilde{t}) &= -\ln\left(\int J \wedge J \wedge J\right) \\ &= -\ln\left(\int a_1 a_2 a_3 R_1 R_2 R_3 - \int b_r b_s b_t E_r E_s E_t\right) \\ &= -\ln\left(9 a_1 a_2 a_3 - 9 \sum_r b_r^3\right) \\ &= -\ln\left[\frac{9}{2}(\tilde{t}_1 + \tilde{t}_1^*)(\tilde{t}_2 + \tilde{t}_2^*)(\tilde{t}_3 + \tilde{t}_3^*) - \frac{9}{2}\sum_r (t_r + t_r^*)^3\right] \,. \end{aligned} \tag{6.7}$$

Hence $K(t, \tilde{t}) = K(t + t^*, \tilde{t} + \tilde{t}^*)$ and therefore $R(t) = R(\tilde{t}) = 0$. From this it follows immediately that superpotential terms which contain moduli only must vanish due to R charge conservation.

The combined superpotential for chiral matter and moduli (which give zero contribution) matches the orbifold superpotential nicely. It reads [54]

$$W = 9U_1U_2U_3 + 9(S_r)^3\,, \tag{6.8}$$

where U and S now denote the blown-up counterparts of (6.1a) and (6.1b), respectively. Thus, all charged matter fields must have R charge $\frac{2}{3}$. Note that this superpotential takes no instantonic corrections into account, which on the orbifold correspond to couplings over different fixed points.

R symmetry breakdown

Let us now focus on those superpotential terms which are allowed on the orbifold, but break the R symmetry in the blow-up phase. As mentioned earlier, interesting candidates for such couplings are T^6, U^6, U^3S^3, and U^6T^3. For the sixth order couplings we observe the following facts:

- The coupling U^6 has R charge 4 on the blow-up and thus leaves only a $\mathbb{Z}_2^R$ symmetry unbroken. The argument for this works as follows:

  For a generic discrete R symmetry we choose the charge normalization such that $R(\theta) = 1$, $R(W) = 2$, and $R(\text{chiral matter}) = \frac{2}{3}$. However, let us now choose a case where under a $\mathbb{Z}_6^R$ symmetry θ has charge 3, W has charge 6, and chiral matter (in the following denoted by Φ) has charge 2. A coupling Φ^6 integrated over $\mathrm{d}^2\theta$ then has charge 6. We can write the surviving symmetry generator as

  $$T_6 \equiv e^{2\pi \mathrm{i}\frac{1}{6}q}\,, \tag{6.9}$$

  where q is the charge under the surviving symmetry. Looking at (6.9) we deduce that a $\mathbb{Z}_6$ symmetry is left over. However,

  $$T_6 \cdot \theta = e^{2\pi \mathrm{i}\frac{3}{6}}\theta = -\theta\,, \tag{6.10}$$

  and therefore the R-part of the surviving symmetry is merely a $\mathbb{Z}_2$.

- On the orbifold these couplings of type U^6 seem to be allowed, since $U_1U_2U_3$ has $R^i = (-1,-1,-1)$ and U_i^3 has $R^i = (-3,0,0)$ and thus can be added without violating (6.2). However, it has been argued several times in the literature [27, 67] that untwisted couplings are only allowed at cubic order. In [65] it was argued that this is due to *Rule 5*. Indeed, this can be verified in an explicit CFT calculation[1]. Therefore, the discussion of U^6 appears irrelevant. But, since only untwisted couplings up to order three are allowed, it seems that the untwisted sector is subject to an effective $U(1)_R$ symmetry.

[1] We thank Damian Kaloni Mayorga Pena for pointing this out and performing the calculation.

- The above discussion concerning R charge of U^6 works analog for U^3S^3, for which there is no reason to vanish. For example, one non-zero contribution to the superpotential is given by

$$C_{U^3S^3} = \epsilon_{IJKLMN}\epsilon_{abc}U_1^{I,a}U_1^{J,b}U_1^{K,c}S_r^L S_s^M S_t^N f^{rst}\,, \tag{6.11}$$

  where capital letters denote E_6 indices and a,b,c denote $SU(3)$ indices. f^{rst} encodes the selection rules for couplings over different fixed points. Both ϵ tensors are totally antisymmetric. This is possible, as the tensor product $\mathbf{27}^6$ contains both a symmetric and an antisymmetric singlet. The contribution of the symmetric E_6 contraction vanishes. Also note that couplings with $r = s = t$ vanish, so that all contributing terms of type U^3S^3 can be interpreted as instantonic corrections. In total, there are 351 non-vanishing couplings of this kind.

