Heterotic MSSM on Blown-Up Orbifold with Unbroken Hypercharge

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Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

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Contents

Chapter 1

Introduction

Today there are two physical theories which can describe almost all experimental observations. On the one hand there is *general relativity* (GR), a classical field theory of gravity. On the other hand the *standard model* (SM) is a quantum field theory that describes the fundamental particles and their strong, weak and electromagnetic interactions. Unifying these theories into one fundamental framework is probably the most important challenge in modern physics. Whereas the scale of GR is set by the Planck mass, $M_{\mathrm{P}} \approx 10^{19}$ GeV, the SM is valid only up to the electro-weak breaking scale, $M_{\mathrm{EW}} \approx 100$ GeV. Thus, we need an extension of the SM which is valid at higher scales up to the Planck scale.

One such extension is given by *supersymmetry* (SUSY). It predicts one superpartner for each SM particle with different spin and this way removes quadratic divergences from scalar mass terms. Since these superpartners have not been observed yet, SUSY must be broken at a scale roughly above the SM scale. Furthermore, the *minimal supersymmetric extension of the standard model* (MSSM) provides unification of gauge couplings at a scale of $M_{\mathrm{GUT}} \approx 10^{16}$ GeV.

The gauge coupling unification is necessary for the realization of a *grand unified theory* (GUT). The idea of a GUT is to embed the SM gauge group, $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, into a bigger group. The best candidates for such a group are SU(5) and SO(10), see [1]. Since GUTs predict proton decay they must be broken at a scale around M_{GUT}.

As a further step one can impose SUSY to be a local symmetry. This implies invariance under local coordinate transformations which is the setup of GR. Thus, such theories are called *supergravity* (SUGRA). Unfortunately SUGRA's are non-renormalizable theories and hence cannot be a fundamental theory which should be valid at all scales.

In *string theories* the concept of the particle being the fundamental ingredient is replaced by a string whose typical length is of the order of the string scale $M_{\mathrm{S}} \approx 10^{17}$GeV[1]. Different oscillatons and windings of the string can represent

[1]This fixing of the string scale is only valid for the heterotic string which we consider here.

different particles with different quantum numbers. One of the advantages of string theories is that they are free of ultraviolet divergences and hence can act as a fundamental description of gravity.

We are in particular interested in the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic superstring which automatically provides $N = 1$ SUGRA in $d = 10$ with an $\mathrm{E}_8 \times \mathrm{E}_8$ gauge theory. To obtain the MSSM we can compactify the heterotic string on a *Calabi–Yau* (CY) manifold with an appropriate gauge symmetry breaking. Another possibility is compactification on an *orbifold* which can be seen as a singular limit of a CY. This thesis deals with the comparison of these two paths.

Outline

This theses is organized as follows. In chapter 2 we present the heterotic string and its compactification on an orbifold. The $\mathbb{Z}_{2,\mathrm{free}}$ symmetry of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, which allows new ways of symmetry breaking, is discussed and one particular MSSM model is shown. In chapter 3 we demonstrate how to resolve orbifold singularities. This procedure is called *blow-up*. We first resolve local singularities and then glue them together to obtain a smooth compact manifold and study its properties. In chapter 4 we discuss heterotic model building on the resolved CY manifold. We are particularly interested in finding a matching of the orbifold and the CY models. Finally we also present an exact MSSM model on the blown-up $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2,\mathrm{free}}$ orbifold. Here the $\mathbb{Z}_{2,\mathrm{free}}$ allows us to break an SU(5) GUT to the SM without breaking the hypercharge, which was not possible in other blow-up scenarios.

