Orbifold Compactifications with Flux Backgrounds

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Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Referent:	Prof. Dr. Hans Peter Nilles
Korreferent :	Prof. Dr. Rainald Flume

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Contents

1 Introduction

The Standard Model of Particle Physics [1] is a theory describing nature at the highest experimentally produced energies. The excellent agreement between its predictions and observed data establishes it as one of the most exact theories of mankind.

On the other hand, the model is not complete for it lacks several essential ingredients. For example, a lot of parameters like the masses of quarks and leptons or the exact values of their charges have to be inserted by hand. Furthermore, to our present understanding, a complete model of nature should incorporate gravity, i.e. it should also be a theory of quantum gravity.

Looking at the internal structure of the theory from a more esthetic point of view one sees that not all possible symmetries are realized. To be more precise, one can introduce a symmetry between fermions and bosons called supersymmetry [2]. The adoption of such a symmetry also opens the way for a solution to the hierarchy problem. Therefore the minimal supersymmetric extension of the Standard Model MSSM [3] has been considered a partial solution.

Superstrings [4] are the most prominent candidates for a theory beyond the Standard Model of Particle Physics and its minimal supersymmetric extension. In contrast to quantum field theory, string theory gives up the notion of locality. According to this theory, the fundamental ingredients of the universe are no more point particles but rather one dimensional string-like objects. Quantizing the classical relativistic string leads to a spectrum showing excitations which can be interpreted as the photon and the graviton. Therefore, Schwarz and Scherk (1974) [5] suggested that String Theory is a theory that includes gravity.

One soon realizes that Superstring Theories are only consistent in 10 space time dimensions. This is a first prediction of string theory but on the other hand we have to find a way to explain the gap between this prediction and the observed four dimensional world. A way to get rid of the 6 extra dimensions is compactification and has been known through the work of Oskar Klein and Theodore Kaluza since the 1920s. The first "Kaluza-Klein"-theory was originally proposed as a framework to unify gravity with electromagnetism through the introduction of one extra-dimension. These type of theories were given up soon because of the existence of an unobserved scalar particle and were not considered again until the 1970s. At that time people got interested into higher dimensional supergravity theories. Also the low energy approximation of string theory is a supergravity theory.

But what do we mean by the term compactification? The easiest way to imagine this is to find an analogy in the three dimensional world we know. Imagine a garden hose looked at from far away. We only see a line, i.e. a one dimensional object. On the other hand an ant sitting on the garden hose will "see" a two dimensional "world".

Similarly, String Theory predicts a world where on every space time point there are sitting 6 curled up extra-dimensions. The form and shape of these dimensions is constrained by the effective four dimensional theory they produce. To get a supersymmetric effective theory resembling our world, one needs "Calabi-Yau" manifolds which are complicated mathematical objects not fully understood, yet.

Let us for illustrative purposes think of the compactified extra dimensions as a six dimensional torus. Then the shape of the torus and the sizes of its cycles represent degrees of freedom present in the ten dimensional theory. If we pass to the four dimensional limit of this theory these degrees of freedom have to become trasparent in the four dimensional Lagrangian one way or the other. As it turns out they appear as massless scalar fields which are called moduli. The fact that they are massless can be understood from the observation that it costs no energy to deform the size and shape of the curled up extra dimensions. This is unsatisfactory for a number of reasons which will be explained later, and therefore one tries to create a potential for the moduli - or in other words give them masses. This is done through the mechanism of "flux compactifications".

In this work I will give an introduction to supersymmetric theories in various dimensions and their compactifications on Calabi-Yau manifolds. Later I will concentrate on turning on background fluxes which will also be the main topic of this thesis.

2 Supersymmetric compactifications

2.1 Supersymmetry multiplets

There are altogher five known string theories which are the type IIA, type IIB, heterotic $E_8 \times E_8$, heterotic $SO(32)$ and the type I string theory [6]. The type IIA theory is also called the nonchiral-theory as it predicts two supersymmetries with opposite chiralities, whereas type IIB is known as the chiral theory. The heterotic string is build out of a left moving bosonic string theory and a right moving type II string theory. It has been known as the phenomenologically most interesting theory as it incorporates large gauge groups, either $SO(32)$ or $E_8 \times E_8$. There is also another string theory, called type I, which is based on open superstrings. All these theories are anomaly free. Their low energy effective action can be identified with the various Supergravity theories in 10 dimensions and the supergravity theory in eleven dimensions is considered to be the low energy limit of another theory, called M-theory. Therefore it seems appropriate to study Supersymmetry and Supergravity [7], [8].

2.1.1 Supersymmetry in ten dimensions

The physical states in a supergravity theory are given by representations of the supersymmetry algebra

$$\{Q, \bar{Q}\} = 2P^M \cdot \gamma_M, \quad M = 0, \cdots, 9 \tag{2.1}$$

(where we do not take into account the central charges). The minimal amount of supersymmetry in D dimensions is given by the dimension of the smallest irreducible spinor representation of the Lorentz group $SO(1, D-1)$.

This amounts to finding representations for the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \tag{2.2}$$

in D dimensions [9], leading to the following formula for the dimension of the Dirac spinor:

$$d_\gamma = \begin{cases} 2^{D/2} & d \text{ even} \\ 2^{(D-1)/2} & d \text{ odd} \end{cases}$$

Our next attempt will be to answer the question in what dimensions Majorana- or Weyl-spinors are possible. In order to do that we need generalized versions of these spinors in arbitrary dimensions. The equivalent of the γ_5-matrix is defined to be

$$\gamma_{D+1} = i^k \gamma^0 \cdots \gamma^{D-1}, \tag{2.3}$$

where the relation between k and D is $D = 2k+2$ in the case of even D, and $D = 2k+1$ in the case of odd D. One can easily check the following relations

$$\gamma_{D+1}^2 = 1, \quad \{\gamma_{D+1}, \gamma^\mu\} = 0. \tag{2.4}$$

From this one can see that the Dirac representation splits into two Weyl representations with eigenvalues $\gamma_{D+1} = \pm 1$ and dimensions 2^k each. It turns out that for even k ($D = 2, 6, 10, \cdots$) each Weyl representation is its own conjugate and for odd k ($D = 4, 8, 12, \cdots$) the two Weyl representations are conjugate to each other. Furthermore one can deduce that an odd dimensional space admits no Weyl spinors at all.

For Majora-spinors a similar generalization can be implemented. Consider the generators of $SO(1, D-1)$

$$\Sigma^{\mu\nu} := \frac{1}{4}[\gamma^\mu, \gamma^\nu]. \tag{2.5}$$

One can find exactly two matrices B_1 and B_2 such that

$$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *}, \tag{2.6}$$

for any of these two matrices. This allows one to define a spinor $B^{-1}\Psi^*$ which transforms the same way under the Lorentz group as the ordinary Dirac spinor Ψ. The Majorana condition can be stated as

$$B^{-1}\Psi^* = \Psi. \tag{2.7}$$

Taking the conjugate of the above equation shows $\Psi = B^*\Psi^* = B^*B\Psi$, which implies that the Majorana condition is only onsistent if

$$B^*B = 1. \tag{2.8}$$

On the other hand concrete representations of B_1 and B_2 lead to

$$B_1^*B_1 = (-1)^{k(k+1)/2}, \quad B_2^*B_2 = (-1)^{k(k-1)/2}. \tag{2.9}$$

Furthermore a Majorana condition can be imposed on a Weyl spinor only if the Weyl spinor is conjugate to itself.

Now the crucial property of a ten dimensional space - which can be shown using the above remark together with equations (2.8), (2.9) - is that it allows Weyl-, Majorana- as well as Majorana-Weyl spinors. A Majorana-Weyl spinor in 10 dimensions has 16 real components. Consequently the minimal supersymmetry in ten dimensions has 16 real supercharges.

Looking at massless states, which always form representations of $SO(D-2)$, we see that the corresponding supercharges form spinor representations of SO(8). The $\underline{16}$ of $SO(1, 9)$ decomposes under $SO(8)$ into

$$\underline{16} \rightarrow 8_v + 8_s, \quad or \quad \underline{16} \rightarrow 8_v + 8_c, \tag{2.10}$$

where 8_v and 8_c are the two Weyl spinors of $SO(8)$, complex conjugate to each other.

As explained in [10] tensor products of these representations form the irreducible $N = 1$ supermultiplets in ten dimensions.

Multiplet	Bosons	Fermions
vector	$\hat{A}_M$	$\lambda^{\pm}$
graviton	$\hat{g}_{MN}, \hat{B}_{MN}, \hat{\Phi}$	$\Psi^{+}_{M}, \lambda^{+}$
gravitino	$l, \hat{C}_{MN}, \hat{A}_4$	$\Psi^{+}_{M}, \lambda^{+}$
gravitino	$\hat{A}_M, \hat{C}_{MNP}$	$\Psi^{-}_{M}, \lambda^{-}$

Table 1: N=1 supermultiplets in ten dimensions

It is important to stress here that the above multiplets do not form a supergravity theory, but as it will turn out, they rather contain all fields appearing in the various supergravity theories.

In the above table $\hat{g}_{MN}$ denotes the metric (35 of $SO(8)$), $\Psi^{\pm}_{M}$ is the gravitino (56_s of $SO(8)$), $\lambda^{\pm}$ is a spinor (8_s of $SO(8)$), $\hat{\phi}$ is the dilaton, l is a scalar and $\hat{A}$, $\hat{B}$, $\hat{C}$ are totally antisymmetric tensor fields. The representations to which these tensor fields belong are 8_v of $SO(8)$ for $\hat{A}_M$, 28 of $SO(8)$ for $\hat{C}_{MN}$, 56_v of $SO(8)$ for $\hat{C}_{MNP}$ and 35_c of $SO(8)$ for $\hat{A}_{MNPQ}$. The $\pm$ superscripts denote the chirality of the spinors. Our convention is to use hats on antisymmetric tensor fields whenever these live in dimensions ten and eleven. All representations are counted on-shell which means that the fields should satisfy their equations of motion.

2.1.2 N=1 Super-Yang-Mills in ten dimensions

The supergravity theory with one supersymmetry in ten dimensions coupled to a Yang-Mills theory is the field theory limit of the Heterotic and Type I Superstrings [6]. It's field content is the vector together with the graviton multiplet of table 1.

As it turns out $N = 1$, $D = 10$ supergravity is not anomaly free. Consequently there cannot be a string theory that is approximated by this supersymmetric field theory. On the other hand the coupling of this theory to the ten dimensional supersymmetric Yang-Mills multiplet leads under certain circumstances to theories free of both gravitational and mixed gauge-gravitational anomalies. Coupling the $N = 1$ supergravity to $N = 1$ Yang-Mills, consistent with local supersymmetry and gauge invariance, leads to extra contributions to the field strength of B_{MN}. These extra terms form a functional of the Yang-Mills gauge field A_M which requires that B_{MN} transforms non trivially under Yang-Mills transformations. This makes possible the Green-Schwarz anomaly cancellation mechanism leading to an anomaly-free $D = 10$ supergravity-Yang-Mills theory for the following choices of the gauge group [8]:

$$SO(32),\ E_8 \times E_8,\ E_8 \times [U(1)]^{248},\ [U(1)]^{496}. \tag{2.11}$$

The first two choices correspond to the field theory limits of the $SO(32)$ and $E_8 \times E_8$ heterotic strings.

The bosonic action is given by [9]

$$S_{het} = \int d^{10}x \sqrt{-G} e^{-2\hat{\Phi}} (\hat{R} + 4\partial_\mu \hat{\Phi} \partial^\mu \hat{\Phi} - |\hat{H}_3|^2 + \frac{1}{g^2} \mathrm{Tr} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}). \tag{2.12}$$

The fields are defined as

$$\hat{H}_3 = d\hat{B}_2 - \frac{1}{g^2}\omega_3, \quad \omega_3 = \mathrm{Tr}(\hat{A} \wedge d\hat{A} + \frac{2}{3}\hat{A} \wedge \hat{A} \wedge \hat{A}), \tag{2.13}$$

$$d\omega_3 = \mathrm{Tr}\hat{F}_2 \wedge \hat{F}_2. \tag{2.14}$$

2.1.3 $N = 1$ Supergravity in $D = 11$

The supersymmetry algebra of this supergravity theory is generated by a single Majorana spinor $Q_a,\ a = 1, \cdots 32$ [10]. The field content of the massless $d = 11,\ N = 1$ gravity multiplet is characterized by irreducible representations of the little group $SO(9)$:

- $\hat{g}_{MN}$, the 11-dimensional graviton. On-shell this constitutes a 44 of $SO(9)$.
- $\Psi^a_M,\ a = 1, \cdots 32$, an 11-dimensional massless vector-spinor, i.e. the gravitino field. On-shell this is a 128 of SO(9).
- $\hat{A}_{MNP}$, a 3rd rank antisymmetric tensor. On-shell this is an 84 of SO(9).

Eleven-dimensional supergravity is the field theory limit of M-theory [11], which is in turn a strong coupling limit of superstring theory.

2.1.4 Type IIA theory

The type IIA theory is the low energy approximation of type IIA string theory. It can be deduced from dimensional reduction of $d = 11$ supergravity: The Majorana spinor Q_a (32 components) splits into two 10-dimensional Majorana-Weyl spinors $Q^1_\alpha, Q^2_{\dot{\alpha}}$ (16 components each) with opposite chirality. The field content of this nonchiral theory can be obtained as in the following:

$$\hat{g}_{MN} \rightarrow \begin{pmatrix} \hat{g}^{10}_{MN} & \hat{A}_M \\ 0 & \hat{\Phi} \end{pmatrix}$$

One sees that a ten dimensional massless vector $\hat{A}_M$ arises from the $\hat{g}_{MN}$ components of the metric, while the 10-dimensional dilaton $\hat{\Phi}$ arises from the $\hat{g}_{11\ 11}$ component.

The eleven dimensional antisymmetric tensor field $\hat{A}_{MNP}$ breaks up into a ten dimensional 3rd rank antisymmetric tensor $\hat{C}_{MNP}$ and an antisymmetric tensor $\hat{B}_{MN}$ (from $\hat{A}_{MN11}$) of rank two. The gravitino Ψ^a_M undergoes the following splitting:

$$\Psi^a_M \to \Psi^\alpha_M,\ \tilde{\Psi}^{\dot{\alpha}}_M,\ \lambda^\alpha,\ \tilde{\lambda}^{\dot{\alpha}} \tag{2.15}$$

This gives two ten dimensional Majorana-Weyl vector-spinors of opposite chirality and two ten dimensional Majorana-Weyl spinors of reverse chirality.

The bosonic action of the type IIA theory is given by [9]

$$S_{10} = \int \left[e^{-2\hat{\Phi}}(-\frac{1}{2}\hat{R} * 1 + 2d\hat{\Phi} \wedge *d\hat{\Phi} - \frac{1}{4}\hat{H}_3 \wedge *\hat{H}_3) - \frac{1}{2}(\hat{F}_2 \wedge *\hat{F}_2 + \hat{F}_4 \wedge *\hat{F}_4) + L_{top} \right] \tag{2.16}$$

Here the field strengths are defined as

$$\hat{F}_2 = d\hat{A}_1, \quad \hat{F}_4 = d\hat{C}_3 + \hat{B}_2 \wedge d\hat{A}_1, \quad \hat{H}_3 = d\hat{B}_2,\ , \tag{2.17}$$

and the expression for L_{top} is

$$L_{top} = -\frac{1}{2}\left[\hat{B}_2 \wedge d\hat{C}_3 \wedge d\hat{C}_3 - (\hat{B}_2)^2 \wedge d\hat{C}_3 \wedge d\hat{A}_1 + \frac{1}{3}(\hat{B}_2)^3 \wedge d\hat{A}_1 \wedge d\hat{A}_1 \right]. \tag{2.18}$$

2.1.5 Type IIB theory

There is also a second $N = 2,\ D = 10$ supergravity theory which is 'chiral', i.e. both supersymmetry generators have the same handedness. This theory is called type IIB supergravity and is the low energy effective field theory of type IIB superstring theory.It can not be obtained by reduction or truncation of a higher dimensional theory.

Its on-shell multiplet has the following field content [10]:

- $\hat{g}_{MN}$, the 10-dimensional vielbein.
- $\hat{A}_{MNPQ}$, a self-dual 4th rank antisymmetric tensor.
- $\hat{B}_{MN},\ \hat{C}_{MN}$, 2nd rank antisymmetric tensors.
- $\Phi,\ l$, real scalars.
- $\Psi^\alpha_M,\ \tilde{\Psi}^\alpha_M$, Majorana-Weyl vector-spinors of the same chirality.
- $\lambda^{\dot{\alpha}},\ \tilde{\lambda}^{\dot{\alpha}}$, Majorana-Weyl spinor of the same chirality.

At this point one can see that the field content of the two $N = 2,\ D = 10$ theories is made up of multiplets of the $N = 1$ theory. More precisely type IIA comprises the $N = 1$ graviton multiplet and the gravitino multiplet with the same chirality. On the other hand type IIB contains the graviton multiplet together with the gravitino multiplet of opposite chirality.

If one thinks of this in the language of string theory the difference goes this way: The bosonic fields of the two theories are identical in the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector while the difference occurs from the Ramod-Ramond (RR) sector.

Next we will present actions for the two supergravity theories as this will be important for our discussion of flux compactifications.

The type IIB theory is special in the sense that it admits a self-dual field strength $\hat{F}_5$. A naive expression for the kinetic term of the field $\hat{A}_4$ like $\hat{F}_5 \wedge *\hat{F}_5$ where $\hat{F}_5$ is the self dual field strength ($\hat{F}_5 = *\hat{F}_5$) would vanish identically. For this reason we will write down an action from which the self duality condition does not follow and has to be inserted by hand in oder to get the correct equations of motion. The action is given by [9]

$$\begin{aligned} S_{IIB} \;\; = \;\; & \int e^{-2\hat{\Phi}}(-\frac{1}{2}\hat{R} * 1 + 2d\hat{\Phi} \wedge *d\hat{\Phi} - \frac{1}{4}\hat{H}_3 \wedge *\hat{H}_3) \\ & -\frac{1}{2}(d\hat{l} \wedge *d\hat{l} + \hat{F}_3 \wedge *\hat{F}_3 + \frac{1}{2}\hat{F}_5 \wedge *\hat{F}_5) \\ & -\frac{1}{2}\hat{A}_4 \wedge \hat{H}_3 \wedge d\hat{C}_2 \end{aligned}$$

The field strengths are defined as

$$\hat{H}_3 = d\hat{B}_2, \quad \hat{F}_3 = d\hat{C}_2 - \hat{l}d\hat{B}_2, \quad \hat{F}_5 = d\hat{A}_4 - \frac{1}{2}\hat{C}_2 \wedge d\hat{B}_2 + \frac{1}{2} \wedge d\hat{C}_2. \tag{2.19}$$

2.1.6 N=2 supersymmetry in four dimensions

As we will see later, compactifications of type II theories give rise to $N = 2$ supersymmetry in four dimensions. Therefore this section serves as an introduction to four dimensional supersymmetry theories.

The $N = 1$ supersymmetry algebra of massless states in the frame $P^M = (E, 0, 0, E)$ is

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E, \qquad \{Q_2, \bar{Q}_{\dot{2}}\} = 0.$$

Defining a vacuum $|\Omega >$, one sees that $\bar{Q}_{\dot{2}}$ produces states of zero norm

$$< \Omega|Q_2\bar{Q}_{\dot{2}}|\Omega >= 0 \tag{2.20}$$

That is equivalent to setting $\bar{Q}_{\dot{2}}$ equal to zero. We then get as operator algebra a pair of creation and annihilation operators:

$$a = \frac{1}{2\sqrt{E}} Q_1, \qquad a^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}}. \tag{2.21}$$

As the creation operator $a^\dagger$ transforms like $(0, \frac{1}{2})$ under the Lorentz group, one gets the following massless $N = 1$ supersymmetry representations:

$$\begin{aligned} |\Omega> \qquad & \text{helicity} \quad \lambda, \\ a^\dagger|\Omega> \qquad & \text{helicity} \quad \lambda + \frac{1}{2}. \end{aligned}$$

Thus $N = 1$ super-multiplets contain only two states whose spin differ by 1/2. The relevant representations are listed in table 2.

Multiplet	Bosons	Fermions
chiral	Φ	Ψ
vector	A_μ	λ
gravitino	A_μ	Ψ_μ
graviton	$g_{\mu\nu}$	Ψ_μ

Table 2: $N = 1$ super-multiplets in four dimensions

Now we turn our attention to $N = 2$ multiplets. As in the ten dimensional case these can be obtained by combining various $N = 1$ multiplets. Putting together two chiral multiplets one obtains the so called hyper multiplet. This leads to two spinors and four real scalars. A vector and a chiral multiplet make up the $N = 2$ vector multiplet containing a vector, two gaugini and one complex scalar. Last but not least the gravitino and the graviton multiplet of $N = 1$ are the two ingredients of the graviton multiplet.