- Since the t's and $\tilde{t}$'s, which are some linear combination of the orbifold T's after gauge symmetry breaking, have R charge 0 on the blow-up, couplings of the form T^n will always break the R symmetry. However, as discussed above, the superpotential for the moduli actually vanishes. This can also be seen from the orbifold perspective. Consider the coupling

$$\begin{aligned} T^6 \equiv& f^{rstuvw}T^a_{r,i}T^b_{s,j}T^c_{t,k}T^d_{u,l}T^e_{v,m}T^f_{w,n}\cdot\epsilon_{abc}\epsilon_{def} \\ =&\lambda^6\epsilon_{ijk}\epsilon_{lmn}\,, \end{aligned}$$

  where we have used (6.3) in the second equality. This requires that $i \neq j \neq k \neq i$ and $l \neq m \neq n \neq l$, but then the total orbifold R charge of this coupling is $R^i = (0,0,0)$, which clearly violates (6.2).

Couplings of type U^6T^3

The discussion of couplings of the form U^6T^3 is more involved. Several cases have to be distinguished since gauge invariance allows for different kinds of index contractions. As discussed before, the E_6 tensor product $\mathbf{27}^6$ contains a symmetric as well as an antisymmetric singlet. In $SU(3)$, the tensor products $\mathbf{3}^3$ and $\mathbf{3}\times\overline{\mathbf{3}}$ both contain a singlet. After giving a vev to the T's the discussed coupling has the generic form

$$C_9 \equiv \epsilon_{IJKLMN}U^I_{\alpha,a_1}U^J_{\beta,a_2}U^K_{\gamma,a_3}U^L_{\delta,a_4}U^M_{\kappa,a_5}U^N_{\lambda,a_6}\cdot\delta_i^{b_1}\delta_j^{b_2}\delta_k^{b_3}\cdot\lambda_r\lambda_s\lambda_t f^{rst}\,, \tag{6.12}$$

where again capital letters denote E_6 indices and a_i and b_i are $SU(3)$ indices. Greek indices as well as i, j, and k label the three different triplet states in the orbifold spectrum (2.53). As before, r, s, and t label fixed points and f^{rst} again encodes the selection rules. In total, there are 10773 couplings of this kind, each belonging to one of the following cases:

(i) ϵ_{IJKLMN} is symmetric (thus, in this case we omit capital indices) and U's and T's are contracted antisymmetrically among themselves. Then (6.12) becomes

$$\epsilon_{b_1b_2b_3}\epsilon^{a_1a_2a_3}\epsilon^{a_4a_5a_6}C_9 = \lambda^3\epsilon_{ijk}(U_{\alpha,a_1}U_{\beta,a_2}U_{\gamma,a_3}\epsilon^{a_1a_2a_3})(U_{\delta,a_4}U_{\kappa,a_5}U_{\lambda,a_6}\epsilon^{a_4a_5a_6}) \,, \quad (6.13)$$

which is clearly antisymmetric in (α,β,γ) and (δ,κ,λ). However, this is forbidden by orbifold R charge conservation. Therefore, this case gives no contribution to the discussed coupling.

(ii) ϵ_{IJKLMN} is symmetric and the T's are contracted symmetrically with three of the U's while the remaining three U's are contracted antisymmetrically. The coupling then becomes

$$\delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\epsilon^{a_4a_5a_6}C_9 = \lambda^3 U_{\alpha,i}U_{\beta,j}U_{\gamma,k}(U_{\delta,a_4}U_{\kappa,a_5}U_{\lambda,a_6}\epsilon^{a_4a_5a_6}) \,, \quad (6.14)$$

which gives a non-zero contribution for $\alpha=\beta=\gamma$, since δ, κ, and λ all have to be different.

(iii) ϵ_{IJKLMN} is antisymmetric and U's and T's are contracted antisymmetrically among themselves. In this case we have

$$\lambda^3\epsilon_{IJKLMN}\epsilon_{ijk}(U^I_{\alpha,a_1}U^J_{\beta,a_2}U^K_{\gamma,a_3}\epsilon^{a_1a_2a_3})(U^L_{\delta,a_4}U^M_{\kappa,a_5}U^N_{\lambda,a_6}\epsilon^{a_4a_5a_6}) \,, \quad (6.15)$$

which is symmetric in (α,β,γ) and (δ,κ,λ). We deduce that there can be non-vanishing contributions in a number of cases, e.g. for $\alpha=\beta=\gamma$ while $\delta\neq\kappa\neq\lambda\neq\delta$.