Chapter 2

Heterotic Strings on Orbifolds

Heterotic string theory [3,4] is probably the phenomenologically most interesting among all string theories. In the low energy limit it provides besides SUGRA [5] an $E_8 \times E_8$ or $SO(32)$ *super Yang–Mills* (SYM) theory, the former of which nicely contains some useful GUT groups and a hidden sector for SUSY breaking dynamics. To obtain a theory in $d = 4$ with $N = 1$ SUSY the heterotic string can be compactified e.g. on an orbifold [6–9] such that the gauge group gets broken and chiral matter appears. Some of these theories contain the MSSM [10] and could hence connect string theory to our world. In the past years a huge amount of heterotic MSSM models [16–20] was found with various phenomenological properties.

2.1 The Heterotic String

The simplest consistent quantized string theory one can write down is the bosonic string in $d = 26$. Alone it is phenomenologically unsuitable since it cannot describe fermionic degrees of freedom (dofs) and since it has a tachyon so the real ground state of this theory is unknown. These problems can be avoided in a superstring theory. Here one imposes supersymmetry on the world sheet (WS) and obtains after quantization a $N = 1$ spacetime (ST) supersymmetric theory with critical dimension $d = 10$. This means that after the GSO projection imposed by modular invariance the bosonic dofs from the Neveu–Schwarz (NS) sector and the fermionic dof's from the Ramond (R) sector appear in supermultiplets.

Now when one looks at closed strings where the left- and right-moving string excitations are somehow decoupled one has to take the tensor product of the Hilbert space with itself which results in an $N = 2$ theory in $d = 10$.

One can use the decoupling of the left- and right-movers and try to construct a string theory where they behave totally differently. The idea of heterosis is to have a right-moving superstring in $d = 10$ and a left-moving bosonic string with $d = 26$. We will see that this way we automatically obtain a $E_8 \times E_8$ or $SO(32)$

gauge theory in the low-energy sector. There is also an analogous fermionic description in which the left mover is a $d = 10$ bosonic string with additional fermions to cancel the conformal anomaly, see e.g. [2].

Mode Expansion

The right-movers on the worldsheet are supermultiplets containing a WS scalar $X_R^\mu(\sigma_-)$ and a WS Majorana Weyl spinor $\psi_R^\mu(\sigma_-)$ which both carry a spacetime index $\mu = 0\ldots 9$. The left-movers contain the partners of the right-moving bosons $X_L^\mu(\sigma_+)$ and 16 additional bosons $X_L^I(\sigma_+)$, $I = 1\ldots 16$. After fixing the worldsheet metric the action is given by

$$S = \frac{1}{\pi}\int \mathrm{d}^2\sigma \left(2\partial_+ X^\mu \partial_- X_\mu + \mathrm{i}\psi_R^\mu \partial_- \psi_{R,\mu} + 2\partial_+ X^I \partial_- X_I\right) , \tag{2.1}$$

where $X^\mu = X_R^\mu + X_L^\mu$ and $X^I = X_L^I$. In order to quantize the theory, we can perform a mode expansion. Let us start with the right movers. The bosons can be expanded in Fourier modes respecting the periodicity on the cylindric worldsheet,

$$X_R^\mu(\sigma_-) = x_R^\mu + p_R^\mu(\tau - \sigma) + \frac{\mathrm{i}}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu e^{-2\mathrm{i}n(\tau-\sigma)} . \tag{2.2}$$

The fermions allow for two different periodicity conditions, $\psi(\sigma_- + \pi) = \pm\psi(\sigma_-)$. The periodic ones correspond to the R sector while the antiperiodic ones build up the NS sector. These two sectors have to be studied separately since they result in completely different Hilbert spaces. In the R sector the worldsheet fermions can be expanded in integer modes denoted by d_n^μ,

$$\psi_R^{(\mathrm{R})\mu}(\sigma_-) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2\mathrm{i}n(\tau-\sigma)} . \tag{2.3}$$

In the NS sector the antiperiodicity leads to half integer modes b_r^μ,

$$\psi_R^{(\mathrm{NS})\mu}(\sigma_-) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2\mathrm{i}r(\tau-\sigma)} . \tag{2.4}$$