Multiplet	Bosons	Fermions
hyper-multiplet	$4 \times \phi$	$2 \times \Psi$
vector	A_μ, Φ	$2 \times \lambda$
graviton	$g_{\mu\nu},\ A^0_\mu$	$2 \times \Psi_\mu$

Table 3: $N = 2$ super-multiplets in four dimensions

2.2 Kaluza-Klein theories

In order to be able to describe string compactifications we first need to make a visit to general Kaluza-Klein reductions. Kaluza and Klein developed their method originally to unify electromagnetism with gravity. They achieved this through postulating one extra dimension, i.e. a 5th dimension which had the topology of a circle.

Consider pure gravity in five dimensions which is described by the action

$$S = \frac{1}{2\pi\kappa^2}\int d^4x d\theta \sqrt{-\widehat{g}}\widehat{R}. \tag{2.22}$$

Our convention for use of coordinates will be:

$$z^M = \{x^\mu, y\}, \tag{2.23}$$

with $M = 0\cdots 4$ and $\mu = 0\cdots 3$. Furthermore, we assume a circular extra dimension of radius R, such that

$$\theta = \frac{y}{R}, \qquad 0 \leq \theta \leq 2\pi \tag{2.24}$$

The metric splits in the following way:

$$\widehat{g}_{MN} \longrightarrow g_{\mu\nu} + g_{\mu 4} + g_{44}, \tag{2.25}$$

$$\widehat{g}_{MN} = \Phi^{-1/3}\left[\begin{array}{cc} g_{\mu\nu} + \kappa^2\Phi A_\mu A_\nu & \kappa\Phi A_\mu \\ \kappa\Phi A_\nu & \Phi \end{array}\right].$$

The periodicity condition in the 5th direction is expressed through a Fourier expansion:

$$g_{\mu\nu}(x,\theta) = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x)e^{in\theta}, \tag{2.26}$$

$$A_\mu(x,\theta) = \sum_{n=-\infty}^{\infty} A_{\mu n}(x)e^{in\theta}, \tag{2.27}$$

$$\Phi(x,\theta) = \sum_{n=-\infty}^{\infty} \Phi_n(x)e^{in\theta} \tag{2.28}$$

As the metric should be real we also obtain the following condition:

$$g^*_{\mu\nu n} = g_{\mu\nu -n} \tag{2.29}$$

Integrating over the extra dimension and keeping only the zero modes (i.e. the $n = 0$ terms), one arrives at the action:

$$S = \int d^4x\sqrt{-g_0}\left[\frac{1}{\kappa^2}Rg_0 - \frac{1}{4}\Phi_0 F_{\mu\nu 0}F^{\mu\nu 0} - \frac{1}{6\kappa^2\Phi_0^2}\partial^\mu\Phi_0\partial_\mu\Phi_0\right] \tag{2.30}$$

Here indices are raised and lowered by $g_{\mu\nu 0}$ and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength of the vector-like field A_μ. This way electromagnetism is recovered from pure 5 dimensional gravity. It is important to notice the appearance of an extra scalar field in the effective theory. One can see that the vacuum expectation value of Φ determines the gauge coupling constant.

Including terms with $n \neq 0$ one sees that $\Phi(x, y)$ splits into infinitely many massive fields $\Phi_n(x)$, also known as the Kaluza-Klein tower. Their appearance can be realized through investigating the equation of motion of the scalar field Φ in 5d:

$$\Box_5 \Phi(x, y) = 0 \tag{2.31}$$

$$\Rightarrow (-\partial_i^2 - \partial_5^2 + \partial_0^2)\Phi(x, y) = 0 \quad i = 1, 2, 3 \tag{2.32}$$

$$\Leftrightarrow (\Box_4 - \partial_5^2)\Phi(x, y) = 0 \tag{2.33}$$

$$\Leftrightarrow \sum_n (\Box_4 + \frac{n^2}{R^2})\Phi_n(x) e^{iny/R} = 0 \tag{2.34}$$

Thus one sees that the mass of each Φ_n is $m_n^2 = \frac{n^2}{R^2}$. In ordinary derivations of effective four dimensional field theories one neglects these massive modes or they are removed through integrating out. The only fields remaining are thus the massless zero modes which from now on will be refered to as moduli fields and will play an important role in this thesis.

Let us now come to the symmetries of the effective theory. In the above example the $U(1)$ symmetry is a consequence of an isometry in the extra dimension. This approach can be generalized to arbitrary dimensions where the isometry group of the compactified extra dimensions turns out to be the gauge group in four dimensions. This way even non-abelian gauge-fields can be obtained.

For a much more detailed survey on Kaluza-Klein theories see [12].

2.3 Calabi-Yau compactifications

In the previous sections we developed some aspects of supersymmetry in higher dimensions. As we go on and compactify the extra dimensions while trying to conserve some supersymmetry we will find out that the right amount of supersymmetry is obtained through complex compactifications.

More precisely the complex manifolds should be Calabi-Yau spaces [13]. The following is a motivation for these kinds of compactifications.

2.3.1 Supersymmetric Vacua

A first step towards string phenomenology is the construction of supersymmetric Minkowski vacua. More precisely one is interested in unbroken $N = 1$ supersymmetry as these theories have the most promising phenomenological properties.

Here we will outline how one can get such vacua in the case of the heterotic string. The procedure can be applied to the type II theories with slight modifications where one obtains $N = 2$ supersymmetry in four dimensions.

A vacuum Ω is supersymmetric once $< \Omega|\delta U|\Omega >$ vanishes for every fermionic field U. Here δU is the variation of U under supersymmetry transformations $\{Q, U\}$ where Q is the conserved supercharge. Taking the low energy effective action of the heterotic string - i.e. Type I supergravity - the supersymmetry transformation of the gravitino Ψ_M is

$$\delta\Psi_M = \nabla_M \epsilon + (\Gamma \cdot H)_M \epsilon + \cdots ; \quad M = 0 \cdots 9. \tag{2.35}$$

At this point some remarks are nececcary. The term $(\Gamma \cdot H)_M$ denotes contraction with the H-field and can be generalized to the type II cases through $\sum_p (\Gamma \cdot F_p)_M$ where the F_p denote the p-form field strengths. The definition of the contraction is

$$(\Gamma \cdot F_p)_M \sim \Gamma_{MN_1 \cdots N_P} (F_p)^{N_1 \cdots N_p} + \Gamma_{N_1 \cdots N_{p-1}} (F_p)_M^{N_1 \cdots N_{p-1}}. \tag{2.36}$$

In equation (2.35) we have neglected the dependence on the dilaton ϕ of some terms as we assume the dilaton to be a constant on the internal space. Using this approximation and setting the H-form equal to zero - which is the simplest ground state consistent with Lorentz invariance - one can further show that the supersymmetric variations of the dilatino vanish. In this case equation (2.35) reduces to

$$\nabla_M \epsilon = 0. \tag{2.37}$$

There is still one more fermionic field missing - the gluino. As the vanishing of supersymmetric variations of the gluino is more complicated and does not influence the internal manifold we will neglect this issue. So we are left with equation (2.37).

From this one can follow the integrability condition $[D_M, D_N]\epsilon = 0$ which is the same as

$$R_{MNPQ} \Gamma^{PQ} \epsilon = 0. \tag{2.38}$$

We will assume that the ten dimensional space splits as

$$M = T^4 \times K_6, \tag{2.39}$$

where T^4 is either de Sitter space, anti de Sitter space or Minkowski space and K_6 is a compact six dimensional manifold. Looking at the curvature tensor of this space

$$R_{\mu\nu\alpha\beta} = (R/2)(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}), \tag{2.40}$$

(R is the four-dimensional Ricci scalar) we find that the solution of equation (2.38) with $M, N = 1, \cdots, 4$ implies that the space T^4 is Minkowski with $R = 0$.

On a Minkowski space (2.37) implies that ϵ is independent of the four uncompactified coordinates. Writing ϵ as a direct product

$$\epsilon = \theta \otimes \eta, \tag{2.41}$$

where θ is a space-time spinor while η is an internal one, the resulting equations are

$$\partial_\mu \theta = 0, \qquad \mu = 0, \cdots, 3, \tag{2.42}$$

$$\nabla_m \eta = 0, \qquad m = 1, \cdots, 6. \tag{2.43}$$

The first equation is not very spectacular as it just means that our space-time spinor should be constant. But the second equation is striking as it is one of several equivalent definitions of a Calabi-Yau manifold. As a Calabi-Yau manifold admits a Ricci flat metric we can also see that the resulting background with all other fields vanishing is a solution of the ten-dimensional Einstein equations. In order to understand these aspects better we will give an introduction to complex manifolds.

2.3.2 Complex manifolds

In order to be able to describe complex manifolds ([17], [15]) we have to explain the notion of the almost complex structure. Let M be an even dimensional real manifold. The tangent space T_pM is spanned by $2m$ vectors

$$\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{2m}}.\} \tag{2.44}$$

Now suppose that one has a tensor field J on M such that on every point p $J_p := J(p) \in T_pM \otimes T_p^*M$ and

$$J_p^2 = -id_{T_pM}. \tag{2.45}$$

Such a tensor field is called an almost complex structure.

As $J_p : T_pM \to T_pM$ is a linear map one can define

$$J_p(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, \quad i = 1 \cdots m. \tag{2.46}$$

With respect to the new coordinates

$$\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^m}.\} \tag{2.47}$$

J_p takes the form

$$J_p = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.48}$$

At each point p one can define holomorphic and antiholomorphic tangent vectors

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\{\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\}, \tag{2.49}$$

$$\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\{\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\} \tag{2.50}$$

for the complexified tangent space

$$T_pM^{\mathbb{C}} \equiv T_pM \otimes \mathbb{C}. \tag{2.51}$$

Here i runs from 1 to m.

With respect to this coordinate system J takes the form

$$J_p = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}. \tag{2.52}$$

The subspace of $T_pM^{\mathbb{C}}$ on which J_p acts as iI_m is called the holomorphic tangent space and is denoted by $T_p^{1,0}M$ whereas the orthogonal subspace is called antiholomorphic and is denoted by $T_p^{0,1}M$.

Until now we have assumed that the tensor field J has the above properties only at each point. However, if we can find charts such that J takes the canonical form (2.52) on each open set the resulting manifold is called a complex manifold and J is called the "complex structure".

In this case one also says that the almost complex structure J is "integrable". One may ask the question of whether a given almost complex structure is integrable. This question leads us to the Newlander-Nierenberg theorem. It says that an almost complex structure is integrable if and only if the Nijenhuis tensor which - choosing local coordinates - is given by

$$N_{ij}^k = J_i^l(\partial_l J_j^k - \partial_j J_l^k) - J_j^l(\partial_l J_i^k - \partial_i J_l^k) \tag{2.53}$$

vanishes identically.

As we will need the notion of a complex structure later we have used it in our definition of complex manifolds. But there is another equivalent definition which does not make use of the term complex structure at all:

Suppose a differentiable manifold admits a holomorphic atlas. That means that there are charts $\{(U_i, \varphi_i)\}$ of the form $\varphi_i : U_i \simeq \varphi(U_i) \subset \mathbb{C}^n$ such that the transition functions $\varphi_{ij} := \varphi_i \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j)$ are holomorphic. Then this manifold is called a complex manifold of dimension n.

As a last remark we want to say that on a complex manifold the exterior derivative splits the following way

$$d = \partial + \bar{\partial}, \tag{2.54}$$

where the operators ∂ and $\bar{\partial}$ are called "Dolbeault operators" and are defined by

$$\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}. \tag{2.55}$$

2.3.3 Kähler manifolds and U(N) holonomy

Kähler properties are of essential importance in the study of compactification manifolds as well as in the description of their moduli spaces, i.e. parameter spaces of deformations of the original manifold.

In order to be able to describe them we have to clarify the notion of a "hermitian metric" first. We say that a Riemannian metric g of a complex manifold M is hermitian, if it satisfies

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{2.56}$$

at each point $p \in M$ and for any $X, Y \in T_pM$ (here J_p denotes the complex structure).

Furthermore the following property follows:

$$g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(J_pX, X) = 0. \tag{2.57}$$

One sees that the vector J_pX is orthogonal to X with respect to the metric. So once we have found a hermitian metric (a complex manifold always admits such a metric) we can think of J_p as a "rotation around 90 degrees".

Now consider the extension of g_p to a symmetric form on $T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}}$:

$$g_p(X \otimes \lambda, Y \otimes \mu) := (\lambda\mu) \cdot g_p(X, Y) \tag{2.58}$$

Choosing holomorphic coordinates defined previously (see (2.49), (2.50)) we get the components of g with respect to this basis:

$$\begin{aligned}
g_{ij}(p) &= g_p\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) \\
g_{i\bar{j}}(p) &= g_p\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) \\
g_{\bar{i}j}(p) &= g_p\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}\right) \\
g_{\bar{i}\bar{j}}(p) &= g_p\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right)
\end{aligned} \tag{2.59}$$

From (2.56), we find that

$$g_{ij} = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = g(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}) = -g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = -g_{ij} \tag{2.60}$$

hence $g_{ij} = 0$. While deriving this result we have used $J\frac{\partial}{\partial z^i} = i\frac{\partial}{\partial z^i}$ as $\frac{\partial}{\partial z^i} \in T^{1,0}M$.

One can also show that $g_{\bar{i}\bar{j}} = 0$. The only non vanishing components are $g_{\bar{i}j}$ and $g_{i\bar{j}}$.

Next, define a tensor field ω through

$$\omega_p(X, Y) = g_p(J_p X, X) \qquad X, Y \in T_p M. \tag{2.61}$$

As ω is antisymmetric

$$\omega(X, Y) = g(JX, Y) = g(J^2 X, JY) = -g(JY, X) = -\omega(Y, X), \tag{2.62}$$

it is actually a two-form which is called the “Kähler form” of a hermitian metric g. Extending the domain from T_pM to $T_pM^{\mathbb{C}}$ and calculating the components of ω we find

$$\omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = g(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = ig_{ij} = 0. \tag{2.63}$$

Thus $\omega_{ij} = 0$. Similar calculations give

$$\omega_{\bar{i}\bar{j}} = 0 \qquad \omega_{i\bar{j}} = -\omega_{\bar{i}j} = ig_{i\bar{j}}. \tag{2.64}$$

Altogether we can write

$$\omega = ig_{i\bar{j}} dz^i \wedge d\bar{z}^j. \tag{2.65}$$

Kähler forms are always real forms. As an example consider flat space where ω is given by

$$\omega = \frac{i}{2}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{2.66}$$

Finally, we have arrived at the point to define the notion of a Kähler manifold. This is a hermitian manifold (M, g) whose Kähler form ω is closed:

$$d\omega = 0. \tag{2.67}$$

One can show that this is equivalent to finding a scalar function ϕ such that locally

$$\omega = -i\partial\bar{\partial}\phi. \tag{2.68}$$

The scalar function is called the Kähler potential.

Before proceeding further we first introduce the Levi-Cevita connection as it will enable us to give an equivalent definition of the Kähler property.

From the above equations ((2.65) and (2.68)) one can easily deduce that the only nonzero components of the affine connection $\Gamma^{\mu}_{\nu\lambda} = g^{\mu\kappa}(\partial_\nu g_\kappa \lambda + \partial_\lambda g_\kappa \nu - \partial_\kappa g_\nu \lambda)/2$ are

$$\Gamma^{i}_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{i}l}\partial_{\bar{b}} g_{\bar{c}d}. \tag{2.69}$$

Using this we can calculate the covariant derivative of J. Using the standard form (equation (2.52)), we find

$$(\nabla_k J)_i^j = \partial_k i\delta_i^j - i\delta_l^j \Gamma^l_{ki} + i\delta_i^l \Gamma^j_{kl} = 0. \tag{2.70}$$

Vice versa one can prove that the vanishing of the covariant derivative of J implies the Kähler condition (equation (2.67)).

Now we want to give a description of Kähler manifolds in terms of holonomy. We recall that the holonomy group of a manifold is the rotation group of tangent vectors which are being parallel transported along a closed curve. In complex geometry the holonomy group is always a subgroup of $SO(2N)$ where N is the complex dimension of the manifold.

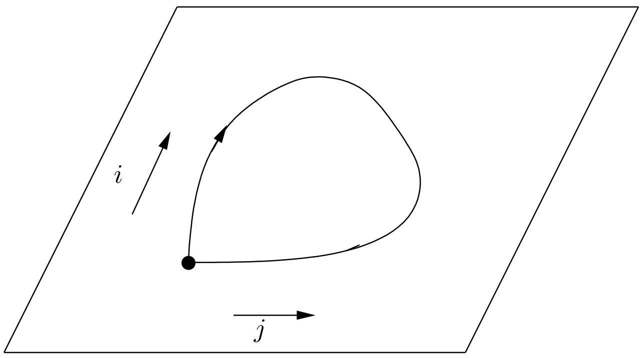


Figure 1: Holonomy

We want to find this group in the case of Kähler spaces. In order to do that let us first give another interpretation of J. In the standard form (2.52) one can think of J as a matrix that assigns the value i or $-i$ to states in the $\mathbf{N}$ or $\bar{\mathbf{N}}$ of U(N). One could also say that it is invariant under $U(N)$. What is the meaning of covariant constancy in this new light? Suppose that under parallel transport around a closed curve a holomorphic tangent vector - that is a vector in $T_p^{1,0}M$ and therefore a state in the $\mathbf{N}$ - is taken to a vector with holomorphic and antiholomorphic parts. This would be a contradiction to the covariant constancy of J. Therefore the holonomy group is at most $U(N)$.

2.3.4 Calabi-Yau manifolds and SU(N) holonomy

Having discussed manifolds with $U(N)$ holonomy we can now ask under what conditions the holonomy group of such a manifold is restricted to $SU(N)$.

This question is intimately related to properties of the Riemann tensor. On a complex manifold of dimension 2N the relation $[D_\mu, D_\nu]V^\lambda = R_{\mu\nu}{}^{\lambda}{}_{\kappa}V^\kappa$ (Greek indices label real dimensions) shows that in parallel transport around a small loop tangent vectors undergo rotation by the $SO(2N)$ matrices $R_{(\mu\nu)}$. If the holonomy group is $U(N)$ the $R_{\mu\nu}$ are $U(N)$ sub matrices of $SO(2N)$. One also says that the spin connection has values in $U(N)$. To find metrics of $SU(N)$ holonomy we have to make the $U(1)$ part of the spin connection vanish. The $U(1)$ generator is the complex structure J. In our familiar complex basis its nonzero components are $J^i_j = i\delta^i_j, J^{\bar{i}}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}$. Extracting the $U(1)$ part of $R(\mu\nu)$ is therefore the same as calculating $F_{\mu\nu} = trJR_{(\mu\nu)} = R_{\mu\nu\lambda\kappa}J^{\kappa\lambda}$. This is the $U(1)$ part of the rotation undergone by tangent vectors that are transported around a small loop in the $\mu\nu$ plane. It is also called the field strength of the $U(1)$ part of the spin connection.

For the nonzero components of $F_{\mu\nu}$ we find

$$F_{i\bar{j}} = -F_{\bar{j}i} = R_{i\bar{j}}{}^{\lambda}{}_{\kappa}J^\kappa_\lambda = iR_{i\bar{j}}{}^{l}{}_{l} - iR_{i\bar{j}}{}^{\bar{l}}{}_{\bar{l}}. \tag{2.71}$$

On the other hand we also get

$$R_{i\bar{j}}{}^{l}{}_{l} = R_{i\bar{j}\bar{k}l}g^{\bar{k}l} = -R_{i\bar{j}l\bar{k}}g^{l\bar{k}} = -R_{i\bar{j}}{}^{\bar{k}}{}_{\bar{k}}. \tag{2.72}$$

Using the definition of the Ricci tensor and the identity $R^l_{i\ \bar{j}\ l} = -R^l_{i\bar{j}\ l}$, one can easily see the following relation:

$$F_{i\bar{j}} = -2iR_{i\bar{j}}. \tag{2.73}$$

This shows that a metric of $SU(N)$ holonomy is the same thing as a Ricci-flat Kähler metric. The two-form $F_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ defines an element of the cohomology group $H^2(M;\mathbb{R})$. This element is called the first chern class $c_1(M)$ of the manifold M.

Until now we have just established the conditions for $SU(N)$ holonomy. The existence of the necessary ricci-flat Kähler metric is a much more complicated issue. Here we will present the central results due to Calabi and Yau. For an almost complete proof see [16].

Let M be a compact Kähler manifold with complex structure J, Riemannian metric g, Kähler form ω, and Ricci form ρ.

Theorem(Calabi-Yau). Let $\rho' \in 2\pi c_1(M)$ be a closed real (1,1). Then there is a unique Kähler metric g' on M with Kähler form ω' cohomologous to ω and with Ricci form ρ'.

Corollary.If $c_1(M) = 0$, then M has a unique Ricci-flat Kähler metric g' with Kähler form ω' cohomologous to ω.

As we will not encounter the above properties of Calabi-Yau manifolds directly during our journey we will present some equivalent conditions.