(iv) In the last possible scenario ϵ_{IJKLMN} is antisymmetric and the T's are contracted symmetrically with three of the U's while the remaining three U's are contracted antisymmetrically. The relevant coupling then reads

$$\lambda^3\epsilon_{IJKLMN}U^I_{\alpha,i}U^J_{\beta,j}U^K_{\gamma,k}(U^L_{\delta,a_4}U^M_{\kappa,a_5}U^N_{\lambda,a_6}\epsilon^{a_4a_5a_6}) \,, \quad (6.16)$$

which is antisymmetric in (α,β,γ) and symmetric in (δ,κ,λ). Therefore, there is a contribution to the superpotential if and only if $\delta=\kappa=\lambda$.

This completes our discussion of R symmetry breaking via higher order superpotential couplings. Note that the breaking could be induced both via purely localized couplings at single fixed points and via finite volume effects, i.e. twisted couplings over three different fixed points.

6.2 Conclusion

In this thesis we investigated anomaly cancellation and R symmetries in $E_8 \times E_8$ heterotic string theory models. We studied the compactification of heterotic strings on orbifolds, in particular the $T^6/\mathbb{Z}_3$ orbifold, and its resolved Calabi–Yau manifold. The most important result of this discussion is that Green–Schwarz anomaly cancellation does not require a universal axion coupling to different gauge group factors and the Lorentz group in generic compactifications. Although this result has been established in the literature for some time, it was recently under dispute. We first demonstrated non-universality by deriving the four-dimensional anomaly polynomial for a compactification in the presence of finite-volume four-cycles which resolve the orbifold singularities. Afterwards, we gave an explicit example by constructing a string model on the $T^6/\mathbb{Z}_3$ orbifold and its resolution. By computing all relevant anomaly coefficients we proved that anomaly universality is not a necessary condition for the Green–Schwarz mechanism to work, thus clarifying some confusion which recently arose in the literature.

Another important result of this work concerns R symmetries on orbifolds and their smooth counterparts. In the aforementioned model, we were able to identify a $\mathbb{Z}_6^R$ symmetry in the orbifold phase and match it to the remnant R symmetry in the blow-up phase using a GLSM description. The investigation of R symmetries and their origin is a very interesting topic for future research. In the last part of this thesis we commented on the potential connection between R symmetries on orbifolds and smooth Calabi–Yau manifolds. In particular, we compared superpotential terms on the $\mathbb{Z}_3$ orbifold and its blow-up in standard embedding for untwisted fields, twisted fields, and moduli to find those terms which might break a continuous $U(1)_R$ symmetry to a discrete $\mathbb{Z}_N^R$. The prime goal of future work will be to obtain a fundamental understanding of R symmetries as remnants of the local Lorentz group.

Appendix A

A.1 Resolution of $\mathbb{C}^3/\mathbb{Z}_3$

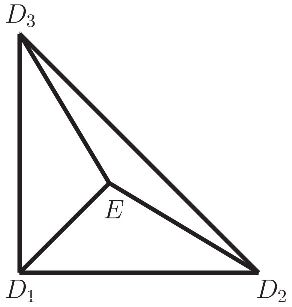


Figure A.1: *Two-dimensional projection of the toric diagram of Res*$(\mathbb{C}^3/\mathbb{Z}_3)$.

The toric diagram of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_3)$, depicted in Figure A.1, contains three three-dimensional cones: (D_1, E, D_2), (D_1, E, D_3), and (D_2, E, D_3). Thus, there is a single linear equivalence relation between the ordinary divisors D_i and the exceptional divisor E, given by

$$0 \sim 3D_i + E\,, \qquad i = 1\dots 3\,. \tag{A.1}$$

From this it follows that the only non-vanishing triple-intersection number is given by $E^3 = 9$.