Let us discuss the left-movers. They form a 26 dimensional bosonic string but it is necessary to distinguish between the 10 spacetime dimensions and the 16 remaining ones. The spacetime left-movers are the partners of the right-moving bosons

$$X_L^\mu(\sigma_+) = x_L^\mu + p_L^\mu(\tau + \sigma) + \frac{\mathrm{i}}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2\mathrm{i}n(\tau+\sigma)} . \tag{2.5}$$

Since the heterotic string theory is a closed string theory the coordinates have to satisfy the boundary condition

$$X(\sigma = \pi) = X(\sigma = 0)\,. \tag{2.6}$$

Although the remaining left-movers have the same mode expansion they are denoted by a capital latin index I. They span the gauge dimensions.

$$X_L^I(\sigma_+) = x_L^I + p_L^I(\tau + \sigma) + \frac{\mathrm{i}}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^I e^{-2\mathrm{i}n(\tau+\sigma)}\,, \qquad I = 1\ldots 16\,. \tag{2.7}$$

For a non-compact dimension the momenta of the left- and right-movers are forced to satisfy

$$p_L = p_R := p/2\,. \tag{2.8}$$

But for the gauge dimensions no right-moving momentum exists so the left-moving one would have to be zero if these dimesions were not compact. One can avoid this by compactifying them on a torus T^{16}. A torus can be described by a lattice

$$\Lambda_{16} = \left\{ 2\pi \sum n_i e_i \,|\, n_i \in \mathbb{Z} \right\}\,, \tag{2.9}$$

spanned by 16 linearly independent vectors $e_i \in \mathbb{R}^{16}$. This lattice can act on the vector space by translation and one can easily see that it has a group structure. It can be modded out which means we identify points in $\mathbb{R}^{16}$ which differ by a lattice vector and the result is a torus $T^{16} = \mathbb{R}^{16}/\Lambda_{16}$. With the string being compactified on a torus, the X_L^I take values on T^{16}. It can be described e.g. by X_L^I taking values on the covering space $\mathbb{R}^{16}$ while taking the lattice group action into account. This in particular means that one can have closed strings where the starting and end point differ by a lattice vector and hence are mapped onto each other.

$$X^I(\sigma = \pi) = X^I(\sigma = 0) + 2\pi \sum w_i e_i^I\,, \qquad W_i \in \mathbb{Z}\,. \tag{2.10}$$

Here we automatically obtain the concept of winding numbers w_i. Every string state has a winding number which is topologically stable since the heterotic string is oriented. In the mode expansion the only way to realize (2.10) while fulfilling the equations of motion (eom) is to have a term linear in σ. For the mode expansion (2.7) it follows that the momentum has to be twice a lattice vector $p_L^I = 2w_i e_i^I$. After quantization the single valuedness of the wave function dictates the momentum to be quantized such that it takes values in the dual lattice

$$\Lambda^* := \left\{ \, v \in \left(\mathbb{R}^{16}\right)^* \;|\; v_I\; w^I \in \mathbb{Z} \;\;\forall\; w \in \Lambda \right\}\,. \tag{2.11}$$

Furthermore, modular invariance of the one-loop vacuum-to-vacuum amplitude requires the lattice to be self-dual ($\Lambda = \Lambda^*$) and even ($v^2 \in 2\mathbb{Z}\ \forall v \in \Lambda$) so the requirement of the momentum being in the lattice and the dual lattice at the same time is always fulfilled. Fortunately, there exist two even self-dual lattices in 16 dimensions, the root lattices of SO(32) and of $\mathrm{E}_8 \times \mathrm{E}_8$, denoted by Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ respectively. They lead to the two distinct heterotic string theories of which only the former will be considered.