One can show that having a manifold of $SU(3)$ holonomy is the same as a manifold admitting a nowhere vanishing covariantly constant spinor field. Furthermore one can find a nowhere vanishing holomorphic three-form $\Omega \in H^{(3,0)}$.

The vanishing of the first chern class is a topological property and therefore Calabi-Yau manifolds can be constructed without knowing the details of the Kähler-metric. Concrete examples are given through intersections of hypersurfaces in projective spaces.

2.3.5 The Moduli space of Calabi-Yau manifolds

In the previous section we found that the vacua we are searching for are Calabi-Yau. This is a very restrictive condition on the compactification space but, as we will see in this section, having chosen the topology of such a space is not the end of the story. Once the topology is fixed there are still parameters left which can be adjusted freely. In the case of Calabi-Yau manifolds the remaining parameters are the complex structure and the Kähler form which will be denoted by complex structure and Kähler moduli.

In string theory there is no mechanism to create a potential for these moduli generically, i.e there is no way to fix the compactification manifold completely. Therefore it costs no energy to walk from a Calabi-Yau manifold to a nearby one with the same topology. These deformations present further degrees of freedom which appear as scalar fields in the effective four dimensional theory.

Our first attempt will be to explain the complex structure moduli space. Having a complex manifold one can locally choose holomorphic coordinates such that the nonzero components of the complex structure J are

$$J^i_j = i\delta^i_j, \qquad J^{\bar{i}}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}. \tag{2.74}$$

Deformations of the complex structure can be described as

$$J \to \tilde{J} = J + \tau. \tag{2.75}$$

We want to show that the only non vanishing indices of τ are $\tau^i_{\bar{j}}$ and $\tau^{\bar{i}}_j$. Suppose $\tau^i_j \neq 0$, then we have

$$(J^i_j + \tau^i_j)(J^j_k + \tau^j_k) = -\delta^i_k + 2i\tau^i_k + \text{ terms of order } \tau^2. \tag{2.76}$$

This is a contradiction to the requirement (see section 2.3.2) $\tilde{J}^2 = -id$ unless $\tau^i_j = 0$. The same can be shown for $\tau^{\bar{i}}_{\bar{j}}$.

Now, for $\tilde{J}$ to be a complex structure the Nijenhuis tensor has to vanish $N(\tilde{(J)}) = 0$. Inserting $\tilde{J}$ into the definition of N (equation (2.53)) one can easily see that the only term which does not vanish trivially is

$$N^i_{\bar{j}\bar{k}} = \bar{\partial}_{\bar{j}}\tau^i_{\bar{k}} - \bar{\partial}_{\bar{k}}\tau^i_{\bar{j}} \tag{2.77}$$

and its complex conjugate $N^{\bar{i}}_{jk}$.

Once we view $\tau^i_{\bar{j}}$ as a $(0,1)$ form with values in the holomorphic tangent bundle $\mathcal{T}$, the $N^i_{\bar{j}\bar{k}} = 0$ translates to

$$\bar{\partial}\tau^i = 0. \tag{2.78}$$

Next, we want to determine which deformations correspond to trivial ones. We call a deformation trivial if it is obtained by an infinitesimal change of coordinates. Under such a transformation J transforms by the transformation law of tensors in general relativity. Using this one can conclude that in our case τ transforms as

$$\tau^i \to \tau^i + \epsilon\bar{\partial}v^i, \tag{2.79}$$

where v describes the infinitesimal change of coordinates and is a function of holomorphic as well as of antiholomorphic coordinates. Thus if we define deformations which vary by a trivial one to be equivalent we see - using equations (2.78) together with (2.79) - that deformations of the complex structure of a complex manifold M correspond to elements of $H^1(\mathcal{T})$, the first Dolbeault cohomology group of M with values in the holomorphic tangent bundle $\mathcal{T}$.

Having arrived at this point, we may ask how many scalar fields we get in the effective four dimensional theory constituting the complex structure moduli. This is the same as asking for the number of cohomology classes in $H^1(\mathcal{T})$. It turns out that in the case of Calabi-Yau manifolds -exactly the type of manifolds we are interested in - there is an elegant way to obtain this number. Recall that one of the consequences of vanishing first chern class was the existence of a unique globally defined nowhere vanishing holomorphic three-form Ω. Using this form we can define

$$\chi_{ij\bar{k}} = \Omega_{ijl}\tau^l_{\bar{k}}, \tag{2.80}$$

which shows that an element $\tau^i_{\bar{j}}$ of $H^1(\mathcal{T})$ is equivalent to a $(2,1)$ form. On the other hand using $\Omega^{lmi} = g^{l\bar{l}'}g^{m\bar{m}'}g^{i\bar{i}'}\bar{\Omega}_{\bar{l}'\bar{m}'\bar{i}'}$ (here $\bar{\Omega}$ is the dual $(0,3)$-form) one can establish the equivalence of a given χ with an element of $H^1(\mathcal{T})$

$$\tau^i_{\bar{j}} = \Omega^{lmi}\chi_{lm\bar{j}}. \tag{2.81}$$

Thus, on a manifold of $SU(3)$ holonomy, $H^1(\mathcal{T})$ coincides with $H^{(2,1)}$.

The meaning of this is that the Hodge number $h^{(2,1)}$, defined as the dimension of $H^{(2,1)}$, is the same as the number of complex parameters that enter in defining the complex structure of our manifold M.

Considering variations of the Kähler form one finds immediately that this time there are exactly $h^{(1,1)}$ parameters involved, the dimension of $H^{(1,1)}$. They are responsible for $h^{(1,1)}$ scalar fields v_i in the effective four dimensional theory called Kähler moduli. One can think of these moduli as sizes of certain cycles in the compactified dimensions. In contrast to fields arising from complex structure deformations, which are complex, ordinary Kähler moduli are real as the Kähler form is a real form (see section 2.3.3). Now, the $h^{(1,1)}$ Kähler moduli do not come alone. Consider the ten dimensional B-field which splits according to

$$\hat{B}_2 = B_2 + b^i \omega_i. \tag{2.82}$$

Here ω_i are the $h^{(1,1)}$ two-forms defined in section 2.3.3, B_2 is a four dimensional two-form and b^i are scalars which only depend on the four space time dimensions. Therefore dimensional reduction of a $D = 10$ supergravity theory will give rise to $h^{(1,1)}$ scalar fields b^i at the four dimensional level. These are the model dependent axions. They pair up with the real Kähler moduli forming $h^{(1,1)}$ complex scalar fields

$$t^i = b^i + iv^i. \tag{2.83}$$

The definition of the complexified Kähler form is now straight forward

$$\omega_c = t^i \omega_i. \tag{2.84}$$

This way all degrees of freedom of the original higher dimensional theory appear as scalar modes in our effective world.

Further investigations show that the spaces spanned by the Kähler and complex structure moduli are Kähler. The metrics on these moduli spaces determine the kinetic terms of moduli fields in the four dimensional theory. The technical details of these topics are explained in appendix A.

3 Flux compactifications

We have reached the main topic of this thesis. If we recall our survey until now we find two novel physical implications of string compactifications for our effective four dimensional world. The first is supersymmetry which is imposed by the chose of our compactificaton space. Supersymmetry is a rather welcome outcome as by now there is extensive use of it in modern phenomenological models. The other impact on four dimensional theories is the existence of scalar moduli fields which were investigated in the last section. These fields are not observed in the real world as, being massless, they would mediate long range interactions. Therefore we seek a mechanism to stabilize these moduli or in other words to give them masses. One of these mechanisms is flux compactification.

3.1 What are fluxes?

As we discussed in section 2.1 in supergravity theories which are low energy approximations of string theory bosonic matter is represented by differential forms. To each $p-1$-form A_p we can associate a p-field strength F_p. Historically the field strength of the B-field is called H, so that we have

$$\hat{H} = d\hat{B}. \tag{3.1}$$

Now, by a flux we mean a nontrivial dependence of the field strength on the coordinates of the internal manifold. More precisely, we mean that the integral of the field strength over a cycle of the internal manifold is nonzero. In such a situation one says that there is nonzero "flux". It is important to note that these fluxes are quantized in string theory. Any flux should satisfy

$$\frac{1}{(2\pi\sqrt{\alpha'})^{p-1}} \int_{\Sigma_p} \hat{F}_p \in \mathbb{Z} \tag{3.2}$$

for any p-cycle Σ_p.

Next, we will classify fluxes important for the type II and heterotic theories. In order to do that we need a standardized notation and therefore follow in our presentation reference [17] closely. Using Poincare duality we find that 3-cycles come in pairs (A_K and B^K) while there is the same number of 2- and 4- cycles (A_a and B^a). For a definition of these cycles see A, for now we will only use them to classify our fluxes.

$$\frac{1}{(2\pi)^2\alpha'} \int_{A_K} \hat{H}_3 = m^K, \quad \frac{1}{(2\pi)^2\alpha'} \int_{B^K} \hat{H}_3 = e_K, \quad K = 1, \cdots, \frac{b^3}{2} \tag{3.3}$$

$$\frac{1}{(2\pi)^2\alpha'} \int_{A_K} \hat{F}_3 = m^K_{RR}, \quad \frac{1}{(2\pi)^2\alpha'} \int_{B_K} \hat{F}_3 = e_{RRK}, \tag{3.4}$$

$$\frac{1}{2\pi\sqrt{\alpha'}}\int_{A_a}\hat{F}_2 = m^a_{RR}, \quad \frac{1}{(2\pi\sqrt{\alpha'})^3}\int_{B^a}\hat{F}_4 = e_{RRa}, \quad a = 1, \cdots, b^2. \tag{3.5}$$

On the level of forms we find three-forms α_K, β^K, two forms w_a and four forms $\tilde{w}^a$ obeying the following relations

$$\int_{A_L}\alpha_K = \int \alpha_K \wedge \beta^L = \delta^L_K, \quad \int_{B^K}\beta^L = \int \beta^L \wedge \alpha_K = -\delta^L_K \tag{3.6}$$

$$\int_{A_a} w_b = \int w_a \wedge \tilde{w}^b = \delta^b_a, \quad \int_{B^a}\tilde{w}^b = \int \tilde{w}^b \wedge w_a = -\delta^b_a \tag{3.7}$$

Using the above relations we can deduce the expansion of the field strengths in terms of the forms just defined.

$$\frac{1}{(2\pi)^2\alpha'}\hat{H}_3 = m^K\alpha_K - e_K\beta^K, \quad \frac{1}{(2\pi)^2\alpha'}\hat{F}_3 = m^K_{RR}\alpha_K - e_{RRK}\beta^K \tag{3.8}$$

$$\frac{1}{2\pi\sqrt{\alpha'}}\hat{F}_2 = m^a_{RR}w_a, \quad \frac{1}{(2\pi\sqrt{\alpha'})^3}\hat{F}_4 = -e_R Ra\tilde{w}^a \tag{3.9}$$

$$2\pi\sqrt{\alpha'}\hat{F}_0 = m^0_{RR}, \quad \frac{1}{(2\pi\sqrt{\alpha'})^5}\hat{F}_6 = e_{RR0}\mathrm{Vol}_6 \tag{3.10}$$

From now on we will neglect all dependence on α', i.e. we will set $2\pi\sqrt{\alpha'} = 1$.

3.2 Why do we need fluxes?

As mentioned before fluxes provide a potential for some of the moduli fields, i.e. give masses to some of the fields. This mechanism was originally proposed by Scherk and Schwarz in [18] where it was also shown that fluxes enable a novel method to break supersymmetry.

The difference to original Kaluza-Klein reductions is that there one only keeps the zero modes of the moduli while truncating those modes dependent on the internal coordinates. This amounts to the expansion of fields in harmonic forms on the internal space.

Let us consider the different scheme proposed by Scherk and Schwarz and improved by others ([**?**]). We will consider for simplicity a toy model [20] where there is only one extra dimension and only one scalar field λ. The action for this model shall be

$$S_\lambda = \int(-\hat{R} * 1 - \frac{1}{2}\hat{d}\hat{\lambda} \wedge *\hat{d}\hat{\lambda}). \tag{3.11}$$

One can find a global symmetry given by

$$\lambda \to \lambda + a, \tag{3.12}$$

where a is a constant. Assuming that the topology of our space is $S^1 \times \mathbb{R}^{1,3}$ we can write for the scalar field the Ansatz

$$\hat{\lambda}(x, y) = \lambda(x) + my, \tag{3.13}$$

where by x we denote the four space-time coordinates and y is a coordinate on the compactified dimension.

One can easily see that the integration of the "field strength" $(\hat{d}\hat{\lambda})_y = \lambda(x) + my$ gives a nontrivial value. Thus we have turned on a "flux".

The next step is to perform the ordinary Kaluza-Klein reduction in order to derive the effective four dimensional theory. The result is

$$S = \int (-R * 1 - \frac{1}{6\phi^2} d\phi \wedge *d\phi - \frac{\phi}{4} F \wedge *F - \frac{1}{2} D\lambda \wedge *D\lambda - \frac{m^2}{2\phi} * 1). \tag{3.14}$$

From this action one can infer several important conclusions. The first observation is that invariance under the global transformation (3.12) in the original theory has turned to a local gauge invariance due to the appearance of the covariant derivative $D\lambda = d\lambda - mA$. A by far more important conclusion is that we have created a potential for the modulus ϕ. Of course this potential is not an ideal one for it does not admit a stable minimum, but it points the way to a completely new research area : flux induced moduli stabilization.

3.3 Supergravity in four dimensions

The last section showed that it is possible to generate a nontrivial potential for the moduli through turning on fluxes. However, in a general supergravity theory, the scalar potential is made up of two ingredients, the superpotential and the Kähler potential.

Recall that the most general Lagrangian that can be built from chiral superfields Φ^i takes the form

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\Phi^i, (\Phi^\dagger)^j + [\int d^2\theta W(\Phi^i) + h.c.]. \tag{3.15}$$

Here K is a polynomial in the chiral superfields Φ^i and their conjugates while W is as usual the superpotential. K is called the Kähler potential and determines the kinetic terms of fields appearing in a chiral multiplet. To be more precise, if we take the scalar components of Φ^i to be A^i their kinetic term is given by

$$K_{ij^*} \partial_m A^i \partial^m A^{*j}, \tag{3.16}$$

where K_{ij^*} is defined as

$$K_{ij^*} = \frac{\partial}{\partial A^i} \frac{\partial}{\partial A^{*j}} K. \tag{3.17}$$

The exact form of the Kähler potential can be derived from explicit Kaluza-Klein reduction and the scalar fields, whose kinetic terms are determined by the Kähler potential, turn out to be the moduli of the compactification manifold. Obtaining the Kaehler potential this way gives the same result as directly taking the Kähler potential of the moduli space belonging to a given compactification geometry. In other words this means that once one knows the topology of the compactification manifold one can take K to be the Kähler potential of its moduli space.

Having explained the notion of Kähler potential we can now write down the scalar potential of a $N = 1,\ D = 4$ supergravity theory

$$V = e^K \left(\sum_{i,j} K^{ij^*} D_i W D_{\bar{j}} W - 3|W|^2 \right). \tag{3.18}$$

The vacuum expectation value of this potential is identical with the cosmological constant. Supersymmetric solutions are given by

$$D_i W = 0 \quad \text{for all } i, \tag{3.19}$$

where D_i is given by

$$D_i = \partial_i + K_i. \tag{3.20}$$

If these conditions are satisfied one can see from the form of the scalar potential that the solution amounts to an anti-de-Sitter space.

Broken supersymmetry arises when one or more equations of the set (2.35) can not be satisfied. In such a case the gravitino aquires a mass

$$m_{\text{gravitino}} =< e^{G/2} >, \tag{3.21}$$

where the brackets denote expectation values and G is defined to be

$$G = K + \log W + \log W^*. \tag{3.22}$$

3.4 Flux induced superpotentials

We have seen through a simple example that turning on fluxes can lead to a nontrivial potential for the moduli. This potential can be calculated by explicit Kaluza-Klein reduction where one integrates over the extra dimensions. Having obtained the scalar potential one goes on to extract from it the superpotential ([22]). Of course this is a way to obtain concrete formulas for the reduced theory, but recent progress has shown that the use of more sophisticated methods can give direct formulas for the superpotential (see [21]).

Collaborated work of both approaches has led to specific formulas for the various string theories. Here we will present the heterotic and the type II theories.

Let us first come to the heterotic string as it admits the easiest formula. Denoting by Ω the holomorphic three form, by M the compactified extra dimensions and using the definitions of the preceding section the superpotential is given by ([23])

$$W_{het} = \int_M \Omega \wedge \hat{H}_3 = m^K \mathcal{G}_K - e_K \mathcal{Z}^K, \tag{3.23}$$

where $\mathcal{G}_K$ and $\mathcal{Z}^K$ are the Calabi-Yau periods defined in appendix A.

One can see that such a superpotential only takes notice of the complex structure moduli encoded in $\mathcal{G}_K$ and $\mathcal{Z}^K$. Also, there is no dependence on the dilaton which will remain a flat direction.

Next, we come to the Type IIB case. Here the corresponding formula is more involved and takes care of the three-form flux $\mathcal{F}_3$:

$$W_{IIB} = \int_M (F_3 - iSH_3) \wedge \Omega = (m^K_{RR} - iSm^K)\mathcal{G}_K - (e^K_{RR} - iSe_K)\mathcal{Z}_K. \tag{3.24}$$

This kind of superpotential takes care of the dilaton and of complex structure moduli, but it still lacks any dependence on the Kähler moduli.

The Type IIA superpotential on the other hand admits dependence on both Kähler and complex structure moduli. It consists of two parts. The part dependent on the complex structure is the same as the heterotic superpotential (3.23). The part dependent on the Kähler moduli is given by

$$\begin{aligned} W^k &= \int_M e^{-\omega_c} \wedge \hat{F} \\ &= e_{RR0} + \int_M \omega_c \wedge \hat{F}_4 \\ &- \frac{1}{2}\int_M \omega_c \wedge \omega_c \wedge \hat{F}_2 - \frac{m^0_{RR}}{6}\int_M \omega_c \wedge \omega_c \wedge \omega_c. \end{aligned}$$

Inserting the definitions in section 3.1 we get

$$W^K_{IIA} = e_{RR0} + e_{RRa}t^a + \frac{1}{2}\kappa_{abc}t^a t^b m^c_{RR} - \frac{m_{RR0}}{6}\kappa_{abc}t^a t^b t^c, \tag{3.25}$$

where κ_{abc} are the intersection numbers of the manifold defined in appendix A.

One can see that dependent on the choice of the compactification manifold there are terms of linear , quadratic and cubic order in the Kähler moduli.

As we will see in the next section flux compactifications provide a framework for relaxing the Calabi-Yau condition and considering more general vacua, i.e. vacua which are not supersymmetric Minkowski. Therefore once relaxing the strict restrictions on the compactified space we can also introduce new terms in the superpotential. Relaxing the

Kähler condition will lead to the following term which will appear as a contribution to all superpotentials discussed above:

$$W = \dots + \int_M \Omega \wedge d\omega_c. \tag{3.26}$$

This additional term is only nonzero if the manifold is non-Kähler and $d\omega_c \neq 0$ which is in turn achieved through the introduction of "metric fluxes". Introducing such a term is one way to make the heterotic and Type IIB superpotential dependent on the Kähler moduli.

Concrete examples of the above formula will be given in the next Chapter where we will also investigate the details of the dependence on Kähler and complex structure moduli.

3.5 Backreaction on geometry

Until now we have neglected an important issue which we will take account of in this section. Once one turns on fluxes - for example through giving a non-vanishing value to a field strength F_p on the internal manifold - this gives rise to a non-vanishing energy-momentum tensor which back reacts on the geometry. Calculations show that this backreaction leads to a non-trivial warp factor ([24], [25]) and in order to obtain supersymmetric solutions one has to consider compactification geometries with torsion. The heuristic Einstein equations in the presence of a p-form flux are

$$R_{mn} \sim (F_p^2)_{mn}. \tag{3.27}$$

From this one can see that in the limit of small fluxes the Calabi-Yau condition is still valid. Nevertheless one has to take into account that the fluxes are quantized according to Dirac quantization conditions (see [26]). If we express the fluxes as $F_p = m_i \omega_p^i$, where ω_p^i are harmonic p-forms on the internal manifold and m_i parameterize the fluxes, the quantization condition implies that there is no continuous limit $m_i = 0$ in string theory. If one keeps in mind that the m_i are integer multiples of the string scale α' then the situation improves in supergravity as supergravity is the $\alpha' \to 0$ limit of string theory. Therefore in the rest of the work we will neglect the backreaction of fluxes.

Now, one might ask to what extend fluxes change the geometry of the four dimensional space time. This question has been investigated in reference [27]. In our presentation we will follow Grana [17]. For the contribution of the internal and space time components of fluxes to the four dimensional energy momentum tensor it was found

$$T_{int} = \frac{p-1}{2p} F_p^2 \geq 0 \tag{3.28}$$

and

$$T_{ext} = \frac{9-p}{2p} F_p^2 \geq 0. \tag{3.29}$$

Therefore all fluxes give strictly positive contributions except for the one-form and nine-form fluxes which give a vanishing contribution.