A.2 Resolution of $T^6/\mathbb{Z}_3$

The $T^6/\mathbb{Z}_3$ orbifold has 27 isolated quotient singularities. On each of these singularities we put one compact exceptional divisor denoted by $E_{\alpha\beta\gamma}$, where $\alpha, \beta, \gamma = 1\dots 3$ label the fixed points on the three T^2 tori. In addition, there are nine divisors R_{jk} inherited from T^6 and nine ordinary divisors $D_{i\alpha}$, where $i, \alpha = 1\dots 3$ label the three tori and fixed points per torus, respectively. In the global resolution, the equivalence relation (A.1) becomes

$$R_i \sim D_{i\alpha} + \sum_{\beta,\gamma} E_{\alpha\beta\gamma}\,. \tag{A.2}$$

Since all exceptional divisors are compact and do not intersect the R_{jk}, the intersection ring is of a very simple form. One can show that all non-vanishing intersection numbers are given by

$$R_1 R_2 R_3 = 9\,, \qquad \text{and} \qquad E^3_{\alpha\beta\gamma} = 9\,. \tag{A.3}$$

We can now use the expression for the Euler number of a given divisor S [47],

$$\chi(S) = c_2(X) \cdot S + S^3\,, \tag{A.4}$$

to calculate the second Chern classes of the inherited and exceptional divisors. Applying (A.4) to $D_{i\alpha}$ and $E_{\alpha\beta\gamma}$ and plugging into (A.2) gives

$$c_2 \cdot R_i = 0\,, \qquad \text{and} \qquad c_2 \cdot E_{\alpha\beta\gamma} = -6\,. \tag{A.5}$$

A.3 Differential Forms

A complex (p,q)-form $\omega_{p,q}$ is a totally antisymmetric tensor with p holomorphic and q anti-holomorphic indices. It is usually expanded in the coordinates of a given complex manifold as

$$\omega_{p,q} = \frac{1}{p!q!}\omega_{i_1\ldots i_p j_1\ldots j_q}\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\bar{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}\,, \tag{A.6}$$

where the wedge-product of k one-forms is defined by

$$\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_k} = \sum_{P\in S_k}\mathrm{sgn}(P)\mathrm{d}z^{i_{P(1)}}\wedge\cdots\wedge\mathrm{d}z^{i_{P(k)}}\,. \tag{A.7}$$

For r-forms on a real n-dimensional manifold M with coordinates x_i, the exterior derivative d maps r-forms to $(r+1)$-forms as follows,

$$\mathrm{d}\omega_r = \frac{1}{r!}\left(\frac{\partial}{\partial x^j}\omega_{i_1\ldots i_r}\right)\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\,. \tag{A.8}$$

Furthermore, if M has a metric g and Levi–Cevita tensor $\epsilon_{i_1\ldots i_n}$ there is a natural operator $*$ called *Hodge duality operator*, which maps r forms to $(n-r)$-forms according to

$$*\,\omega_r = \frac{\sqrt{|g|}}{r!(n-r)!}\omega_{i_1\ldots i_r}\epsilon^{i_1\ldots i_r}{}_{j_{r+1}\ldots j_n}\mathrm{d}x^{j_{r+1}}\wedge\cdots\wedge\mathrm{d}x^{j_n}\,. \tag{A.9}$$

From this and our definition of the wedge-product follows that the product

$$\omega_r\wedge *\,\eta_r = \frac{\sqrt{|g|}}{r!}\omega_{i_1\ldots i_r}\eta^{i_1\ldots i_r}\mathrm{d}x^1\wedge\ldots\mathrm{d}x^n\,, \tag{A.10}$$

is symmetric and an n-form. Note also that the invariant volume element of M is given by

$$*\,1 = \frac{\sqrt{|g|}}{n!}\epsilon_{i_1\ldots 1_n}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_n} = \sqrt{|g|}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\,. \tag{A.11}$$

Returning to the case of complex differential forms, the exterior derivative separates into the so-called *Dolbeault operators* $\mathrm{d} = \partial + \bar{\partial}$. Each of them is nilpotent, i.e. $\partial^2 = \bar{\partial}^2 = 0$. With these operators one defines the *Dolbeault cohomology* as

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\bar{\partial}\text{-closed } (p,q)\text{-forms in } M}{\bar{\partial}\text{-exact } (p,q)\text{-forms in } M} \,. \tag{A.12}$$

the dimension of $H^{p,q}_{\bar{\partial}}(M)$ is the *Hodge number* $h^{p,q}$. In addition, define the adjoint Dolbeault operators ∂^* and $\bar{\partial}^*$ and the Laplacians

$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial \qquad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \,. \tag{A.13}$$

Then the $\Delta_{\bar{\partial}}$-harmonic (p,q)-forms are in one-to-one correspondence with the cohomology classes $H^{p,q}_{\bar{\partial}}(M)$. This fact is of central importance to the notion of *index theorems*.