Energy Momentum Tensor

String theory is a superconformal field theory, i.e. the action is invariant under superconformal transformations. Thus the dof's which appear in the mode expansion are not all physical. In other words, negative or zero norm states appear and have to be decoupled from the physical ones. This can be done by going to the lightcone gauge which removes the non physical dof's. In this situation the eom of the metric and the WS gravitino can be imposed as equations for the string modes which remove the unphysical dof's. These equations read

$$T_{++} = 0\,, \qquad T_{--} = 0\,, \qquad J_- = 0\,. \tag{2.12}$$

$T_{\alpha\beta}$ denotes the WS energy momentum tensor and J the WS supercurrent. In terms of the fields they read

$$T_{++} = -\partial_+ X_L^\mu \partial_+ X_{L,\mu} - \partial_+ X_L^I \partial_+ X_{L,I}\,, \tag{2.13a}$$

$$T_{--} = -\partial_- X_R^\mu \partial_- X_{R,\mu} - \frac{\mathrm{i}}{2}\psi_R^\mu \partial_- \psi_{R,\mu}\,, \tag{2.13b}$$

$$J_- = \psi_R^\mu \partial_- X_{R,\mu}\,. \tag{2.13c}$$

Note that the and left- and rightmover equations are completely decoupled. We expand these fields in Fourier modes which are called *super-Virasoro generators*

$$T_{++} = \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-2\mathrm{i}n(\tau+\sigma)}\,, \tag{2.14a}$$

$$T_{--} = \sum_{n\in\mathbb{Z}} L_n e^{-2\mathrm{i}n(\tau-\sigma)}\,, \tag{2.14b}$$

$$J_-^{(\mathrm{R})} = \sum_{n\in\mathbb{Z}} F_n e^{-2\mathrm{i}n(\tau-\sigma)}\,, \tag{2.14c}$$

$$J_-^{(\mathrm{NS})} = \sum_{r\in\mathbb{Z}+1/2} G_r e^{-2\mathrm{i}r(\tau-\sigma)}\,. \tag{2.14d}$$

In the right moving sector we distinguish between the R and NS sectors since the periodicity differs between them. Combining equations (2.2)-(2.7), (2.13) and (2.14), the super Virasoro generators can be expressed in terms of the modes and momenta of the string fields. In this thesis the equations for the zero modes

L_0 and $\tilde{L}_0$ are of particular interest since they are the mass equations for the string states. We will encounter them in the blow-up models when identifying the blow-up modes.

Quantization

In order to quantize the theory the fields and hence the modes become quantum-mechanical operators which act on a Hilbert space. From the canonical commutation relations for the fields one obtains the commutation relations for the modes,

$$[\alpha, \tilde{\alpha}] = 0\,, \tag{2.15a}$$
$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = n\delta_{n+m,0}\eta^{\mu\nu}\,, \tag{2.15b}$$
$$\left[\tilde{\alpha}^I_m, \tilde{\alpha}^J_n\right] = n\delta_{n+m,0}\delta^{IJ}\,, \tag{2.15c}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = n\delta_{n+m,0}\eta^{\mu\nu}\,. \tag{2.15d}$$

Due to their Grassmann property the fermionic modes fulfill anticommutation relations,

$$\{b^\mu_r, b^\nu_s\} = \delta_{r+s,0}\eta^{\mu\nu}\,, \tag{2.16a}$$
$$\{d^\mu_m, d^\nu_n\} = \delta_{m+n,0}\eta^{\mu\nu}\,. \tag{2.16b}$$

The reality of X and the Majorana property of ψ impose conditions on the modes.

$$\alpha_n = \alpha^\dagger_{-n}\,, \qquad \tilde{\alpha}_n = \tilde{\alpha}^\dagger_{-n}\,, \qquad d_r = d^\dagger_{-r}\,, \qquad b_n = b^\dagger_{-n}\,. \tag{2.17}$$