From this one can show ([17]) that compactifications to de Sitter space are completely ruled out while a Minkowski solution is only allowed in the presence of one-form flux. Furthermore in compactifications leading to anti-de-Sitter space the cosmological constant is directly related to the square of fluxes. However, these conditions can be relaxed if one includes localized sources with negative tension such as orientifold planes. In that case the condition for a Minkowski vacuum is modified and localized sources should obey the following equation

$$\sum_i \int (T_i^{\text{flux}} + T_i^{\text{loc}}) = 0. \tag{3.30}$$

Among the objects which can contribute to the localized sector there are D_p-branes. Investigations show that for $p < 7$ these branes have a posivitve contribution to the stress-energy-tensor. In order to maintain the Minkowski condition in such a case orientifold planes have to be included.

4 Toroidal compactifications

In this chapter we present the easiest setup for flux compactifications following reference [28]. The geometry of the extra dimensions is a factorized torus $T^6 = \otimes_{i=1}^{3} T_i^2$ which will also be important for orbifold compactifications presented later. This example is particularly suited for studying the structure of the Kähler potential and the various superpotentials arising in different string theories. Furthermore, an orientifold projection will be imposed on the type IIA and type IIB models analyzed here, as this is necessary for $N = 1$ supersymmetry (see appendix B). Before proceding we shall first discuss compactifications on T^2 .

4.1 T^2 compactifications

A two dimensional compactification is not a realistic model. Nevertheless, as the six dimensional torus we will study is the product of three two-tori, many aspects of the higher dimensional case can be seen more easily through studying this model.

A two dimensional torus is obtained from identifying points on a lattice in $\mathbb{C}$. This is shown in figure 2.

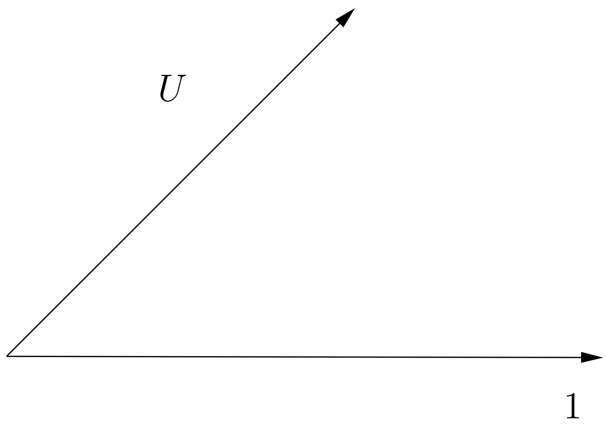


Figure 2: Torus Lattice

The lattice is spanned by complex numbers v_1, v_2 which can also be thought as vectors on $\mathbb{R}^2$. In the picture we have normalized these vectors and taken v_2 to be 1 and v_1 to be U. U is known as the complex structure of the torus and parameterizes the different two-tori. There is also a Kähler modulus which can be thought of as the size of the torus. It turns out that the moduli space associated with this complex structure is $\frac{SU(1,1)}{U(1)}$ and the same result also holds in the case of the complexified Kähler modulus as well as for the dilaton.

The Kähler manifold associated with the moduli space is then

$$M_{T^2} = \left[\frac{SU(1,1)}{U(1)}\right]_S \times \left[\frac{SU(1,1)}{U(1)}\right]_T \times \left[\frac{SU(1,1)}{U(1)}\right]_U . \tag{4.1}$$

The Kähler potential of such a space is given by

$$K = -\log(S + S^*) - \log(T + T^*) - \log(U + U^*). \tag{4.2}$$

In the above formula S stands for the complexified dilaton

$$S = e^{-\phi} + ia, \tag{4.3}$$

where a is the model independent axion arising from the four dimensional two-form $B_{\mu\nu}$.

4.2 The T^6 - Basic setup

As discussed in the last chapter fluxes (and also moduli) are expanded in a basis of harmonic three-forms. In our particular example these forms appear with one leg on each sub-torus and are given by

$$\begin{aligned}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \;\; ; \;\; \beta^0 = dy^1 \wedge 9dy^2 \wedge dy^3, \\
\alpha_1 &= dy^1 \wedge dy^2 \wedge dy^3 \;\; ; \;\; \beta^1 = dy^1 \wedge dx^2 \wedge dx^3, \\
\alpha_2 &= dy^1 \wedge dx^2 \wedge dy^3 \;\; ; \;\; \beta^2 = dx^1 \wedge dy^2 \wedge dx^3, \\
\alpha_3 &= dy^1 \wedge dy^2 \wedge dx^3 \;\; ; \;\; \beta^3 = dx^1 \wedge dx^2 \wedge dy^3,
\end{aligned} \tag{4.4}$$

where $y^i = x^{i+3}$.

The above harmonic forms satisfy the relations

$$\int_{T^6} \alpha_K \wedge \beta^L = \delta_K^L, \tag{4.5}$$

see section 3.1.

We also define closed two forms and their dual four-forms relevant for the expansion of the Kähler form as well as for the two-form and four-form fluxes:

$$\omega_i = -dx^i \wedge dy^i \;\; ; \;\; \tilde{\omega}_i = dx^i \wedge dy^j \wedge dx^k \wedge dy^k \;\; ; \; i \neq j \neq k, \tag{4.6}$$

satisfying $\int_{T^6} \omega_i \wedge \tilde{\omega}_j = \delta_{ij}$.

The complexified Kähler form ω_c is given by the general formula (2.84) whereas the holomorphic three-form is given by

$$\Omega = (dx^1 + iU_1 dy^1) \wedge (dx^2 + iU_2 dy^2) \wedge (dx^3 + iU_3 dy^3). \tag{4.7}$$

Here the U_i are the three complex structure moduli associated with the three sub-tori.

4.3 Kähler potential

The scalar Kähler potential can be obtained by analyzing the moduli space of the compactification geometry.

The factorized torus is a product of three two-tori. Therefore in case of the factorized torus the scalar Kähler manifold associated with the moduli space is

$$M = \left[\frac{SU(1,1)}{U(1)}\right]_S \times \prod_{i=1}^{3} \left[\frac{SU(1,1)}{U(1)}\right]_{T_i} \times \prod_{i=1}^{3} \left[\frac{SU(1,1)}{U(1)}\right]_{U_i}. \tag{4.8}$$

From this manifold one can derive the Kähler potential to be

$$K = -\log(S + S^*) - \sum_{i=1}^{3} \log(T_i + T_i^*) - \sum_{i=1}^{3} \log(U_i + U_i^*). \tag{4.9}$$

4.4 Superpotentials

The following sections will give formulas for the type II superpotentials arising from the given geometry. Superpotentials arising from heterotic compactifications won't be analyzed here as there will be an extra chapter for a heterotic model.

4.4.1 Type IIA superpotential

The model at hand is a IIA orientifold on $T^6/[\Omega_p(-1)^{F_L}\sigma_A]$. Ω_p is the world-sheet parity operator, $(-1)^{F_L}$ is the space-time fermionic number for left-movers and σ_A encodes the action on the geometry as seen in the following. σ_A is defined through its action on the Kähler form and the holomorphic three-form as defined in appendix B. In the special case of the factorized torus this translates to

$$\sigma_A(x^i) = x^i \qquad \sigma_A(y^i) = -y^i. \tag{4.10}$$

From this we can see that there are O6-planes that span space-time and the x^i directions, where by Od-planes we mean d-dimensional planes which stay fixed under the orientifold action.

As discussed in appendix B for type IIA orientifolds a new three-form is introduced which is a modification of the usual holomorphic three-form. In our case its expansion is

$$\Omega_c = C_3 + i\mathrm{Re}(C\Omega) = iS\alpha_0 - i\sum_{i=1}^{3} U_i\alpha_i, \tag{4.11}$$

where C_3 being even under σ_A can be expanded in the α_K.

The NS H_3, the RR forms F_2 and F_6 are odd under the orientifold involution. On the other hand F_0 and F_4 are even. From this one concludes the general expansion of flux in terms of closed forms

$$H_3 = \sum_{L=0}^{3} = -e_L\beta^L; \quad F_0 = m^0_{RR}; \tag{4.12}$$

$$F_2 = \sum_{i=1}^{3} m^a_{RR}\omega_a; \quad F_4 = \sum_{i=1}^{4} -e^a_{RR}\tilde{\omega_a}; \tag{4.13}$$

$$F_6 = e_{RR0}\alpha_0 \wedge \beta_0. \tag{4.14}$$

The Type IIA superpotential can now be obtained using the above definitions and the equations (3.23) and (3.25)

$$\begin{aligned} W \quad = \quad & e_{RR0} - ie_0S - \sum_{i=1}^{3}\left[ie^i_{RR}T_i + ie_iU_i\right] \\ & -m^1_{RR}T_2T_3 - m^2_{RR}T_1T_3 - m^3_{RR}T_1T_2 - im^0_{RR}T_1T_2T_3. \end{aligned} \tag{4.15}$$

One can see that here the dependence on the complex structure moduli and the dilaton is at most linear. If the superpotential had been calculated without orientifold involution then it would have contained terms up to cubic order in the U_i.

4.4.2 Type IIB superpotential

Our example will be a Type IIB orientifold with O3-planes following reference [28]. The model at hand is an orientifold on $T^6/[\Omega_P(-1)^{F_L}\sigma_B]$, where the action of σ_B on the Kähler form and holomorphic three-form is given as $\sigma_B(J) = J$ and $\sigma_B(\Omega) = -\Omega$. This translates to

$$\sigma_B(x^i) = -x^i, \tag{4.16}$$

$$\sigma_B(y^i) = -y^i. \tag{4.17}$$

From this we can see that there are O3-planes spanning space-time. Next, we will determine the allowed fluxes. As the RR three-form is odd under the orientifold involution, its expansion in terms of closed forms is given by

$$F_3 = m^K_{RR}\alpha_K - e_{RRK}\beta^K. \tag{4.18}$$

The NS three-form is also odd under the orientifold involution which implies the expansion

$$H_3 = m^K \alpha_K - e_K \beta^K. \tag{4.19}$$

Using equation (3.24) and the above expansions the superpotential can be determined to be

$$\begin{aligned} W \;=\;& e_{RR0} + i\sum_{i=1}^{3} m_{RRi}U_i \\ & - e_{RR1}U_2U_3 - e_{RR2}U_1U_3 - e_{RR3}U_1U_2 - im_{RR0}U_1U_2U_3 \\ & + S\left[-ie_0 + \sum_{i=1}^{3} m_iU_i + ie_1U_2U_3 + ie_2U_1U_3 + ie_3U_1U_2 - m_0U_1U_2U_3\right]. \end{aligned} \tag{4.20}$$

As one can see this potential is independent on the T_i and the highest order terms are cubic in the U_i. This is somehow complementary to the Type IIA potential which is no surprise as the two theories are related through mirror symmetry.

5 Orbifold compactifications

The torus is the simplest way to compactify the six extra dimensions but is also very restrictive as there are no topologically distinct six dimensional tori. Therefore we will introduce Orbifold compactifications which open a vast area for performing string compactifications leading to very different four dimensional models. Orbifolds are singular limits of Calabi-Yau three-folds and as such they can be topologically very different. In this thesis we will concentrate on toroidal orbifolds because these are the easiest to handle and therefore the most relevant for string phenomenology.

5.1 Orbifolds - A definition

In this section we will give a definition for orbifolds which is especially appropriate for physicists. In mathematics the definition is a more general one. In order to define an orbifold we have to provide three ingredients

- A smooth base manifold Y which is complex
- A finite group G
- The action of the group G on Y $\qquad G : Y \to Y$

Now, we are ready to present our definition. An orbifold is the space arising from identifying points on Y which differ through the action of an element of G. This space will be denoted by Y/G from now on.

In physics the relevant orbifolds [29] are those where the base manifold Y is a six dimensional torus T^6 and the discrete group G is $\mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Z}/(m\mathbb{Z} \times n\mathbb{Z})$.

Here a cyclic group $\mathbb{Z}/n\mathbb{Z}$ will be just denoted by $\mathbb{Z}_N$. The group action on T^6 can be thought of as rotations on each of the three complex planes of the torus. As an example consider the orbifold $T^6/\mathbb{Z}_3$. Choose three sub-tori such that the action of an element g of $\mathbb{Z}_3$ is diagonal with respect to this basis.

The one-cycles of each sub-torus are given by

$$
\begin{aligned}
e_x^a &= \sqrt{2}, \\
e_y^a &= \sqrt{2} e^{2\pi i/3} \qquad \text{for } a = 1, 2, 3.
\end{aligned}
\tag{5.1}
$$

One can think of the e_x^a as vectors in $\mathbb{R}^2$. Complex coordinates can be defined as

$$
z^1 = x^1 + iy^1, \quad z^1 = x^2 + iy^2, \quad z^3 = x^3 + iy^3. \tag{5.2}
$$

The $\mathbb{Z}_3$ group consists of the elements $\{1, \theta, \theta^2\}$ where θ is a rotation by 120. The action of the group on the torus is given by its action on the three complex coordinates

$$\theta^k : z^a \mapsto e^{2\pi i k v^a} z^a, \tag{5.3}$$

where $a = 0, \cdots, 3$ and $k = 0, 1, 2$. v is called the "twist vector" and is given by

$$v = (0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}). \tag{5.4}$$

One easily generalizes this example for the case of the cyclic $\mathbb{Z}_N$ group consisting of the elements

$$\mathbb{Z}_N = \{\theta^k | k = 0, \cdots, N-1\}. \tag{5.5}$$

Here θ is generated using (5.3) through the twist vector

$$v = (\frac{a_1}{N}, \frac{a_2}{N}, \frac{a_3}{N}), \tag{5.6}$$

where $a_1, a_2, a_3 \in \mathbb{Z}$. Product groups $\mathbb{Z}_N \times \mathbb{Z}_M$ are described similarly using

$$\mathbb{Z}_N \times \mathbb{Z}_M = \{\theta_1^k \circ \theta_2^l | k = 0, \cdots, N-1 \text{ and } l = 0, \cdots, M-1\}. \tag{5.7}$$

Orbifolds usually also contain fixed point sets under the orbifold action. These are points which stay fixed modulo lattice translations and are defined through

$$z^a = \theta^k z^a + n_x^a e_x^a + n_y^a e_y^a, \tag{5.8}$$

where $a = 1, 2, 3$ and $n_x^a, n_y^a \in \mathbb{Z}$. As these are points of the underlying torus T^6 we will generally denote the set of all fixed points under a group element $g \in G$ by T_g^6.

5.2 Orbifold Cohomology - The untwisted sector

The cohomology of orbifolds, as motivated by string theoretic calculations, consists of several sectors, the untwisted and the various twisted sectors. By a g-twisted sector we mean the fixed point set under the action of the group element g, i.e. T_g^6. The untwisted sector which arises from the fixed point set under the identity action, is therefore given by the whole torus T^6.

Each sector gives rise to cohomology elements with certain hodge numbers. Here we will be only interested in the untwisted sector, a complete treatment including all sectors will be given in section 7.1. The action of orbifold group elements on differential forms is given by the pullback:

$$dz^a \mapsto e^{2\pi i v^a} dz^a, \quad d\bar{z}^a \mapsto e^{-2\pi i v^a} d\bar{z}^a \tag{5.9}$$

The cohomology which arises from the untwisted sector is defined to consist of those cohomology elements of T^6 which stay invariant under the above orbifold action.

5.3 Conditions for N=1 Supersymmetry

We are interested in the low energy physics arising from orbifold constructions of the heterotic string. As the effective theory should incorporate $N = 1$ supersymmetry the orbifold should obey several conditions which were derived in reference [30]

- The holomorphic three-form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ should be invariant under the orbifold action
- There should be no fix-torus in the first twisted sector

The first condition is equivalent to the statement that the a_i from (5.6) should obey $a_1 + a_2 + a_3 = 0$ mof N. The second condition can be also written as $T^6_\theta \not\supseteq T^2$ in the case of $\mathbb{Z}_N$-orbifolds.

6 A Heterotic Orbifold model with background fluxes

We will present an orbifold model with background fluxes turned on. We will analyze the untwisted sector of this model and try to understand moduli stabilization in this sector. As the heterotic string does not admit perturbative terms dependent on the Kähler moduli we have to find non-perturbative contributions or introduce metric fluxess. Furthermore, there are some constrains on the fluxes, as these will turn out to be structure constants of the underlying gauged supergravity theory.

6.1 N=4 gauged supergravity from the torus

Compactification of the low energy effective action of the Heterotic String on the Torus with various fluxes turned on has been studied very well in [19]. There the bosonic part of the four dimensional effective action has been extracted through methods of ordinary dimensional reduction. A generalized Scherk-Schwarz-mechanism gives masses to moduli fields through turning on H-flux and giving a twist to the Torus. It turns out that the flux parameters have two functions. On the one hand they are the parameters of the effective scalar potential and on the other hand they are structure constants of the effective gauged $N = 4$ supergravity. This puts constrains on the fluxes which are basically Jacobi identities.

The two relevant fluxes are:

1. H-flux:

$$\hat{H}_3 = dB_2 + H_{flux}, \tag{6.1}$$

where H_{flux} is of the form $\bar{H}_{MNP}, \quad N, M, P = 1, ..., 6$

2. metric fluxes:

$$d\eta^P = -\frac{1}{2}\lambda^P_{NM}\eta^M \wedge \eta^N \quad ; \qquad M, N, P = 1, ..., 6 \tag{6.2}$$

where η^P are the tangent 1-forms. The λ^P_{MN} are structure constants of the Lie algebra generated by the isometries Z_M,

$$[Z_M, Z_N] = \lambda^P_{MN} Z_P. \tag{6.3}$$

They satisfy

$$\lambda^P_{[MN}\lambda^S_{R]P} = 0, \tag{6.4}$$

$$\lambda^P_{PN} = 0. \tag{6.5}$$

The metric fluxes contribute to the superpotential as they make the manifold non-Kähler and thus contribute flux terms given by $\int \Omega \wedge d\omega_c$ as was discussed in section 3.4.

6.2 Heterotic on the Orbifold $T^6/\mathbb{Z}_6$

We consider compactification of the heterotic string on the $\mathbb{Z}_{6II}$ orbifold. The lattice vectors of the underlying torus form the root lattice Λ of the Lie algebra $G_2 \times SU(3) \times SO(4)$ which has a $\mathbb{Z}_{6II} = \mathbb{Z}_3 \times \mathbb{Z}_2$ discrete symmetry. The corresponding twist vector is

$$v_6 = \frac{1}{6}(-1, -2, 3). \tag{6.6}$$

The Orbifold is defined as $M = T^6/\mathbb{Z}_{6II}$. This orbifold has no fixed torus in the first twisted sector and therefore fulfills the second condition for $N = 1$ supersymmetry. Furthermore it has two $N = 2$ sectors as there are fixed tori in the twisted sectors corresponding to the θ^2- and θ^3-actions.

In the following we will calculate the cohomology of the orbifold in the untwisted sector.

6.2.1 Invariant 3-cycles

Under the twist action the 6 fundamental 1-cycles corresponding to the lattice vectors of the three T^2 transform as:

$$\begin{aligned} \theta e_1 &= -e_1 - 3e_2 \\ \theta e_2 &= e_1 + 2e_2 \\ \theta e_3 &= -e_3 - e_4 \\ \theta e_4 &= e_3 \\ \theta e_5 &= -e_5 \\ \theta e_6 &= -e_6 \end{aligned} \tag{6.7}$$

We find the following invariant 3-cycles:

$$\begin{aligned} [c_1] &= -e_{136} + e_{146} + 3e_{246} \\ [c_2] &= 2e_{136} + e_{146} + 3e_{236} \\ [c_3] &= -e_{135} + e_{145} + 3e_{245} \\ [c_4] &= 2e_{135} + e_{145} + 3e_{235} \end{aligned} \tag{6.8}$$

6.2.2 Complex Structure Moduli Space

Now we can construct a symplectic basis of harmonic 3-forms invariant under the orbifold action which are linear combinations of forms dual to the above cycles. We get:

$$\begin{aligned}
\alpha_0 &= -\frac{1}{4}dy^1\wedge dy^2\wedge dy^3 - \frac{1}{12}dx^1\wedge dx^2\wedge dy^3 \\
&\quad -\frac{1}{6}dx^1\wedge dy^2\wedge dy^3 \\
&\quad -\frac{1}{4}dy^1\wedge dx^2\wedge dy^3 \\
\beta^1 &= 3(dy^1\wedge dy^2 - dx^1\wedge dx^2 \\
&\quad -dy^1\wedge dx^2)\wedge dy^3 \\
\beta^0 &= (3dy^1\wedge dy^2 + dx^1\wedge dx^2 \\
&\quad +2dx^1\wedge dy^2 + 3dy^1\wedge dx^2)\wedge dx^3 \\
\alpha_1 &= -\frac{1}{4}(dy^1\wedge dy^2 - dx^1\wedge dx^2 \\
&\quad -dy^1\wedge dx^2)\wedge dx^3
\end{aligned} \tag{6.9}$$

These forms are given up to a symplectic rotation and satisfy

$$\begin{aligned}
\int_M \alpha_i\wedge\beta^j &= \delta_i^j, \\
\int_M \alpha_i\wedge\alpha_j &= 0, \\
\int_M \beta^i\wedge\beta^j &= 0.
\end{aligned} \tag{6.10}$$

Now according to equation A.14 in appendix A the complex structure moduli space is a Kähler manifold with Kähler potential

$$K^{ks} = -\log i[\bar{Z}^k\mathcal{G}_k - Z^k\bar{\mathcal{G}}_k], \tag{6.11}$$

with

$$Z^k = \int_M \Omega\wedge\beta^k, \qquad \mathcal{G}_k = \int_M \Omega\wedge\alpha_k, \tag{6.12}$$

where Ω is the holomorphic 3-form.