Bibliography

[1] H. Georgi and S. Glashow, "Unity of All Elementary Particle Forces", *Phys.Rev.Lett.* **32** (1974) 438–441.

[2] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String", *Phys.Rev.Lett.* **54** (1985) 502–505.

[3] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String", *Nucl.Phys.* **B256** (1985) 253.

[4] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory", *Phys.Lett.* **B149** (1984) 117–122.

[5] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on Orbifolds", *Nucl.Phys.* **B261** (1985) 678–686.

[6] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on Orbifolds. 2.", *Nucl.Phys.* **B274** (1986) 285–314.

[7] L. E. Ibanez, H. P. Nilles, and F. Quevedo, "Orbifolds and Wilson Lines", *Phys.Lett.* **B187** (1987) 25–32.

[8] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings", *Nucl.Phys.* **B258** (1985) 46–74.

[9] P. S. Aspinwall, "Resolution of orbifold singularities in string theory", `arXiv:hep-th/9403123 [hep-th]`. To appear in 'Essays on Mirror Manifolds 2'.

[10] S. Groot Nibbelink, M. Trapletti, and M. Walter, "Resolutions of C**n/Z(n) Orbifolds, their U(1) Bundles, and Applications to String Model Building", *JHEP* **0703** (2007) 035, `arXiv:hep-th/0701227 [hep-th]`.

[11] S. Nibbelink Groot, T.-W. Ha, and M. Trapletti, "Toric Resolutions of Heterotic Orbifolds", *Phys.Rev.* **D77** (2008) 026002, `arXiv:0707.1597 [hep-th]`.

[12] S. Nibbelink Groot, "Heterotic orbifold resolutions as (2,0) gauged linear sigma models", *Fortsch.Phys.* **59** (2011) 454–493, `arXiv:1012.3350 [hep-th]`.

[13] M. Blaszczyk, S. Groot Nibbelink, and F. Ruehle, "Gauged Linear Sigma Models for toroidal orbifold resolutions", `arXiv:1111.5852 [hep-th]`.

[14] A. Schellekens and N. Warner, "Anomalies, Characters and Strings", *Nucl.Phys.* **B287** (1987) 317.

[15] R. Blumenhagen, G. Honecker, and T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles", *JHEP* **0506** (2005) 020, `arXiv:hep-th/0504232 [hep-th]`.

[16] S. Groot Nibbelink, H. P. Nilles, and M. Trapletti, "Multiple anomalous U(1)s in heterotic blow-ups", *Phys.Lett.* **B652** (2007) 124–127, `arXiv:hep-th/0703211 [HEP-TH]`.

[17] M. Blaszczyk, N. G. C. Bizet, H. P. Nilles, and F. Ruehle, "A perfect match of MSSM-like orbifold and resolution models via anomalies", *JHEP* **1110** (2011) 117, `arXiv:1108.0667 [hep-th]`.

[18] M. Blaszczyk, S. Nibbelink Groot, F. Ruehle, M. Trapletti, and P. K. Vaudrevange, "Heterotic MSSM on a Resolved Orbifold", *JHEP* **1009** (2010) 065, `arXiv:1007.0203 [hep-th]`.

[19] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, "Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds", *Phys.Rev.* **D84** (2011) 106005, `arXiv:1106.4804 [hep-th]`. 19 pages. References Added.

[20] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, "Heterotic Line Bundle Standard Models", `arXiv:1202.1757 [hep-th]`.

[21] C. Lüdeling, F. Ruehle, and C. Wieck, "Non-Universal Anomalies in Heterotic String Constructions", *Phys.Rev.* **D85** (2012) 106010, `arXiv:1203.5789 [hep-th]`.

[22] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono, *et al.*, "TABLES OF Z(N) ORBIFOLD MODELS",.

[23] L. Susskind, "The Anthropic landscape of string theory", `arXiv:hep-th/0302219 [hep-th]`.

[24] D. Lust and S. Theisen, "Lectures on string theory", *Lect.Notes Phys.* **346** (1989) 1–346.