We immediately see that the modes act as ladder operators for an infinite series of harmonic oscillators. The positive modes can be interpreted as annihilation and the negative modes as creation operators which motivates the definition of the Hilbert space $\mathcal{H}$ which is a direct product of a left and a rightmoving part respecting the condition (2.8). A physical state $|\phi\rangle \in \mathcal{H}$ is defined by the conditions

$$L_n|\phi\rangle = \tilde{L}_n|\phi\rangle = 0\,, \quad \forall n > 0, \tag{2.18a}$$
$$F_n|\phi\rangle_{(\mathrm{R})} = 0\,, \quad \forall n \geq 0, \tag{2.18b}$$
$$G_r|\phi\rangle_{(\mathrm{NS})} = 0\,, \quad \forall r \geq 0\,. \tag{2.18c}$$

The zero mode equations obtain a shift $a_{L/R}$ due to normal ordering of the modes,

$$\big(L_0 - a_R\big)|\phi\rangle = 0\,, \qquad \big(\tilde{L}_0 - a_L\big)|\phi\rangle = 0\,. \tag{2.19}$$

For the bosonic string and hence the left-movers one finds $a_L = 1$. For the right-moving superstring the shift differs between the two sectors, $a_R^{(\mathrm{R})} = 0$, $a_R^{(\mathrm{NS})} = 1/2$. In terms of the modes, equations (2.19) define the mass operators,

$$\frac{M_L^2}{8} = -\frac{p_{L,\mu}p_L^\mu}{8} = \frac{p_I p^I}{2} + \tilde{N} - a_L\,, \tag{2.20a}$$

$$\frac{M_R^2}{8} = -\frac{p_{R,\mu}p_R^\mu}{8} = N - a_R\,, \tag{2.20b}$$

with the number operators

$$\tilde{N} := \sum_{n=1}^{\infty} \left(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n,\mu} + \tilde{\alpha}_{-n}^I \tilde{\alpha}_{n,I}\right)\,, \tag{2.21}$$

$$N^{(\mathrm{R})} := \sum_{n=1}^{\infty} \left(\alpha_{-n}^\mu \alpha_{n,\mu} + d_{-n}^\mu d_{n,\mu}\right)\,, \tag{2.22}$$

$$N^{(\mathrm{NS})} := \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu} + \sum_{r=1/2}^{\infty} b_{-r}^\mu b_{r,\mu}\,. \tag{2.23}$$

The total spacetime mass is the sum $M = M_L + M_R$ and (2.8) implies that $M_L = M_R$. This leads to the level matching condition $\frac{p_I p^I}{2} + \tilde{N} - a_L = N - a_R$ which projects the product of the left- and right-mover Hilbert spaces on the true on-shell Hilbert space.

Massless Spectrum

For phenomenology, it is of particular interest which states appear in the low energy sector of the theory. The masses of the massive string excitations are of the order of the string scale which in heterotic string theory is around $M_S \approx 10^{17}$ GeV so they are all integrated out in a low energy description and we are left with the massless modes.

Let us start with the right-movers. In the NS sector the ground state $|0\rangle_{(\mathrm{NS})}$ is annihilated by all positive oscillators and has negative mass squared $M_R^2 = -4$. The only massless state is $b_{-1/2}^\mu|0\rangle$ and transforms as a $\mathbf{8}_V$ under the little group SO(8). It will be denoted by $|q\rangle$ where $q = (\underline{\pm 1, 0, 0, 0})$ is the SO(8) weight vector. The underline denotes permutations.

The R ground state which the lowering operators annihilate and which is already massless shows a degeneracy due to $[L_0, d_0^\mu] = 0$. We further find that the fermionic zero modes satisfy a Clifford algebra $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$ so the R ground state $|0\rangle_{(\mathrm{R})}$ is a ST fermion. By convention it is chosen as a left-chiral Majorana–Weyl–fermion i.e. a $\mathbf{8}_S$ of SO(8). In a bosonized description it can also be characterized by its weight $|q\rangle$ where now $q = ([\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}])$. The rectangular brackets denote even number of sign flips. Together the R and NS massless states form a $N = 1$ vector multiplet in $d = 10$.