In our case the holomorphic 3-form is

$$\Omega = dz^1\wedge dz^2\wedge dz^3, \tag{6.13}$$

$$dz^1 = dx^1 + iy^1,$$

$$dz^2 = dx^2 + iy^2,$$

$$dz^3 = dx^3 + iU_3 dy^3. \tag{6.14}$$

(6.15)

One can see that the complex structure moduli are fixed for the first two sub-tori due to the orbifold action. This fits nicely with the observation that the number of $(2,1)$-forms invariant under $dz^i \to e^{2\pi i v_6^i} dz^i$ is exactly one. Ω is also invariant under the twist action and therefore we see that even the second condition for $N = 1$ supersymmetry is fulfilled.

Inserting the realizations of Ω, α_k and b^k into (6.12) and (6.11) a short calculation reveals that K^{ks} is given by

$$K^{ks} = -\log(U_3 + U_3^*). \tag{6.16}$$

6.2.3 Kähler Moduli Space

In order to calculate the Kähler moduli space we need to find the invariant $(1,1)$ forms. In the case of our orbifold these are given by

$$\begin{aligned}\omega_1 &= (\frac{\kappa}{2})^{1/3} i dz^1 \wedge d\bar{z}^1, \qquad (6.17)\\ \omega_2 &= (\frac{\kappa}{2})^{1/3} i dz^2 \wedge d\bar{z}^2,\\ \omega_3 &= \frac{1}{\mathrm{Im}\tau_3}(\frac{\kappa}{2})^{1/3} i dz^3 \wedge d\bar{z}^3.\end{aligned}$$

As described in A the Kähler moduli space is itself a Kähler manifold with Kähler potential

$$K^K = -\log\left[\frac{1}{6}\kappa_{ABC}(T+T^*)^A(T+T^*)^B(T+T^*)^C\right]. \tag{6.18}$$

Here, in order to make later calculations easier, we have made the shift $T \to iT$ in oder to obtain the above formula from equation (A.18).

In our case there is only one non-vanishing intersection number:

$$\int_M \omega_1 \wedge \omega_2 \wedge \omega_3 = \kappa. \tag{6.19}$$

Again a short calculation shows that K^K is equivalent to

$$K^K = -\log(T_1 + T_1^*) - \log(T_2 + T_2^*) - \log(T_3 + T_3^*). \tag{6.20}$$

Combining results shows that the complete Kähler potential is given by

$$K = -\log(T_1 + T_1^*) - \log(T_2 + T_2^*) - \log(T_3 + T_3^*) - \log(U_3 + U_3^*). \tag{6.21}$$

This formula is just a truncation of the formula given in (4.9) which is consistent with the fact that the untwisted sector of our orbifold is a truncation of the ordinary torus examined in section 4.

6.3 The Superpotential

The superpotential is given by formula (3.23) . Now we turn on 3-form flux

$$H_{flux} = m^0\alpha_0 + m^1\alpha_1 - e_0\beta^0 - e_1\beta^1, \tag{6.22}$$

where the m^k, e_k are the quantized flux parameters.

Their contribution to the superpotential is

$$W_{H_3} = m^0 U_3(2i+5) - e_0(\frac{5}{12}i - \frac{1}{6}) + m^1 3i - e_1\frac{i}{4}U_3. \tag{6.23}$$

In order to calculate $d\omega^c$ we have to find out what metric flux parameters are consistent with the orbifold action. From equation (6.3) we can infer that giving non vanishing values to λ^P_{MN} with each index on a different sub-torus is consistent with the orbifold action. Thus the non vanishing metric flux parameters are:

$$\lambda^k_{ij} \text{ with } i = \{1,2\},\ j = \{3,4\},\ k = \{5,6\} \tag{6.24}$$

and permutations of the indices i, j, k.

The exterior derivative of the Kähler-form is then given by

$$\omega^c = \sum_{i=1}^{3} T_i dx^i \wedge dy^i, \tag{6.25}$$

$$(d\omega^c)_{LMN} = \lambda^A{}_{[LM}\omega^c_{N]A}. \tag{6.26}$$

Using (3.26) and writing the resulting contributions in a compact form we get:

$$W_\lambda = \int \Omega \wedge d\omega_c = U_3 d_i T^i + c_i T^i, \tag{6.27}$$

where the d_i, c_i are complex numbers including the metric fluxes.

Combining results , the superpotential is given by:

$$\begin{aligned} W &= \int \Omega \wedge (H_3 + d\omega_c) = W_H + W_\lambda \\ &= m^0 U_3(2i+5) - e_0(\frac{5}{12}i - \frac{1}{6}) + m^1 3i - e_1 \frac{i}{4} U_3 + U_3 d_i T^i + c_i T^i. \end{aligned} \tag{6.28}$$

6.3.1 Gaugino condensates

As one can immediately see there is no dependence on the dilaton in the superpotential (6.28), yet. In heterotic string models this dependence is achieved using the non perturbative method of gaugino condensation. In analogy to QCD, which leads to $q\bar{q}$ condensates, in supergravity gauge fermions can condensate at a scale $<\chi\chi>=\Lambda^3$. As long as Λ is small compared to M_p, we assume that gravity will not qualitatively disturb this dynamical mechanism. If we now assume the formation of such a condensate this will lead to another term in the superpotential

$$W = W' + ke^{-cf}, \tag{6.29}$$

where f is the gauge kinetic function

$$f = \frac{1}{g^2} - i\frac{\theta}{8\pi^2}. \tag{6.30}$$

Here θ is the model independent axion. Furthermore we have $4\pi\frac{1}{g^2} \sim (S+S^*)$ where S is the dilaton field including the axion as the imaginary part. In terms of the one-loop beta function coefficient b the constant c can be expressed as $c = \frac{6\pi}{b}$ and the constant k is just proportional to $\frac{1}{\sqrt{\alpha'}}$ where the proportionality constant is of order one. Therefore the contribution to the superpotential due to gaugino condensation can be expressed as

$$W = W' + \frac{1}{\sqrt{\alpha'}} e^{-\frac{6\pi}{b}S}. \tag{6.31}$$

There is yet another scenario which leads to a T-modulus dependence of the gaugino condensate term. This dependence was postulated in [31] in order to assure invariance of the scalar potential under the $SL(2,\mathbb{Z})$ symmetry

$$T \mapsto \frac{aT - ib}{icT + d}, \qquad ad - bc = 1. \tag{6.32}$$

Imposing such a symmetry modifies the standard gaugino condensate term to be

$$W = W' + k\frac{e^{-cS}}{\eta(T)^6}, \tag{6.33}$$

where $\eta(T)$ is the Dedekind function transforming, as

$$\eta(T) \mapsto \sqrt{icT + c}\ \eta(T) \tag{6.34}$$

under $SL(2,\mathbb{Z})$. This kind of dependence has its origins not only in the duality invariance of the superpotential but also in threshold corrections to gauge couplings [32].

6.3.2 Supersymmetric Solutions

We will investigate supersymmetric solutions of our model due to the superpotential (6.28) where a gaugino condensate will be turned on to obtain a dilaton dependence of the superpotential. As the gaugino condensate term we will take (6.31).

Defining

$$Q := m^0(2i + 5) - e_1\frac{i}{4}, \tag{6.35}$$

$$P := -e_0(\frac{5}{12}i - \frac{1}{6}) + m^1 3i, \tag{6.36}$$

the superpotential is given by

$$W = QU_3 + P + U_3 d_i T^i + c_i T^i + ke^{-cS}. \tag{6.37}$$

As discussed in section (3.3) supersymmetric solutions are given by

$$D_I W = 0, \tag{6.38}$$
$$D_I = \partial_I + (\partial_I K), \tag{6.39}$$
$$K = K^{cs} + K^K. \tag{6.40}$$

In our case these equations translate to:

1) $F_S = W + (S + S^*)ke^{-cS} = 0$

2) $F_{U_3} = W - (U_3 + U_3^*)(Q + d_i T^i) = 0$

3) $F_{T_i} = W - (T_i + T_i^*)(U_3 d_i + c_i) = 0$

Here the various fields are given by

$$S = s + i\sigma, \qquad T_i = v_i + ib_i, \qquad U_3 = u + i\xi. \tag{6.41}$$

From 3) we get

$$W = 2v_i(U_3 d_i + c_i) \quad \forall\ i. \tag{6.42}$$

This can be rewritten in the form

$$2v_1(U_3 d_1 + c_1) = 2v_2(U_3 d_2 + c_2) = 2v_3(U_3 d_3 + c_3). \tag{6.43}$$

These are three complex equations and therefore represent 6 constrains on 5 real parameters (U_3 is complex , v_1, v_2, v_3 are real).

We see that there is no solution in general. Now we put some constrains on the fluxes and consider the situation where

$$d_1 = d_2 = d, \quad c_1 = c_2 = c. \tag{6.44}$$

This modification changes our problem to 2 real equations , 4 real parameters ($v_1 = v_2$), thus fixing all moduli up to two flat directions.

Investigation of the equations 1) and 2) reveals that a solution with some fixed moduli and some flat directions is now possible. Similar superpotentials were investigated in [33] where also supersymmetric solutions, with some flat directions left over, were obtained.

Throughout our analysis we have assumed that the gaugino condensate term is of the shape (6.31). It would be interesting to look at the case where instead (6.33) is employed.

7 The twisted sector of an Orbifold

Until now we have not yet given a complete treatment for orbifolds. All what we have done is to calculate the differential forms which are fixed under the orbifold action. As it turns out these are just the elements of one of the "sectors" of the orbifold - the untwisted sector. In order to define consistently the cohomology the twisted sectors have to be taken into account, too.

7.1 The Orbifold cohomology

In this section we will review the definition of orbifold cohomology first given by Chen and Ruan [34], where we will follow the approach introduced by Fantechi and Goetsche [35]. Our abstract definition will be carried out in three steps. The first step is the construction of a vector space consisting of the various twisted sectors of the orbifold, then a grading is introduced, and as a last step the vector space is modded out by the orbifold action to get the orbifold cohomology.

For the definition the symbols introduced in chapter 5.1 are used. First define the fixed point set with respect to an element $g \in G$ to be those $x \in Y$ for which $gx = x$. It can be shown that this set is a manifold. We can also define fixed sets under subgroups H of G and denote them by $Y^{g_1,\cdots,g_n}$ where the subgroup H is generated by $g_1, \cdots, g_n$.

Define the vector space $H^*(Y, G)$ to be

$$H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g, \mathbb{C}). \tag{7.1}$$

It is important to understand this equation thoroughly. $H^*(Y, G)$ consists of all cohomology elements arising from the fixed point sets of Y. This means that every element of $H^*(T^6, G)$ is a differential form living on one of the fixed point manifolds Y^g. Notice that we have not yet defined a grading which is expressed by the * in the definition, that is a $(1, 0)$-form living on Y^g is not necessarily an element of $H^{(1,0)}(Y, G)$. A grading will be defined next.

Let Y be a manifold of dimension D with the action of a finite group G. For $g \in G$ and $y \in Y^g$, let $\lambda_1, \cdots, \lambda_D$ be the eigenvalues of the action of g on $T_{Y,y}$ (the tangent space of Y at the point y). As G is a finite group the $\lambda_i's$ are roots of unity. Write $\lambda_j = e^{2\pi i r_j}$ where r_j is a rational number in the interval $[0, 1[$. The "age" of g in y is the rational number $a(g, y) := \sum_{j=1}^{D} r_j$. In the special case where Y is the six dimensional torus T^6 and G the cyclic group $\mathbb{Z}_\mathbb{N}$ the $\lambda_i's$ correspond to $e^{2\pi i v_i}$ where v is the twist vector defined in section 5.1.

Now we are ready to define a grading on $H^*(Y, G)$. Let $g \in G$ and let Z be a connected component of Y^g, i.e. if Y^g consists of distinct points than Z may be one of these points. Let $\alpha \in H^i(Z)$ and look at α as an element of $H^i(Y^g)$ which will be denoted by α_g. Now assign to α_g the degree $i+2a(g,Z)$ which means that α_g will be understood as an element of $H^{i+2a(g,Z)}(Y,G)$.

We want to show that this grading prescription respects Poincare duality. To see that note that the fixed point set of g^{-1} is the same as that of g and that

$$a(g,\mathbb{Z}) + a(g^{-1},\mathbb{Z}) = \mathrm{codim}_{\mathbb{C}}(Z \subset Y). \tag{7.2}$$

Therefore, from Poincare duality on a fixed point set Z we see that for each element of $H^i(Z,\mathbb{C})$ there is exactly one element of $H^{\mathrm{dim}Z-i}(Z,\mathbb{C})$ such that $i + \mathrm{dim}Z - i + 2a(g,Z) + 2a(g^{-1},Z) = \mathrm{dim}Y$. This is equivalent to the statement that for each element of $H^{i+a(g,Z)}(Y,G)$ there is a dual element in $H^{\mathrm{dim}Y-i-a(g,Z)}(Y,G)$.

To be able to define the cohomology we need as a further ingredient the action of elements of G on elements of $H^*(Y,G)$. If g,h are two elements of G, then $h(Y^g) = Y^{hgh^{-1}}$. Therefore the action on $H^*(Y,G)$ is defined by

$$h(\alpha_g) := (h_*\alpha)_{hgh^{-1}}, \tag{7.3}$$

where by α_g we denote the element in the g-th direct summand of $H^*(Y,G)$.

As the last step in the chain of our definitions we define the orbifold cohomology of the orbifold Y/G. It is nothing else as the invariant part of $H^*(Y,G)$:

$$H^*(Y/G) := H^*(Y,G)^G. \tag{7.4}$$

Here the generators of $H^*(Y,G)^G$ are of the form

$$\gamma := \sum_{f\in G} f^{-1}(\alpha_g), \tag{7.5}$$

for an $\alpha_g \in H^n(Y^g)$. It can be checked that γ is invariant under the action of a general element $h \in G$.

As all these statements have been formulated in an abstract fashion we will give a concrete example in the following section.

7.2 The Cohomology of $T^6/\mathbb{Z}_4$

The orbifold at hand is the torus T^6 modded out by the action generated through the shift vector $v = \frac{1}{4}(1,1,-2)$. That is, the twist action is multiplication by i on the first two sub-tori of the T^6 and multiplication by -1 on the third sub-torus. Expressing this in complex coordinates the above is equivalent to:

$$\begin{array}{ccc} dz_1 \mapsto idz_1 & dz_2 \mapsto idz_1 & dz_3 \mapsto -dz_3 \\ d\bar{z}_1 \mapsto -id\bar{z}_1 & d\bar{z}_2 \mapsto -id\bar{z}_2 & d\bar{z}_3 \mapsto -d\bar{z}_3 \end{array} \tag{7.6}$$

We will now calculate the Hodge numbers starting with the untwisted sector. This amounts to calculating all differential forms living on T^6 which are invariant under the orbifold action. From the above one can immediately see that there are no invariant one-forms. The invariant two-forms are given by

$$dz_1 \wedge d\bar{z}_1, \quad dz_2 \wedge d\bar{z}_2, \quad dz_3 \wedge d\bar{z}_3, \tag{7.7}$$

$$dz_1 \wedge d\bar{z}_2, \quad d\bar{z}_1 \wedge dz_2. \tag{7.8}$$

$$\Rightarrow h^{1,1}_{untw} = 5, \quad h^{2,0}_{untw} = h^{0,2}_{untw} = 0. \tag{7.9}$$

The invariant three-form are

$$dz_1 \wedge dz_2 \wedge d\bar{z}_3, \quad d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3, \tag{7.10}$$

$$dz_1 \wedge dz_2 \wedge dz_3, \quad d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3. \tag{7.11}$$

$$\Rightarrow h^{2,1}_{untw} = h^{1,2}_{untw} = 1, \quad h^{3,0}_{untw} = h^{0,3}_{untw}. \tag{7.12}$$

All other forms are given by Poincare duality. Therefore we get:

$$h^{2,2}_{untw} = 5, \quad h^{3,1}_{untw} = h^{1,3}_{untw} = 0. \tag{7.13}$$

These are the hodge numbers arising from the untwisted sector. Now we turn our attention to the first twisted sector. In order to calculate the cohomology here we need a picture of the fixed point sets under the various group elements of $\mathbb{Z}_4$. Such a picture is given in the following.

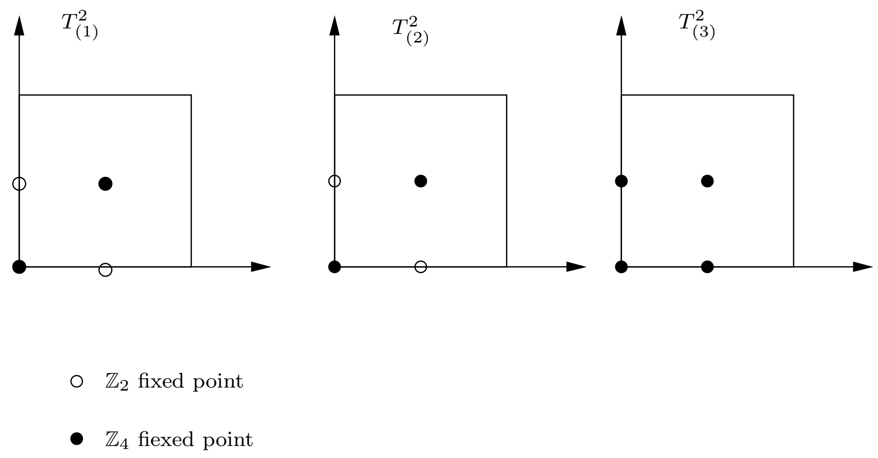


Figure 3: $T^6/\mathbb{Z}_4$

As one can see there are 16 fixed points under the action of $\theta = e^{2\pi i v}$. These are the points

$$T^6_\theta = \{(z_1, z_2, z_3)|z_1, z_2 \in \{0, \frac{1}{2} + \frac{1}{2}i\}, z_3 \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}\}. \tag{7.14}$$

They are marked as black dots in the picture. Now, as these are just points, their cohomology is trivial, i.e. the only non vanishing hodge number is $h^{0,0} = 1$ corresponding to the zero-form 1. This zero-form is invariant under the orbifold action, so each point y of the set T^6_θ gives rise to a cohomology element of the orbifold. In order to determine what hodge number this cohomology element has one has to calculate the age $a(g, y)$:

$$a(\theta, y) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1. \tag{7.15}$$

From this and from the 16 fixed points under the action of θ we see that in the first twisted sector

$$h_\theta^{0+a(\theta,y),0+a(\theta,y)} = h_\theta^{1,1} = 16. \tag{7.16}$$

The third twisted sector is dual to the first one as $\theta^3 = \theta^{-1}$. Therefore $T^6_{\theta^3} = T^6_\theta$. The age can either be calculated in the standard fashion or formula (7.2) may be employed which in our case gives

$$a(\theta^3, y) = 3 - a(\theta, y) = 2, \tag{7.17}$$

as $\text{codim}_{\mathbb{C}}(T^6_\theta \subset T^6) = 3$. This leads to

$$h^{0+a(\theta^3,y),0+a(\theta^3,y)} = h_{\theta^3}^{2,2} = 16, \tag{7.18}$$

which are the Poincare dual cohomology elements.

In the second twisted sector the fixed point set is given by

$$T^6_{\theta^2} = \{(z_1, z_2, z_3)|z_1, z_2 \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}, z_3 \in \mathbb{C}\}. \tag{7.19}$$

This time the connected components of the fixed point set are two-tori. They are of the form $T^2 \times$ point where the points are those with white interior in the picture sitting on the first two sub-tori.