[25] S. Forste, “Strings, branes and extra dimensions”, *Fortsch.Phys.* **50** (2002) 221–403, `arXiv:hep-th/0110055 [hep-th]`.

[26] P. Mayr, H. P. Nilles, and S. Stieberger, “String unification and threshold corrections”, *Phys.Lett.* **B317** (1993) 53–59, `arXiv:hep-th/9307171 [hep-th]`.

[27] A. Font, L. E. Ibanez, H. P. Nilles, and F. Quevedo, “On the concept of naturalness in string theories”, *Phys.Lett.* **B213** (1988) 274.

[28] L. J. Dixon, “Some world sheet properties of superstring compactifications, on orbifolds and otherwise”, (1987) .

[29] J. Giedt, “Z(3) orbifolds of the SO(32) heterotic string. 1. Wilson line embeddings”, *Nucl.Phys.* **B671** (2003) 133–147, `arXiv:hep-th/0301232 [hep-th]`.

[30] L. E. Ibanez and G. G. Ross, “Discrete gauge symmetry anomalies”, *Phys.Lett.* **B260** (1991) 291–295.

[31] T. Banks and M. Dine, “Note on discrete gauge anomalies”, *Phys.Rev.* **D45** (1992) 1424–1427, `arXiv:hep-th/9109045 [hep-th]`. Revised version.

[32] K. Fujikawa, “Path Integral Measure for Gauge Invariant Fermion Theories”, *Phys.Rev.Lett.* **42** (1979) 1195.

[33] K. Fujikawa, “Path Integral for Gauge Theories with Fermions”, *Phys.Rev.* **D21** (1980) 2848.

[34] L. Alvarez-Gaume and P. H. Ginsparg, “The Structure of Gauge and Gravitational Anomalies”, *Annals Phys.* **161** (1985) 423.

[35] C. A. Scrucca and M. Serone, “Anomalies in field theories with extra dimensions”, *Int.J.Mod.Phys.* **A19** (2004) 2579–2642, `arXiv:hep-th/0403163 [hep-th]`.

[36] H. M. Lee, S. Raby, M. Ratz, G. G. Ross, R. Schieren, *et al.*, “Discrete R symmetries for the MSSM and its singlet extensions”, *Nucl.Phys.* **B850** (2011) 1–30, `arXiv:1102.3595 [hep-ph]`.

[37] J. Wess and B. Zumino, “Consequences of anomalous Ward identities”, *Phys.Lett.* **B37** (1971) 95.

[38] B. Zumino, Y.-S. Wu, and A. Zee, “Chiral Anomalies, Higher Dimensions, and Differential Geometry”, *Nucl.Phys.* **B239** (1984) 477–507.

[39] M. B. Green, J. H. Schwarz, and P. C. West, “Anomaly Free Chiral Theories in Six-Dimensions”, *Nucl.Phys.* **B254** (1985) 327–348.

[40] T. Kobayashi and H. Nakano, "'Anomalous' U(1) symmetry in orbifold string models", *Nucl.Phys.* **B496** (1997) 103–131, arXiv:hep-th/9612066 [hep-th].

[41] M. Dine, N. Seiberg, and E. Witten, "Fayet-Iliopoulos Terms in String Theory", *Nucl.Phys.* **B289** (1987) 589.

[42] J. Casas, E. Katehou, and C. Munoz, "U(1) Charges in Orbifolds: Anomaly Cancellation and Phenomenological Consequences", *Nucl.Phys.* **B317** (1989) 171. Revised version.

[43] T. Araki, T. Kobayashi, J. Kubo, S. Ramos-Sanchez, M. Ratz, *et al.*, "(Non-)Abelian discrete anomalies", *Nucl.Phys.* **B805** (2008) 124–147, arXiv:0805.0207 [hep-th].

[44] T. Kobayashi, S. Raby, and R.-J. Zhang, "Searching for realistic 4d string models with a Pati-Salam symmetry: Orbifold grand unified theories from heterotic string compactification on a Z(6) orbifold", *Nucl.Phys.* **B704** (2005) 3–55, arXiv:hep-ph/0409098 [hep-ph].

[45] P. A. Griffiths and J. Harris, *Principles of algebraic geometry.* Wiley classics library. Wiley, New York, NY, 1994.

[46] W. Fulton, *Introduction to toric varieties.* Annals of mathematics studies. Princeton Univ. Press, Princeton, NJ, 1993.