The left-moving ground state is defined as the one which is annihilated by all lowering operators $\tilde{\alpha}_{-n}|0\rangle = 0$, $\forall n > 0$, and has no momentum on the 16-torus $p^I|0\rangle = 0$. Using (2.20a) one finds that this is a tachyon with $M_L^2 = -8$. Fortunately its mass does not match with the tachyon from the right NS-sector such that the lightest states with equal left- and right-mass are the massless ones. Now there are two possibilities to get a massless state. One is to excite an oscillator. With a spacetime oscillator one obtains a state which is a $\mathbf{8}_V$ of SO(8) and in combination with the right-mover builds up the $N = 1$ SUGRA multiplet in $d = 10$:

$$|q\rangle \otimes \alpha^{\mu}_{-1}|0\rangle \to \begin{cases} g_{\mu\nu} & \text{graviton}\,, \\ B_{\mu\nu} & \text{2-form}\,, \\ \phi & \text{dilaton}\,, \\ \psi_\mu & \text{gravitino}\,, \\ \psi & \text{dilatino}\,. \end{cases} \tag{2.24}$$

Group-theoretically one finds the decomposition

$$(\mathbf{8}_V \oplus \mathbf{8}_S) \otimes \mathbf{8}_V = \mathbf{35}_V \oplus \mathbf{28} \oplus \mathbf{1} \oplus \mathbf{56}_C \oplus \mathbf{8}_C\,. \tag{2.25}$$

With an oscillator in the gauge dimensions the states form a vector multiplet which describes a $\mathrm{U}(1)^{16}$ SYM theory. The other possibility is to have non-vanishing momentum around the 16-torus $p^I|P\rangle = P^I|P\rangle$. Equation(2.20a) tells that in this case $N = 0$ and $P^2 = 2$: These P are exactly the roots of $\mathrm{E}_8 \times \mathrm{E}_8$, see appendix A, so in total we obtain $2\cdot(112+128) = 480$ new states. The 16 U(1)'s now play the role of the Cartan algebra and we obtain gauge enhancement to the desired $\mathrm{E}_8 \times \mathrm{E}_8$:

$$\left.\begin{array}{ccc} |q\rangle & \otimes & \alpha^I_{-1}|0\rangle \\ |q\rangle & \otimes & |P\rangle \end{array}\right\} \to \begin{cases} A^a_\mu & \text{gauge bosons}\,, \\ \lambda^a & \text{gauginos}\,. \end{cases} \tag{2.26}$$

2.2 Orbifolds

The simplest way to obtain a $d = 4$ string theory is to compactify a $d = 10$ theory on a torus T^6. It has the advantage that the torus is completely flat and hence the construction of a quantized theory is straight forward (see e.g. the heterotic setup). The generator of $N = 1$ SUSY in $d = 10$ is an 8 component Majorana–Weyl spinor which completely survives the torus compactification. The result is $N = 4$ SUSY in $d = 4$ which is phenomenologically unacceptable since it is not chiral. The next idea is to try a toroidal orbifold. By definition an orbifold is a manifold with a discrete symmetry modded out. Usually these symmetries possess fix points which appear as singularities on the orbifold and allow for new possibilities for string propagation on the space.

Orbifold Geometry

The starting point of constructing a six dimensional orbifold is the torus T^6. It is convenient to parameterize it by three complex coordinates $z_i \in \mathbb{C}\ \ i = 1 \ldots 3$. Again the torus is obtained by modding out a lattice $\Lambda \cong \mathbb{Z}^6$, $\Lambda \subset \mathbb{C}^3$. The coordinates are chosen such that the lattice factorizes. By absorbing the lattice size into the metric we can achieve the lattice base vectors to be of the form,

$$\begin{aligned} e_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} \tau_1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ e_4 &= \begin{pmatrix} 0 \\ \tau_2 \\ 0 \end{pmatrix}, \qquad e_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e_6 = \begin{pmatrix} 0 \\ 0 \\ \tau_3 \end{pmatrix}. \end{aligned} \tag{2.27}$$

The complex numbers τ_i are called *complex structure moduli*. We obtain $\Lambda = \bigoplus_i (\mathbb{Z} \oplus \tau_i \mathbb{Z})$.