The total number of fixed tori is 16. Nevertheless one observes that 12 of these are mapped to each other pairwise through the $\mathbb{Z}_4$-action. Therefore we see that the θ^2-twisted sector splits up into two sub-sectors - the 12 fixed tori which get mapped into each other and the 4 remaining fixed tori which don't - which we denote by a and b. Each fixed torus j with $j \in \{1, \cdots, 12\}$ contributes to the cohomology through

$$dz_3^{(j)}, \quad , d\bar{z}_3^{(j)}, \quad , dz_3^{(j)} \wedge d\bar{z}_3^{(j)}. \tag{7.20}$$

Now we have to form invariant elements out of these. As stated before for every j there is an i with $i \neq j$ such that $\theta(dz_3^{(j)}) = -dz_3^{(i)}$ and $\theta(dz_3^{(i)}) = -dz_3^{(j)}$. Therefore the invariant elements are given by

$$dz_3^{(j)} - dz_3^{(i)}. \tag{7.21}$$

There are 6 such elements. These contribute to the second twisted sector of the orbifold with the age

$$a(\theta^2, y) = \frac{1}{2} + \frac{1}{2} + 0 = 1, \tag{7.22}$$

which leads to

$$h_{\theta^2}^{1+a(\theta^2,y),0+a(\theta^2,y)^a} = h_{\theta^2}^{2,1^a} = 6. \tag{7.23}$$

Similarly one concludes that the contribution of the $dz^{(j)} \wedge d\bar{z}^{(j)}$ gives the hodge number

$$h_{\theta^2}^{1+a(\theta^2,y),1+a(\theta^2,y)^a} = h_{\theta^2}^{2,2^a} = 6. \tag{7.24}$$

We should also not forget about the trivial cohomology elements which are always present. They give

$$h_{\theta^2}^{0+a(\theta^2,y),0+a(\theta^2,y)^a} = h_{\theta^2}^{1,1^a} = 6. \tag{7.25}$$

Now, only the 4 fixed tori which don't pair up remain. On each of these tori the orbifold action acts as $z_3 \mapsto -z_3$. So the invariant cohomology is the cohomology of the invariant space $T^2/\{\mathbb{Z}_2\} = S^2$. There are no nontrivial one-forms on the sphere and the only cohomology elements remaining are the $(1,1)$-forms. Their contribution is

$$h_{\theta^2}^{1+a(\theta^2,y),1+a(\theta^2,y)^b} = 4. \tag{7.26}$$

We should - as always - not forget about the trivial sector

$$h_{\theta^2}^{0+a(\theta^2,y),0+a(\theta^2,y)^b} = 4. \tag{7.27}$$

Combining the two sectors a and b we get

$$h_{\theta^2}^{1,1} = 10, \quad h_{\theta^2}^{2,1} = 6, \quad h_{\theta^2}^{2,2} = 10. \tag{7.28}$$

All results can be written down in a compact form in terms of hodge diamonds:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 5 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 5 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 16 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 16 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 10 & & 0 & \\
0 & & 6 & & 6 & & 0 \\
 & 0 & & 10 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
$$

untwisted $\qquad\qquad$ θ, θ^3- twisted $\qquad\qquad$ θ^2- twisted

As one can see complex structure moduli, i.e. (2,1)-cohomology elements, can only arise from sectors which contain fixed tori. Nevertheless a fixed torus is not always a guarantee for this: As we have seen in our example some fixed tori have the topology of a sphere when considering the full orbifold action.

We will now examine the ring structure of our orbifold as defined in [35]. We will concentrate on those cohomology elements which are graded by hodge number $(1,1)$ and calculate triple intersection numbers for our orbifold. In other words we will calculate products $\alpha\wedge\beta\wedge\gamma$ where $\alpha,\beta,\gamma\in H^{1,1}(T^6/\mathbb{Z}_4)$. The $(1,1)$-graded part of the cohomology is given by

$$H^{1,1}(T^6)^G \quad\oplus\quad \bigoplus_{y\in T^{6\,\theta}} E^{1,1}(y)_\theta \quad\oplus\quad \bigoplus_{Z\in\{T^{6\,\theta^2}\}^\theta} E^{1,1}(Z)_{\theta^2}. \tag{7.29}$$

Here $H^{1,1}(T^6)^G$ is the untwisted sector and the $E^{1,1}(y)_\theta,\ E^{1,1}(Z)_{\theta^2}$ are the cohomology elements arising from the connected components of the various twisted sectors.

In the following the resulting triple intersections will be presented. A definition of the product and a sample calculation will be given in the appendix C.1.

$$E^{1,1}(y)_\theta\cdot E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(y)_\theta = \mathrm{vol}_6$$

$$E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(Z)_{\theta^2}\cdot dz_3\wedge d\bar{z}_3 = \mathrm{vol}_6$$

$$E^{1,1}(y)_\theta\cdot E^{1,1}(y)_\theta\cdot E^{1,1}(y)_\theta = 0$$

$$E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(Z)_{\theta^2} = 0$$

$$E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(Z)_{\theta^2}\cdot E^{1,1}(y)_\theta = 0$$

$$dz_i \wedge d\bar{z}_i \cdot E^{1,1}(y)_\theta \cdot E^{1,1}(y)_\theta = 0$$

$$dz_i \wedge d\bar{z}_i \cdot E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(y)_\theta = 0$$

As one can see non vanishing results are obtained only in the cases where the different $(1,1)$ cohomology elements arise from sectors g, l, k such that $g \cdots l \cdots k = id$. This is a remarkable result as we will explain now. Recall that in the Calabi-Yau case with standard embedding of the spin connection in the gauge group these triple intersection numbers can be interpreted as topological formulas for Yukawa-couplings [6]. Furthermore one knows from conformal field theory on orbifolds that string-Yukawa-couplings are only non vanishing if the three different strings arise from twisted sectors which satisfy the above product relation, which is known as the point selection rule. This way we have found a topological way to calculate orbifold Yukawa-couplings in the case of the standard embedding of the spin connection $SU(3) \hookrightarrow E_8$.

7.3 Resolution of Orbifold singularities

In the last sections we described how to get the hodge numbers for orbifolds. But we did not say anything about their interpretation. In the general Calabi-Yau case non vanishing $h^{1,1}$- and $h^{2,1}$-numbers can be interpreted as Kähler and complex structure moduli. Kähler moduli, for example, correspond to sizes of nontrivial cycles of the manifold. Where are the cycles of the orbifold to which the twisted $(1,1)$-cohomology elements correspond? It turns out that orbifold fixed points are at the same time singular points where curvature is not any longer defined. There is a prescription to blow up these singularities and obtain nontrivial cycles. Nonetheless, a naive blow-up can still lead to remaining singularities. A subseding sequence of blow-ups which at the end leads to a completely smooth space is called the resolution of the singularity. Resolution of orbifold singularities are best studied in the context of toric geometry ([43], [39]) to which we will turn our attention in the following.

7.3.1 Toric geometry

Toric geometry is the geometry of toric varieties. In order to give a definition for a toric variety we have to clarify the meaning of an algebraic variety first. In our exposition we will concentrate on the case where these varieties are complex, although all definitions can be generalized to the case of an arbitrary algebraically closed field k.

Consider the polynomial ring $\mathbb{C}[X_1, \cdots, X_n]$ which can be viewed as $\mathbb{C}$-valued functions on $\mathbb{C}^n$. Now, for each subset S of $\mathbb{C}[X_1, \cdots, X_n]$ one can define its zero locus by

$$Z(S) = \{v \in \mathbb{C}^n | f(v) = 0 \forall f \in S\}. \tag{7.30}$$

Generally, one calls a set V of $\mathbb{C}^n$ an affine algebraic set if $V = Z(S)$ for some S. Furthermore, a nonempty affine algebraic set V is called "irreducible" if it cannot be written as the union of two proper algebraic subsets. An irreducible affine algebraic set is called an "affine variety".

Vice versa, given a subset V of $\mathbb{C}^n$, we call the set of functions which vanish on V the ideal $I(V)$ of V:

$$I(V) = \{f \in \mathbb{C}[X_1, \cdots, X_n] | f(v) = 0 \forall v \in V\}. \tag{7.31}$$

Define the coordinate ring of V to be the quotient of the polynomial ring by this ideal:

$$\mathbb{C}[V] = \mathbb{C}[X_1, \cdots, x_n]/I(V). \tag{7.32}$$

The above definitions can be taken over to the projective case once we recognize that homogeneous polynomials in $\mathbb{C}[X_0, X_1, \cdots, X_n]$ can be viewed as $\mathbb{C}$-valued functions on $\mathbb{P}^n$. Here affine algebraic varieties become projective ones and affine algebraic sets projective sets.

A toric variety X is a variety which contains the algebraic torus $T = \mathbb{C}^* \times \cdots \mathbb{C}^*$ as a dense open subset. Furthermore, the natural action of T on itself can be extended to an action on X. One can show that the points in $X - T$ can be regarded as limit points for the action of T on itself. The resulting space is partially or sometimes even completely compactified.

A nice property of toric varieties is, which is also the reason what makes them so interesting, their relation to simplicial geometry. In toric geometry a variety is described through simplicial cones and their fans. More precisely a n-dimensional simplicial cone describes a n-dimensional toric variety and the sub-cones describe open subspaces of the variety.

Let us now come to some definitions. By a cone we mean a "convex polyhedral cone" which is a set

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s \in V : r_1 \geq 0\}, \tag{7.33}$$

where the $v_1, \cdots, v_s$ form a finite set of vectors in V. The "rays" of these vectors consisting of positive multiples of them are called "generators" for the cone σ. In simplicial geometry one can define another fundamental object, intimately related to σ, this is the dual cone σ^*:

$$\sigma^* = \{u \in V^* | < u, v > \;\; \geq 0 \;\; \forall\, v \in \sigma\}. \tag{7.34}$$

Here V^* is the vector space dual to V. One of the most important properties of this duality transformation is the identity

$$(\sigma^*)^* = \sigma. \tag{7.35}$$

The other main ingredient for describing toric varieties in terms of simplicial geometry is the lattice N. This is a lattice isomorphic to $\mathbb{Z}^n$ for some n in which the above "convex polyhedral cones" are embedded as "strongly convex rational polyhedral cones" and form a "fan" Δ in N. "Rational" means here that the cones are generated by vectors in the lattice, and "strong" convexity that it contains no line through the origin. The definition of a "fan" is a little bit more involved as we have to introduce the notion of "faces" of a cone first. A face τ of σ is the intersection of σ with any supporting hyperplane: $\tau = \sigma \cap u^\perp = \{v \in \sigma | < u, v >= 0\}$ for some u in σ^*. Now a fan is a collection of cones such that every face of a cone in Δ is also a cone in Δ , and the intersection of two cones in Δ is a face of each.

Similar to the definition of the dual cone one can also define a dual lattice M given by

$$M = Hom(N, \mathbb{Z}), \tag{7.36}$$

and the dual pairing is denoted $<,>$. One can think of M as those vectors in $\mathbb{R}^n$ whose scalar products with all vectors $v \in N$ are integer numbers. The intersection of the dual cone with the dual lattice is a commutative semigroup

$$S_\sigma = \sigma^* \cap M = \{u \in M | < u, v > \geq 0 \ \ \forall \ \ v \in \sigma\}. \qquad (7.37)$$

The elements of S_σ correspond to monomials which form the coordinates of the "affine variety"

$$U_\sigma = \mathbb{C}[S_\sigma]. \qquad (7.38)$$

$\mathbb{C}[S_\sigma]$ is a "group ring", which is a commutative $\mathbb{C}$-algebra. This complex vector space has a basis χ^u, with u an element of S_σ. The multiplication is traced back to the addition in S_σ:

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}. \qquad (7.39)$$

If u is the vector (a_1, a_2, a_3) in $\mathbb{Z}^3$, for example, one can think of χ^u as $x_1^{a_1} x_2^{a_2} x_3^{a_3}$. A general commutative $\mathbb{C}$-algebra A can be written as $\mathbb{C}[x_1, \cdots, x_m]/\mathrm{I}$ where I is an ideal generated by the constrains (7.39). The geometric space, called affine variety, associated with A is the variety $V(I)$ of common zeros of the polynomials in I. It is also possible to view $V(I)$ as the space where the polynomial functions of A are well defined.

It is now time to consider some simple examples. Take the lattice N to be $\mathbb{Z}^3$ generated by $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. In this simple case the dual lattice M is same the lattice as N generated by the same vectors. The origin $\{0\}$ is the simplest cone in our lattice N. The dual cone is all of M, the generators are $\pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1)$. Therefore we obtain the following group algebra

$$\mathbb{C}[M] = \mathbb{C}[X, X^{-1}, X, X^{-1}, Z, Z^{-1}]. \qquad (7.40)$$

This is the coordinate ring for the algebraic torus : $U_{\{0\}} = T = (\mathbb{C}^*)^n$, as this is the space on which $\mathbb{C}[M]$ is well defined.

Consider another example. Take the lattice N to be $\mathbb{Z}^2$ and σ shall be a cone in N generated by $2e_1 - e_2$. The cone and its dual are shown in picture 4.

One can see that the generators of S_σ are $e_1^*, e_1^* + e_2^*$ and $e_1^* + 2e_2^*$ from which it follows that

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, Y^2] = \mathbb{C}[U, V, W]/(V^2 - UW).$$

From this we see that U_σ is the zero set of $(V^2 - UW)$ in $\mathbb{C}^3$ which is a conic singularity (Picture 5).

We now turn to some more complicated examples which consist of several cones joining together to form the fan Δ. Each maximal cone corresponds to an affine variety which together form a covering of the toric variety $X(\Delta)$. Here the set of maximal cones is the set of highest dimensional cones. The fact that these affine varieties fit together nicely to form a covering is due to the possibility for two cones to share a face. In fact, if τ is a face of σ, then S_τ contains S_σ, which implies that $\mathbb{C}[S_\sigma]$ is a subalgebra of $\mathbb{C}[S_\tau]$. This in

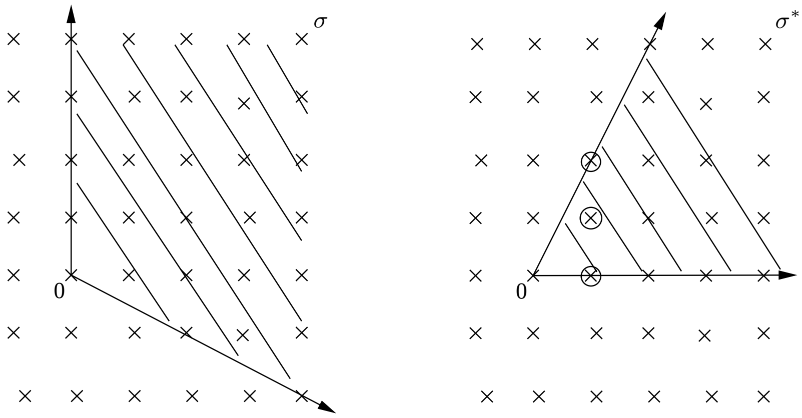


Figure 4: Toric description of a singularity

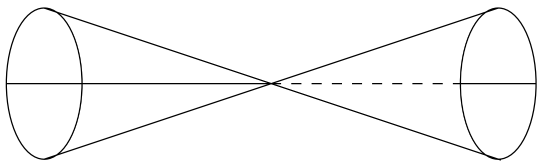
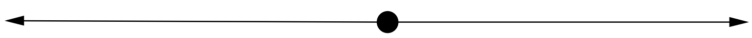


Figure 5: Conic singularity

return gives a map $U_\tau \to U_\sigma$ which makes the gluing procedure between two neighboring cones possible.

As a simplest example look at the simplicial cone Δ consisting of the cones $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. The affine varieties represented by these cones are $\mathbb{C}$, $\mathbb{C}$, and $\mathbb{C}^*$.

The gluing procedure gives the toric variety corresponding to the fan formed by these three cones:



$$\begin{array}{ccccc} \mathbb{C}[X^{-1}] & \hookrightarrow & \mathbb{C}[X, X^{-1}] & \hookleftarrow & \mathbb{C}[X] \\ \mathbb{C} & \hookleftarrow & \mathbb{C}^* & \hookrightarrow & \mathbb{C}^* \end{array}$$

Figure 6: $\mathbb{P}^1$

The patching isomorphism is given by $X \mapsto X^{-1}$ on the overlap which identifies the variety to be $\mathbb{P}^1$.

Now, let's have a look at a two dimensional example which is given by the fan in picture 7.

This example represents the blow-up up $\mathbb{C}^2$ at the origin. Notice, that a blow-up can be performed even at non-singular points on a complex manifold. In the case of $\mathbb{C}^2$ the

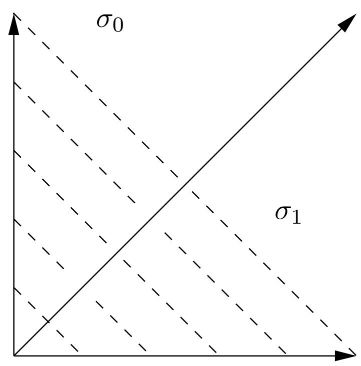


Figure 7: Blwo-up of $\mathbb{C}^2$ at 0

blow-up at the origin is defined to be the sub-variety of $\mathbb{C}^2 \times \mathbb{P}^1$ defined by the equation $XT_1 = YT_0$. Here T_0 and T_1 are homogeneous coordinates of $\mathbb{P}^1$. The open cover of this variety is given by two varieties U_0 and U_1 where T_0 and T_1 are nonzero. Both sets are isomorphic to $\mathbb{C}^2$. The coordinates of the sets are given by X and $T_1/T_0 = X^{-1}Y$ (on U_0), and Y and $T_0/T_1 = XY^{-1}$ on U_0. In fact, this coincides with the toric construction as $U_{\sigma_0} = \mathrm{Spec}(\mathbb{C}[X, X^{-1}Y]) = \mathbb{C}^2$ and $U_{\sigma_1} = \mathrm{Spec}([Y, XY^{-1}]) = \mathbb{C}^2$.

As a last example we will present the toric construction for $\mathbb{P}^2$ as this will appear as a blow-up-cycle in the resolution of orbifold singularities examined in the next section. The fan is two-dimensional and has the shape

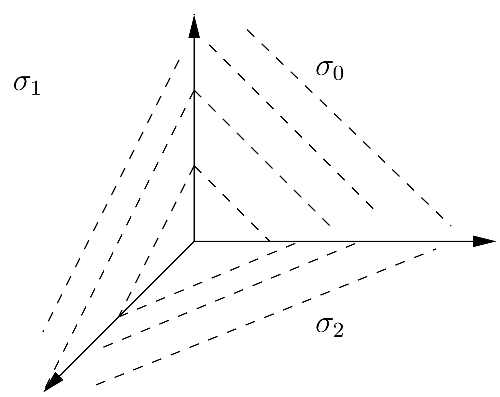


Figure 8: $\mathbb{P}^2$

The dual cones are constructed through rotating the rays forming the borders of each cone by 90 degrees in opposite directions. This way we obtain dual cones in $M = \mathbb{Z}^2$ shown in figure 9. The coordinates for the various U_{σ_i} are (X, Y) for σ_0, $(X^{-1}, X^{-1}Y)$ for σ_1, and (Y^{-1}, XY^{-1}) for σ_2. On $\mathbb{P}^2$ with homogeneous coordinates $(T_0 : T_1 : T_2)$ open subsets U_i are given by $T_i \neq 0$. Dividing by this nonzero coordinate one obtains coordinates for the U_i: T_1/T_i, T_2/T_i and T_3/T_i. If one defines $X = T_1/T_0$ and $Y = T_2/T_0$

one can easily realize that our toric construction coincides with $\mathbb{P}^2$.

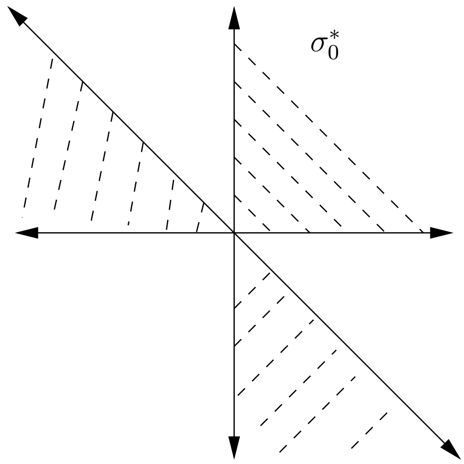


Figure 9: Dual lattice for $\mathbb{P}^2$

7.3.2 Resolution of singular Varieties

As we have seen toric geometry can describe smooth spaces as well as singular ones. Now we will ask for the difference of these two. Is there a way to distinguish between singular and smooth spaces just through the inspection of their toric diagrams? The answer is yes. The condition for non-singularity is that σ^* should not have more than n edges for a n dimensional cone and that the minimal generators along these edges must generate S_σ. In such a case U_σ is isomorphic to $\mathbb{C}^n$.

For a singular example look again at figure 4. There one can beautifully see that the minimal generators along the edges of σ^* do not generate the whole lattice M which is only generated if one includes the marked point in the middle. In order to resolve this singularity one introduces a ray in the original cone in such a way that the minimal generators along the edges of each sub-cone in the new fan generate the interior lattice points. This way we obtain two two-dimensional cones forming a fan together with their faces. The new ray can be interpreted as adding up a divisor, that is, a hypersurface to our geometry.

This procedure will be now applied in the context of orbifold singularities. Our main aim will be to calculate the complete resolution for the $\mathbb{Z}_4$ orbifold.

The $\mathbb{Z}_4$ also contains a $\mathbb{Z}_2$-twisted sector whose singularities are locally of the form $\mathbb{C}^2/\mathbb{Z}_2$. The easiest way to describe such a singularity in terms of toric geometry is to start with the dual lattice M and the dual cone σ^*.