[47] D. Lust, S. Reffert, E. Scheidegger, and S. Stieberger, "Resolved Toroidal Orbifolds and their Orientifolds", *Adv.Theor.Math.Phys.* **12** (2008) 67–183, arXiv:hep-th/0609014 [hep-th].

[48] S. Nibbelink Groot, D. Klevers, F. Ploger, M. Trapletti, and P. K. Vaudrevange, "Compact heterotic orbifolds in blow-up", *JHEP* **0804** (2008) 060, arXiv:0802.2809 [hep-th].

[49] S. Nibbelink Groot, J. Held, F. Ruehle, M. Trapletti, and P. K. Vaudrevange, "Heterotic Z(6-II) MSSM Orbifolds in Blowup", *JHEP* **0903** (2009) 005, arXiv:0901.3059 [hep-th].

[50] E. Calabi, "The space of Kähler metrics", *Proc.Int.Congr.Math.Amsterdam* **2** (1954) 206–207.

[51] S.-T. Yau, "Calabi's Conjecture and some new results in algebraic geometry", *Proc.Nat.Acad.Sci.* **74** (1977) 1798–1799.

[52] V. Bouchard, "Lectures on complex geometry, Calabi-Yau manifolds and toric geometry", arXiv:hep-th/0702063 [HEP-TH].

[53] M. Nakahara, "Geometry, topology and physics", (2003) .

[54] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond", (1998) .

[55] M. B. Green, J. Schwarz, and E. Witten, "Superstring theory. vol. 2: Loop amplitudes, anomalies and phenomenology", (1987) .

[56] N. Arkani-Hamed, T. Gregoire, and J. G. Wacker, "Higher dimensional supersymmetry in 4-D superspace", *JHEP* **0203** (2002) 055, `arXiv:hep-th/0101233 [hep-th]`.

[57] A. Strominger, "Yukawa Couplings in Superstring Compactification", *Phys.Rev.Lett.* **55** (1985) 2547.

[58] E. Witten, "Symmetry Breaking Patterns in Superstring Models", *Nucl.Phys.* **B258** (1985) 75.

[59] E. Witten, "Phases of N=2 theories in two-dimensions", *Nucl.Phys.* **B403** (1993) 159–222, `arXiv:hep-th/9301042 [hep-th]`.

[60] R. Blumenhagen, B. Jurke, T. Rahn, and H. Roschy, "Cohomology of Line Bundles: A Computational Algorithm", *J.Math.Phys.* **51** (2010) 103525, `arXiv:1003.5217 [hep-th]`.

[61] "`cohomCalg` package." Download link, 2010. High-performance line bundle cohomology computation based on [60].
http://wwwth.mppmu.mpg.de/members/blumenha/cohomcalg.

[62] R. Blumenhagen, B. Jurke, and T. Rahn, "Computational Tools for Cohomology of Toric Varieties", *Adv.High Energy Phys.* **2011** (2011) 152749, `arXiv:1104.1187 [hep-th]`. 17 pages, 4 tables/ prepared for the special issue 'Computational Algebraic Geometry in String and Gauge Theory' of Advances in High Energy Physics, cohomCalg implementation available at
http://wwwth.mppmu.mpg.de/members/blumenha/cohomcalg/.

[63] H. P. Nilles, S. Ramos-Sanchez, P. K. Vaudrevange, and A. Wingerter, "The Orbifolder: A Tool to study the Low Energy Effective Theory of Heterotic Orbifolds", *Comput.Phys.Commun.* **183** (2012) 1363–1380, `arXiv:1110.5229 [hep-th]`.

[64] A. Font, L. E. Ibanez, H. P. Nilles, and F. Quevedo, "Degenerate Orbifolds", *Nucl.Phys.* **B307** (1988) 109.

[65] T. Kobayashi, S. L. Parameswaran, S. Ramos-Sanchez, and I. Zavala, "Revisiting Coupling Selection Rules in Heterotic Orbifold Models", *JHEP* **1205** (2012) 008, `arXiv:1107.2137 [hep-th]`.

[66] S. Hamidi and C. Vafa, “Interactions on Orbifolds”, *Nucl.Phys.* **B279** (1987) 465.

[67] A. Font, L. E. Ibanez, H. P. Nilles, and F. Quevedo, “Yukawa Couplings in Degenerate Orbifolds: Towards a Realistic SU(3) x SU(2) x U(1) Superstring”, *Phys.Lett.* **B210** (1988) 101.

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