In order to determine the lattice points of M we have to find all monomials invariant under the twist action. In the $\mathbb{Z}_2$-case these are

$$X_1^2, X_1X_2, X_2^2, \frac{X_1^3}{X_2}, \cdots \tag{7.41}$$

Therefore the invariant ring is $\mathbb{C}[X_1, X_2]^{\mathbb{Z}_2} = \mathbb{C}[X_1^2, X_1X_2, X_2^2]$ which corresponds to the singularity $\mathbb{C}^2/\mathbb{Z}_2$:

$$\mathbb{C}[X_1, X_2]^{\mathbb{Z}_2} = \mathbb{C}[X, Y, Z]/(Z^2 - XY) \sim \mathbb{C}^2/\mathbb{Z}_2. \tag{7.42}$$

The dual cone is generated by the rays $(2, 0)$ and $(0, 2)$. Therefore we obtain the following picture

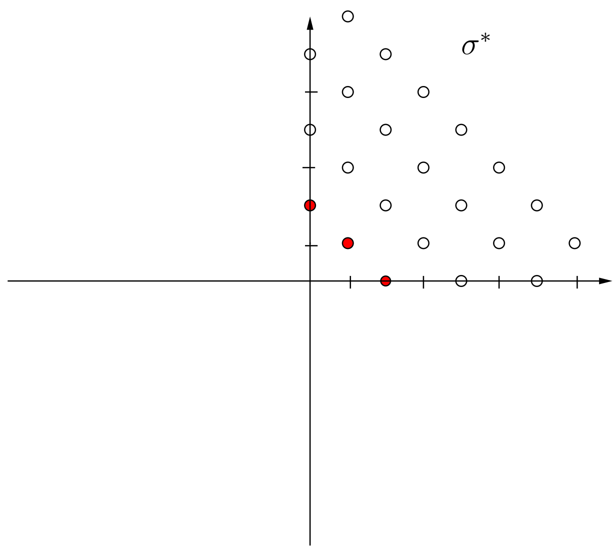


Figure 10: $\mathbb{C}^2/\mathbb{Z}_2$

The lattice N is found by dualizing the lattice M, i.e. finding all vectors whose scalar product with the M-lattice-vectors is an integer. In our case the cone σ is the same as σ^* and this way we find the generators of $N \cap \sigma$ to be

$$(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2}). \tag{7.43}$$

The minimal generators along the edges of σ do not span the lattice N. This represents the singularity. In order to resolve it, one adds a ray through the final generator $(\frac{1}{2}, \frac{1}{2})$ and obtains two cones σ_1 and σ_2 this way.

Dualizing back this setup, one obtains two cones σ_1^* and σ_2^* with

$$\sigma_1^* \cap M = \mathbb{Z}_{\geq 0}m_1 + \mathbb{Z}_{\geq 0}m_2, \tag{7.44}$$

where $n_1 = (1, -1), \quad n_2 = (0, 2)$ and

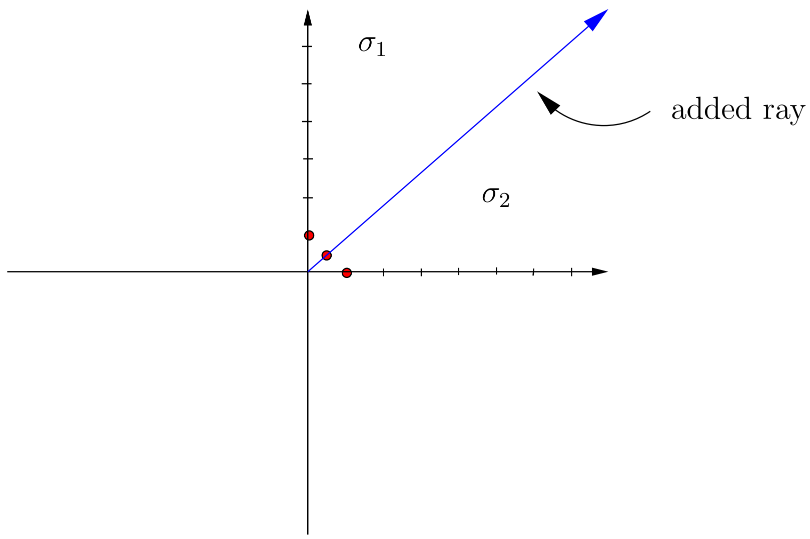


Figure 11: Resolution of $\mathbb{C}^3/\mathbb{Z}_3$

$$\sigma_2^* \cap M = \mathbb{Z}_{geq0} m_3 + \mathbb{Z}_{\geq 0} m_4, \tag{7.45}$$

where $m_3 = (-1, 1)$, $m_4 = (2, 0)$.

This translates to two charts:

$$U_{\sigma_1} = \{\frac{X_1}{X_2}, X_2^2\}, \quad U_{\sigma_2} = \{\frac{X_2}{X_1}, X_1^2\}. \tag{7.46}$$

If we now define

$$V_1 := \frac{X_1}{X_2}, \qquad W_1 := X_2^2, \tag{7.47}$$

$$V_2 := \frac{X_2}{X_1}, \qquad W_2 := X_1^2, \tag{7.48}$$

we get $U_1 \equiv \mathbb{C}[V_1, W_1]$, $U_2 \equiv \mathbb{C}[V_2, W_2]$, and

$$V_2^2 W_2 = W_1, \quad V_1^2 W_1 = W_2, \quad V_1 = V_2^{-1}. \tag{7.49}$$

These are the transition functions of the line bundle $\mathcal{O}(2)$ over $\mathbb{P}^1$. From this we see that the exceptional divisor of this resolution is $\mathbb{P}^1$.

Before proceeding to the $\mathbb{C}^3/\mathbb{Z}_4$ singularity it is convenient to analyze the $\mathbb{C}^3/\mathbb{Z}_3$ case as this is a classical three dimensional example.

The $\mathbb{Z}_3$-action is given by the twist vector

$$v = \frac{1}{3}(1, 1, 1). \tag{7.50}$$

The invariant monomials are therefore

$$X_1^3, X_2^3, X_3^3, X_1^2X_2, X_1^2X_2, X_2^2X_1, X_3^2X_1, X_3^2X_2, X_1^2X_3, X_2^2X_3, X_1X_2X_3, \tag{7.51}$$

and all products of these. The generating lattice vectors for M are then

$$(3,0,0), \quad (0,3,0), \quad (0,0,3), \quad (2,1,0), \quad (2,0,1), \quad (1,2,0), \tag{7.52}$$

$$(0,2,1), \quad (1,0,2), \quad (0,1,2), \quad (1,1,1). \tag{7.53}$$

The simplicial geometry of the corresponding toric variety is :

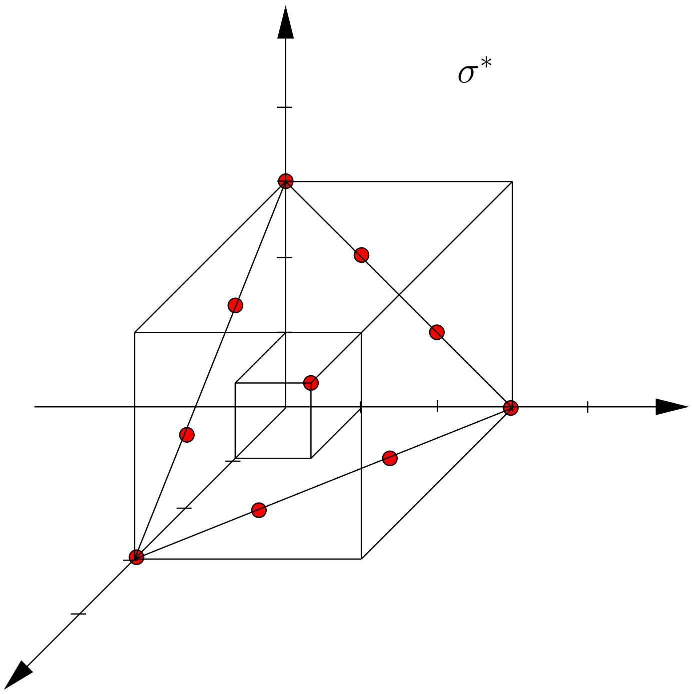


Figure 12: $\mathbb{C}^3/\mathbb{Z}_3$

The dual lattice N is spanned by the vectors

$$(1,0,0), (0,1,0), (0,0,1), \frac{1}{3}(1,1,1), \tag{7.54}$$

the first three of which span the dual cone σ. Again, the lattice is not spanned by the minimal generators along the three edges of σ. In order to cure the problem we paint a ray through the lattice generator $\frac{1}{3}(1,1,1)$ and subdivide the cone into three sub-cones σ_1, σ_2, σ_3:

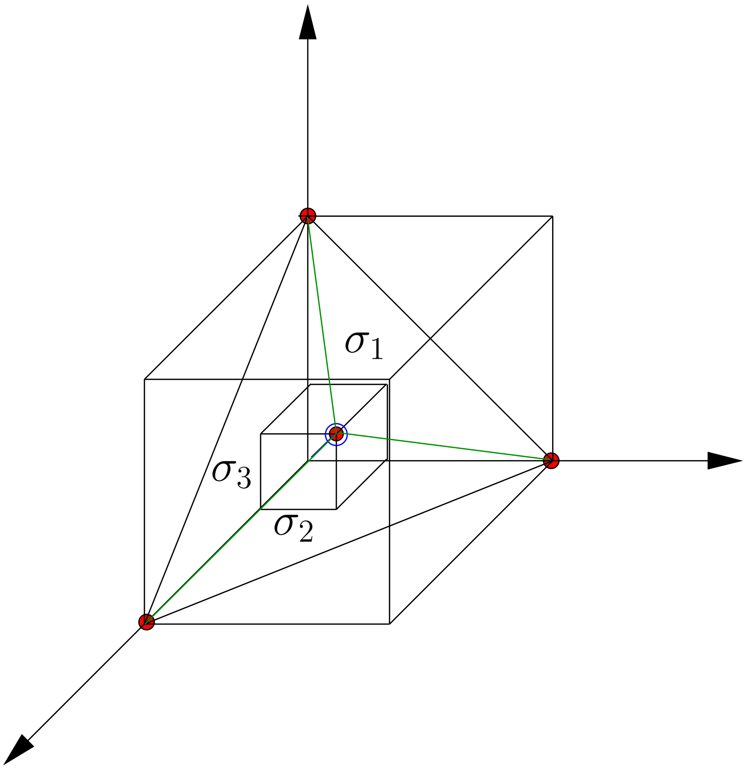


Figure 13: Resolution of $\mathbb{C}^3/\mathbb{Z}_3$

Having obtained the blow-up in the lattice N we can dualize back to the lattice M in order to calculate the transition functions.

It turns out that σ_1^* is generated by

$$(0,-1,1),\ (0,3,0),\ (1,-1,0), \tag{7.55}$$

σ_2^* is generated by

$$(0,0,3),\ (0,1,-1),\ (1,0,-1), \tag{7.56}$$

and σ_3^* is produced by

$$(-1,0,1),\ (3,0,0),\ (-1,1,0). \tag{7.57}$$

Thus, the three charts are

$$U_{\sigma_1} = \{\frac{X_3}{X_2}, X_2^3, \frac{X_1}{X_2}\}, \tag{7.58}$$

$$U_{\sigma_2} = \{X_3^3, \frac{X_2}{X_3}, \frac{X_1}{X_3}\}, \tag{7.59}$$

$$U_{\sigma_3} = \{X_1^3, \frac{X_2}{X_1}, \frac{X_3}{X_1}\}. \tag{7.60}$$

These represent the line bundle $\mathcal{O}(3)$ over $\mathbb{P}^2$, so the exceptional divisor is $\mathbb{P}^2$.

Now we come to our final task: the resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity. In this case, the group action is

$$v = \frac{1}{4}(1, 1, 2), \tag{7.61}$$

from which the invariant monomials follow to be

$$X_1^4, X_1^3X_2, X_1^2X_2^2, X_1^2X_3, X_1X_2X_3, \tag{7.62}$$

$$X_2^4, X_2^3X_1, X_2^2X_3, X_3. \tag{7.63}$$

As vectors these are:

$$(4, 0, 0),\ (3, 1, 0),\ (2, 2, 0),\ (2, 0, 1),\ (1, 1, 1),\ (0, 4, 0), \tag{7.64}$$

$$(1, 3, 0),\ (0, 2, 1),\ (0, 0, 2). \tag{7.65}$$

The corresponding simplicial geometry is

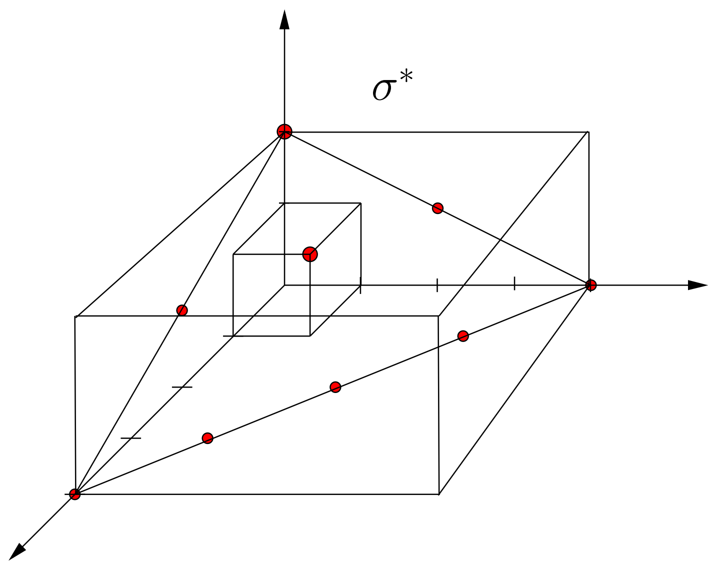


Figure 14: $\mathbb{C}^3/\mathbb{Z}_4$

Through the usual procedure the dual lattice N is obtained:

$$(1,0,0),\ (0,1,0),\ (0,0,1),(\frac{1}{4},\frac{1}{4},\frac{1}{4}),(\frac{1}{2},\frac{1}{2},0), \tag{7.66}$$

and linear combinations. Subdividing the cone σ through the insertion of a ray , three new cones arise:

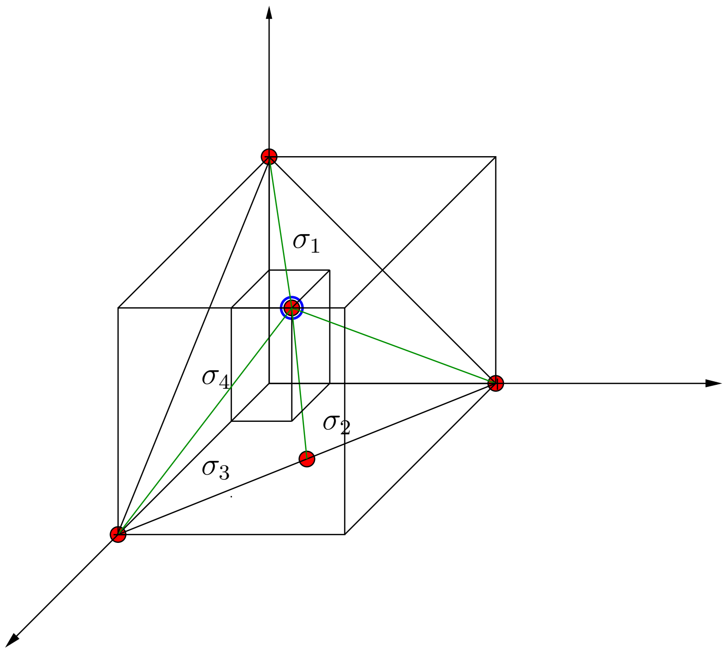


Figure 15: Resolution of $\mathbb{C}^3/\mathbb{Z}_4$

Here, one can see that only one way of triangularization is possible. Therefore, there in only one way to resolve the singularity. In a general setting this is not necessarily always true which implies that generally there are many ways to blow up a singularity.

Dualizing back, σ_1^* is generated by

$$(0,-2,1),\ (1,-1,0),\ (0,4,0), \tag{7.67}$$

σ_2^* by

$$(1,-1,0),\ (0,0,2),\ (0,2,-1), \tag{7.68}$$

σ_3^* by

$$(-1,1,0),\ (0,0,2),\ (2,0,-1), \tag{7.69}$$

and σ_4 is given by

$$(-1,1,0),\ (4,0,0),\ (-2,0,-1). \tag{7.70}$$

We get four patches:

$$U_{\sigma_1} = \{X_2^4, \frac{X_3}{X_2^2}, \frac{X_1}{X_2}\}, \tag{7.71}$$

$$U_{\sigma_2} = \{\frac{X_1}{X_2}, X_3^2, \frac{X_2^2}{X_3}\}, \tag{7.72}$$

$$U_{\sigma_3} = \{X_3^2, \frac{X_2}{X_1}, \frac{X_1^2}{X_3}\}, \tag{7.73}$$

$$U_{\sigma_4} = \{\frac{X_2}{X_1}, X_1^4, \frac{X_3}{X_1}\}. \tag{7.74}$$

The exceptional divisor in this case cannot be seen at first glance. We assign to each patch U_{σ_i} the coordinates u_i, v_i, w_i. For example, for the first patch this means

$$X_2^4 = u_1, \frac{X_3}{X_2^2} = v_1, \frac{X_1}{X_2} = w_1. \tag{7.75}$$

Now, from the invariant monomials one can see that the singularity is described by the following 6 equations in $\mathbb{C}^9$:

$$\begin{aligned}
a_1a_9 - a_4^2 &= 0 && (7.76)\\
a_2a_7 - a_1a_6 &= 0 && (7.77)\\
a_3^2 - a_1a_6 &= 0 && (7.78)\\
a_3a_9 - a_5^2 &= 0 && (7.79)\\
a_3a_9 - a_5^2 &= 0 && (7.80)\\
a_6^2 - a_6a_9 &= 0 && (7.81)
\end{aligned}$$

These equations can be written in a compact way as

$$\vec{A}(a_1, \cdots, a_9) = 0. \tag{7.82}$$

This surface is singular in $(0, \cdots, 0)$:

$$D\vec{A}(a_1, \cdots, a_9)|_{a_1=\cdots=a_9=0} = 0. \tag{7.83}$$

The blown-up space can be described through several charts U_{σ_i} with coordinates u_i, v_i, w_i. For U_{σ_1} we have:

$$a_1 = u_1 \cdot w_1^4, \quad a_2 = u_1 \cdot w_1^3, a_2 = u_1^2 w_1^2 \tag{7.84}$$

$$a_4 = w_1^2 \cdot v_1 \cdot u_1, \quad a_5 = u_1 v_1 w_1, \tag{7.85}$$

$$a_6 = u_1, \quad a_7 = u_1 w_1^2, \quad a_8 = u_1 v_1, \quad a_9 = u_1 v_1^2. \tag{7.86}$$

As the exceptional divisor is the hypersurface which substitutes the point 0 in the blow-up-geometry , we see that in our case it is described by setting $u_1 = 0$. Therefore v_1, w_1 are the coordinates for the exceptional divisor.

Similarly one obtains the divisor in the other patches. At the end we arrive at the four charts

$$U_4^E \quad (\frac{X_2}{X_1}, \frac{X_3}{X_1^2}) \leftrightarrow \quad (\frac{X_1}{X_2}, \frac{X_3}{X_2^2}) \quad U_1^E \tag{7.87}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$U_3^E \quad (\frac{X_2}{X_1}, \frac{X_1^2}{X_3}) \leftrightarrow \quad (\frac{X_1}{X_2}, \frac{X_2^2}{X_3}) \quad U_2^E \tag{7.88}$$

From this we see that our exceptional divisor is the Hirzebruch surface $\mathbb{F}_{-2}$.

7.4 McKay correspondence

We would like to emphasize here that until now we have followed two strategies for constructing the cohomology of orbifolds. In the first section we calculated the hodge numbers through the abstract notion of orbifold cohomology developed by Chen and Ruan. In fact the definitions employed there can be traced back to string theoretic calculations made by Witten, Vafa, Harvey, Dixon in [29] and by Zaslow in [36]. On the other hand methods of toric geometry were used to construct explicit resolutions of the singularities. Using our results of the preceding section one can calculate hodge numbers of the resolved orbifold [37], which is a Calabi-Yau, obtaining the same hodge diamond as in section 7.2. This is a fascinating result: String theory somehow seems to “know” the geometry of the resolved orbifold. This correspondence between the string theoretic calculations and those of algebraic geometry is called McKay correspondence, see [38]. The correspondence is claimed to be even more far reaching as it shall be possible to use string correlation functions to calculate cup products and intersection numbers of the resolved orbifold. However, this has only been proven for HyperKähler manifolds like $K3$-surfaces. A $K3$ two-fold can be obtained as the resolution of $T^4/\mathbb{Z}_2$. Calculating the intersection numbers of this orbifold with the methods introduced in appendix C.1 gives the same result (up to isomorphism) as the algebro-geometric calculation on a $K3$. In the more general context of Calabi-Yau spaces the conjecture is that an isomorphism between the cohomology of the resolved Calabi-Yau and that of the orbifold exists once in the calculation once takes into account all “quantum corrections”.

8 Conclusions

It is time to draw some conclusions. In this thesis I investigated orbifold compactifications in the presence of background fluxes. As was shown in section 6 neglecting backreaction of fluxes on the geometry it possible to derive a superpotential depending on the untwisted moduli in the $T^6/\mathbb{Z}_{6II}$ case. The construction there can be repeated for many other toroidal orbifolds as it only makes use of differential forms in the untwisted sector of the orbifold cohomology.

In sectoin 7 we tried to gain some contronl over the twisted sector. Through the use of the abstract notion of orbifold cohomology it was possible to derive the hodge numbers of the $T^6/\mathbb{Z}_4$ orbifold. These can be interpreted as the moduli (complex structure and Kähler) of the Calabi-Yau manifold whose singular limit is our orbifold. This is due to the fact that in our $\mathbb{Z}_4$-orbifold there was always only one way to blow up the singularities. In a general setting many toric diagrams can represent the resolution of the same singularity, so that many Calabi-Yau spaces can arise with equal singular limits.

Applying the results to flux compactifications it is now possible to calculate the Kähler potential A.18 for the Kähler moduli as the intersection numbers κ_{ABC} can be calculated using toric diagrams of resolved singularities [37]. Also, type IIA superpotnetials are easy to handle as the part depending on the Kähler moduli is dependent on the compactification geometry only through these intersection numbers. On the other hand, orbifolds containing complex structure moduli in the twisted sectors are hard to handle, as in such a case the holomorphic three-form on the resolved Calabi-Yau has to be constructed.

A Moduli Space of Calabi-Yau Manifolds

A.1 The Metric

The discussion we present mainly follows the line of [40]. Due to Yau's theorem the parameter space of Calabi-Yau manifolds can be considered to be the parameter space of Ricci-flat Kähler metrics. Let g_{mn} and $g_{mn} + \delta g_{mn}$ be two Ricci-flat metrics for M, where δg_{mn} is a perturbation. From

$$R_{mn}(g) = 0, \quad R_{mn}(g + \delta g) = 0, \tag{A.1}$$

and the gauge condition

$$\nabla^n \delta g_{mn} = 0 \tag{A.2}$$

one can follow that δg_{mn} satisfies the Lichnerowicz equation

$$\nabla^k \nabla_k \delta g_{mn} + 2R_m{}^P{}_n{}^q \delta g_{pq} = 0, \tag{A.3}$$

to linear order in δg.

The mixed type metric variations $\delta g_{\mu\bar{\nu}}$, and those of pure type $\delta g_{\mu\nu}, \delta g_{\bar{\mu}\bar{\nu}}$ satisfy independently the Lichnerowicz equation. To each of them we can associate elements of cohomology classes. To the variations of mixed type we can associate the real harmonic $(1,1)$-form

$$i\delta g_{\mu\bar{\nu}} dx^\mu \wedge dx^{\bar{\nu}} \tag{A.4}$$

which is harmonic as $g_{\mu\bar{\nu}}$ satisfies equation A.1. Similarly to variations of pure type one associates the $(2,1)$-form

$$\Omega_{\kappa\lambda}{}^{\bar{\nu}} \delta g_{\bar{\mu}\bar{\nu}} dx^\kappa \wedge dx^\lambda \wedge dx^{\bar{\mu}}. \tag{A.5}$$

This form is also harmonic because of the Lichnerowicz equation. One finds that the $(1,1)$-forms correspond to Kähler deformations whereas the $(2,1)$-forms correspond to complex structure variations.

A.2 The (2,1)-forms

One can define a metric on the moduli space of complex structure deformations

$$G_{\alpha\bar{\beta}} = -\frac{\int_M \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{\int_M \Omega \wedge \bar{\Omega}}, \tag{A.6}$$

where

$$\chi_{\alpha\kappa\lambda\bar{\mu}} = -\frac{1}{2}\Omega_{\kappa\lambda}{}^{\bar{n}\bar{u}}\frac{\partial g_{\bar{\mu}\bar{\nu}}}{\partial z^{\alpha}}, \chi_{\alpha} = \frac{1}{2}\chi_{\alpha\kappa\lambda\bar{\mu}}dx^{\kappa}\wedge dx^{\lambda}\wedge dx^{\bar{\mu}}. \tag{A.7}$$

It can be shown that

$$G_{\alpha\bar{\beta}} = -\frac{\partial}{\partial z^{\alpha}}\frac{\partial}{\partial z^{\bar{\beta}}}\log(i\int_M \Omega\wedge\bar{\Omega}), \tag{A.8}$$

from which it follows that the space of complex structures is Kähler with Kähler potential

$$-\log(i\int_M \Omega\wedge\bar{\Omega}). \tag{A.9}$$

Now, we will try to understand this Kähler potential better. In order to do that we will introduce a symplectic basis of harmonic three-forms. Let $(A^a, B_b),\ a, b =, \cdots, b_{21}$ be a canonical homology basis for $H_2(M, \mathbb{Z})$ and let (α_a, β^b) be the dual cohomology basis. Together these shall satisfy

$$\int_{A^b}\alpha_a = \int_M \alpha_a\wedge\beta^b = \delta_a^b,\ \int_{B_a}\beta^b = \int_M \beta^b\wedge\alpha_a = -\delta_a^b. \tag{A.10}$$

Furthermore, we define

$$z^a := \int_{A^a}\Omega, \quad \mathcal{G}_a := \int_{B_a}\Omega \tag{A.11}$$

These are called the "periods" of Ω. One can show that the z^a are projective coordinates for the complex structure and Ω can be regarded as being homogeneous of degree 1 in these coordinates. Also, the holomorphic three-form can be expanded in terms of the above basis

$$\Omega = z^a\alpha_a - \mathcal{G}_a(z)\beta^a. \tag{A.12}$$

The $\mathcal{G}_a$ can be shown to be the entries of the gradient of a function that is homogeneous of degree two

$$\mathcal{G}_a = \frac{\partial\mathcal{G}}{\partial z^a}, \quad \mathcal{G}(\lambda z) = \lambda^2\mathcal{G}(z). \tag{A.13}$$

Expressed in terms of $\mathcal{G}$ the Kähler potential can be written as

$$K = -\log(i\int_M \Omega\wedge\bar{\Omega}) = -\log i(\bar{z}^a\frac{\partial\mathcal{G}}{\partial z^a} - z^a\frac{\partial\bar{G}}{\partial\bar{z}^a}). \tag{A.14}$$

The holomorphic function $\mathcal{G}$ is also called the prepotential. Kähler manifolds whose Kähler potential can be derived from a prepotential are called special Kähler manifolds.

A.3 The (1,1)-forms

Let $e_A,\ A = 1, \cdots, b_{11}$ be a basis for $H^2(M, \mathbb{Z})$. We write the complexified Kähler form in this basis as

$$B + iJ = w^A e_A, \quad w^A = u^A + iv^A. \tag{A.15}$$

Similarly to the complex structure case one can define a metric on the space of variations of the Kähler form

$$G_{A\bar{B}} := -\frac{\partial}{\partial w^A}\frac{\partial}{\partial w^{\bar{B}}} \log \kappa(J, J, J), \tag{A.16}$$

where

$$\kappa(\rho, \sigma, \tau) := \int \rho \wedge \sigma \wedge \tau. \tag{A.17}$$

So, the Kähler potential is given by

$$K = \log(\kappa_{ABC} v^A v^B v^C). \tag{A.18}$$

As in the case of the complex structure one can find a prepotential

$$\mathcal{F}(w) = -\frac{1}{3!}\frac{\kappa_{ABC} w^A w^B w^C}{w^0}, \tag{A.19}$$

where w^0 - which can be identified with the inverse of the dilaton - is introduced in order to make $\mathcal{F}$ homogeneous of degree 2. Again we see that the moduli space has special Kähler geometry.

A.4 N=2 Supergravity

Compactifying type II theories on Calabi-Yau three-folds one obtains $N = 2$ supergravity ([41]) in four dimensions. The moduli of the manifold appear as components of the supergravity multiplets, where the exact arrangement between the moduli and the multiplet depends on the theory at hand. In the type IIA case the ten dimensional fields $\hat{A}_1$, $\hat{B}_2$ and $\hat{C}_3$ can be expanded in the Calabi-Yau harmonics and one obtains

$$\begin{aligned}
\hat{A}_1 &= A^0, \\
\hat{B}_2 &= B_2 + b^i w_i, \\
\hat{C}_3 &= C_3 + A^i \wedge w_i + \xi^a \alpha_a - \tilde{\xi}_b \beta^b
\end{aligned}$$

Here C_3 is a three-form, B_2 a two-form, (A^0, A^i) are one-forms and $b^i, \xi^A, \tilde{\xi}_A$ are scalar fields in $D = 4$. Now, these fields couple to to the Kähler moduli v^i and the complex structure moduli z^a in the following way:

The Kähler moduli v^i combine with the scalars b^i defined in A.4 into complex scalar fields $t^i = b^i + iv^i$ which together with the one-forms A^i from A.4 form the bosonic components of $h^{1,1}$ vector multiplets (A^i, t^i). The complex structure deformations z^a together with the scalars $\xi^a, \tilde{\xi}_a$ (here a runs from 1 to $h^{2,1}$) are members of $h^{(2,1)}$ hypermultiplets. Furthermore, there is the universal hypermultiplet consisting of the fields ϕ, $B_{\mu\nu}$, ξ^0, $\bar{\xi}_0$.

On the other hand the arrangement in the type IIB case is opposite. The Kähler moduli appear as elements of the $h^{1,1}$ hypermultiplets and the complex structure moduli are members of $h^{2,1}$ vector multiplets.

B Type IIA Orientifolds

The moduli space of type IIA theories is a product of two moduli spaces, one containing the Kähler- and the other the complex structure-moduli. The orientifold projection reduces the size of each moduli space [42] and truncates the theory to an $N = 1$ theory. It is defined as $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$ which is the composition of world-sheet parity Ω_p, left-moving fermion number $(-1)^{F_L}$ and an antiholomorphic involution of the Calabi-Yau σ. The involution acts in the following way:

$$\sigma^*\omega = -\omega, \quad \sigma^*\Omega = e^{2i\theta}\bar{\Omega}, \tag{B.1}$$

where θ is some phase. The loci of σ are special Lagrangian three-cycles Σ_n which are wrapped by Orientifold fixed planes ($O6s$).

In order to keep modes of massless ten-dimensional fields invariant under the orientifold projection, they must transform under σ as

$$\sigma^*g_{\mu\nu} = g_{\mu\nu}, \quad \sigma^*B_2 = -B_2, \quad \sigma^*\phi = \phi, \quad \sigma^*C_1 = -C_1, \quad \sigma^*C_3 = C_3. \tag{B.2}$$

B.1 Kähler moduli space

Under the projection, the space of $(1,1)$-forms $H^{1,1}$ decomposes into even and odd subspaces, $H^{1,1} = H^{1,1}_+ \oplus H^{1,1}_-$, of dimensions $h^{1,1}_-$ and $h^{1,1}_+$, respectively. From the transformation of the B-field B.2 and the way the involution acts on the Kähler form B.1 we find $h^{1,1}_-$ surviving complex Kähler moduli t_a:

$$\omega_c = \sum_{a=1}^{h^{1,1}_-} t_a w_a, \qquad t_a = b_a + iv_a, \tag{B.3}$$

with $\{w_a\}$ a basis for $H^{1,1}_-$.

This way the Kähler moduli space of the $N = 2$ theory is reduced to contain only $h^{1,1}_-$ moduli. The Kähler potential from the reduced Kähler moduli space is just inherited from the original theory.

B.2 Complex structure moduli space

As discussed in appendix A to define the complex structure moduli space one chooses a basis for harmonic 3-forms H^3, $\{\alpha_a, \beta^b\}$, where $a, b = 0, \cdots, h^{2,1}$ and $\int \alpha_a \wedge \beta^b = \delta^b_a$. In this basis the holomorphic three-form is given as

$$\Omega = Z^a\alpha_a - \mathcal{G}_b\beta^b. \tag{B.4}$$

Here the complex Z^a are the projective coordinates on the complex structure moduli space, whereas the inhomogeneous coordinates shall be denoted by $z_K, K = 1, \cdots, h^{2,1}$. The complex z_K are joined by the real axionic modes $\xi_K, \tilde{\xi}_K$ defined in A.4 and together with ξ_0, $\tilde{\xi}_0$, the dilaton ϕ and the dual of B_2 they form $4(h^{2,1}+1)$ hypermultiplets. This way one sees that the moduli space is $4(h^{2,1}+1)$-real dimensional.

Under the orientifold involution, there is again a decomposition of harmonic forms into even and odd subspaces:

$$H^3 = H^3_+ \oplus H^3_-, \tag{B.5}$$

where H^3_+ and H^3_- are each $h^{2,1}+1$-realdimensional. The even and odd bases are denoted by $\{\alpha^+_k, \beta^+_\lambda\}$ and $\{\alpha^-_\lambda, \beta^-_k\}$, where $k = 0, \cdots, \tilde{h}$ and $\lambda = \tilde{h}+1, \cdots h^{2,1}$. Here, it should be noted that $\tilde{h}$, which determines how many $\alpha' s$ are even, is basis dependent. From the orientifold involution B.1 one obtains the constrains

$$\mathrm{Im} Z_k = \mathrm{Re}\mathcal{G}_k = \mathrm{Re} Z_\lambda = \mathrm{Im}\mathcal{G}_\lambda = 0. \tag{B.6}$$

We see that for each complex z_k , only one real component survives the projection. Also, the space of axionic fields is truncated to half of its size, namely to $\{\xi_k, \tilde{\xi}_\lambda\}$. This is due to condition B.2 which implies that C_3 has to be even. Altogether the result is, that each quaternionic modulus is truncated to a complex field.

The universal hypermultiplet is also cut in half , as ϕ and one of $\xi_0, \tilde{\xi}_0$ survive. The surviving moduli can be summarized in the object

$$\Omega_c = C_3 + 2i\mathrm{Re}(C\Omega), \tag{B.7}$$

where C is given by

$$C \equiv e^{-D+K^{cs}/2}, \qquad e^D \equiv \sqrt{8} e^{\phi + K^K/2} = \frac{e^\phi}{\sqrt{\mathrm{vol}}}. \tag{B.8}$$

Here K^{cs} is the Kähler potential for complex structure moduli restricted to the surviving modes

$$K^{cs} = -\log(i\int \Omega\wedge\bar{\Omega}) = -\log 2(\mathrm{Im}Z_\lambda \mathrm{Re}\mathcal{G}_\lambda - \mathrm{Re}Z_k \mathrm{Im}\mathcal{G}_k). \tag{B.9}$$

Expanding Ω_c in H^3_+ one obtains the surviving moduli:

$$N_k = \frac{1}{2}\int \Omega_c \wedge \beta_k = \frac{1}{2}\xi_k + i\mathrm{Re}(CZ_k), \tag{B.10}$$

$$T_\lambda = i\int \Omega_c\wedge\alpha_\lambda = i\tilde{\xi}_\lambda - 2\mathrm{Re}(C\mathcal{G}_\lambda). \tag{B.11}$$

The inclusion of the dilaton in the complex modes is responsible for the fact that all $h^{2,1}+1$ scalar fields are physical.

The difference to the Kähler case is that there $h^{1,1}_-$ complex moduli are preserved and the rest removed, whereas in the hypermultiplet case each quaternion is cut in half, always leaving $h^{2,1}+1$ complex moduli.

The Kähler potential for the truncated complex structure moduli space is given by

$$K^Q = -2\log(2\int \mathrm{Re}(C\Omega)\wedge *\mathrm{Re}(C\Omega)). \tag{B.12}$$

C The Ring structure of $H^*(T^6/\mathbb{Z}_4)$

C.1 Definition of the Product

Let us first clarify some notations, for a more detailed survey consult [35]. A morphism $f : T \to S$ of manifolds induces via pullback a degree-preserving ring homomorphism $f^* : H^*(S) \to H^*(T)$. If T is a submanifold of S and $\alpha \in H^*(S)$, we write $\alpha|_T$ instead of $i^*\alpha$, where $i : T \to S$ is the inclusion.

The pushforward $f_* : H^*(T) \to H^*(S)$ on cohomology is given as follows: to every cohomology class α, associate its Poincare dual homology class $\bar{\alpha}$. Then the Poincare dual of $f_*(\bar{\alpha})$ in cohomology is defined to be $f_*(\alpha)$.

The product structure is defined on the space $H^*(Y, G)$. It is a bilinear map

$$\mu : H^*(Y, G) \times H^*(Y, G) \to H^*(Y, G), \tag{C.1}$$

$$\mu(\alpha_g, \beta_h) := \gamma_{gh}. \tag{C.2}$$

Now, all what remains is to define γ_{gh}. This can be done in an easy way once the following condition is satisfied:

$$a(g, U) + a(h, U) - a(gh, U) - \operatorname{codim}(U \subset Y^{gh}) = 0, \tag{C.3}$$

where U is a connected component of Y^H and H the subgroup generated by g and h. In this case γ is determined to be

$$\gamma = i_*(\alpha|_{Y^{g,y}} \cdot \beta|_{Y^{g,h}}) \tag{C.4}$$

and $i : Y^{g,h} \to Y^{gh}$ is the natural inclusion.

One can show that the bilinear map μ sends $H^i(Y, G) \otimes H^j(Y, G)$ to $H^{i+j}(Y, G)$.

C.2 Calculation of Orbifold Yukawa couplings

One can easily check that for the products we will consider condition C.3 is always fulfilled.

1.

$$\begin{aligned}
& E^{1,1}(y)_\theta \cdot E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(y)_\theta \\
= \; & (E^{1,1}(y)_\theta \cdot E^{1,1}(Z)_{\theta^2}) \cdot E^{1,1}(y)_\theta \\
= \; & i_*(1|_{Y^{\theta,\theta^2}} \cdot 1|_{Y^{\theta,\theta^2}}) \cdot E^{1,1}(y)_\theta \\
= \; & i_*(1|_{Y^{\theta^3}} \cdot 1|_{Y^\theta}) = \mathrm{vol}_6.
\end{aligned}$$

2.

$$\begin{aligned}
& E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(Z)_{\theta^2} \cdot dz_3 \wedge d\bar{z}_3 \\
= \;& i_*(1|_{Y^{\theta^2}} \cdot 1|_{Y^{\theta^2}}) \cdot dz_3 \wedge d\bar{z}_3 \\
= \;& dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \cdot dz_3 \wedge d\bar{z}_3 \\
= \;& \mathrm{vol}_6.
\end{aligned}$$

3.

$$\begin{aligned}
& E^{1,1}(y)_{\theta} \cdot E^{1,1}(y)_{\theta} \cdot E^{1,1}(y)_{\theta} \\
= \;& i_*(1|_{Y^{\theta}} \cdot 1|_{Y^{\theta}}) \cdot E^{1,1}(y)_{\theta} \\
= \;& i_*(\underbrace{(dz_3 \wedge d\bar{z}_3)|_{Y^{\theta^2,\theta}}}_{=0} \cdot E^{1,1}(y)_{\theta^2,\theta}) \\
= \;& 0.
\end{aligned}$$

4.

$$\begin{aligned}
& E^{1,1}(Z) \cdot E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(Z)_{\theta^2} \\
= \;& i_*(i_*(1|_{Y^{\theta^2}} \cdot 1|_{Y^{\theta^2}}) \cdot 1|_{Y^{\theta^2}}) \\
= \;& i_*(\underbrace{dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2|_{Y^{\theta^2}}}_{=0} \cdot 1|_{Y^{\theta^2}}) \\
= \;& 0
\end{aligned}$$

5.

$$\begin{aligned}
& E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(y)_{\theta} \\
= \;& i_*(1|_{Y^{\theta^2}} \cdot 1|_{Y^{\theta^2}}) \cdot E^{1,1}(y)_{\theta} \\
= \;& 0.
\end{aligned}$$

6.

$$
\begin{aligned}
& dz_i \wedge d\bar{z}_i \cdot E^{1,1}(y)_\theta \cdot E^{1,1}(y)_\theta \\
= \;& i_*(\underbrace{dz_i \wedge d\bar{z}_i|_{Y^{e,\theta}}}_{=0} \cdot 1|_{Y^\theta}) \cdot E^{1,1}(y)_\theta \\
= \;& 0
\end{aligned}
$$

7.

$$
\begin{aligned}
& dz_i \wedge d\bar{z}_i \cdot E^{1,1}(Z)_{\theta^2} \cdot E^{1,1}(y)_\theta \\
= \;& i_*(dz_i \wedge d\bar{z}_i|_{Y^{e,\theta^2}} \cdot 1|_{Y^{e,\theta^2}}) \cdot E^{1,1}(y)_\theta \\
= \;& i_*(dz_i \wedge d\bar{z}_i|_{Y^\theta} \cdot 1|_{Y^\theta}) \\
= \;& 0
\end{aligned}
$$

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