The lattice and hence the torus can have a discrete symmetry group which acts on the coordinates by rotations. The rotations appearing here are also factorizable and Abelian, i.e. their action is

$$\theta : z_i \to \theta z_i = e^{2\pi \mathrm{i} v_i} z_i\,, \qquad \text{no sum over } i\,, \tag{2.28}$$

with the *twist vector* $v = (v_1, v_2, v_3)$. The group of these rotations is called the *point group* P. The v_i are rational numbers which implies that the point group is finite. In the examples studied here it is $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$. In the first case we can choose one generating element θ which is of order N, $\theta^N = 1$, in the second case we need two of them θ, θ' which are of order N and M, respectively. Together with the lattice translations the point group forms the *space group* $S = P \ltimes \Lambda$. An element of the space group can be obtained by a rotation and a lattice shift, $(\theta, n_i e_i) = g \in S$ with $g : z_i \to \theta z_i + n_{2i-1} + n_{2i}\tau_i$.

Now we define the orbifold as the torus after modding out the point group,

$$\mathcal{O} = T^6/P = \mathbb{C}^3/S\,. \tag{2.29}$$

When we take a vector on the orbifold and parallel transport it along a curve which is closed by some non-trivial point group element θ we find the vector being rotated by $\theta = e^{2\pi \mathrm{i}\, \mathrm{diag}\,(v_1, v_2, v_3)}$. The CY condition requires the holonomy group to be a subgroup of SU(3) which implies the condition,

$$\sum_i v_i \equiv 0 \bmod 1\,. \tag{2.30}$$

It strongly restricts the number of possible orbifold point groups.

Fixed Points

The action of the point group on the torus is not free, i.e. there exist points which are mapped onto themselves. If the rotation θ acts non-trivially on all coordinates we find exactly one fixed point for a given space group element, $z_{\text{fix}} = (\mathbb{1} - \theta)^{-1} n_i e_i$. Otherwise, θ leaves one coordinate untouched and the fixed point equation $z_{\text{fix}} = g z_{\text{fix}}$ can have a one-dimensional space of solutions called fixed torus[1] or fixed (complex) line or fixed brane. The neighborhood of the fixed point or line shows a deficit angle when surrounding it so there must be curvature in form of a delta peak localized on it. Thus the fixed points are singularities. In Chapter 3, we will show how to resolve the singularity and smooth out the curvature. This way the topological structure of the singularity, which is invisible on the orbifold, can be seen.

Orbifold Topology

Many properties of the low-energy effective theory, e.g. the massless chiral spectrum, can be obtained from the topological data of the compactification space. The usual smooth manifold topology defined in terms of homology or cohomology classes and their intersection numbers is not well-defined on the orbifold due to the singularities. It can only be applied to the untwisted sector, but there are also techniques to find the Hodge numbers coming from the twisted sectors. When we resolve the singularities we will see the role of the "twisted topology" by making a connection between the CY and the orbifold as its blow-down limit.

The "untwisted topology" can easily be described by the cohomology of the underlying torus. On a complex torus T^{2n} a basis of (anti-)holomorphic 1-forms is given by $\mathrm{d}z_i$ ($\mathrm{d}\bar{z}_i$), $i = 1 \dots n$. A complete basis[2] of the cohomology ring is obtained by wedging these 1-forms together in all possible ways. Using combinatorics one finds the torus Hodge numbers $h^{p,q} = \binom{n}{p} \cdot \binom{n}{q}$.

Let us take one element of a point group of type (2.28). Its action on the 1-forms is given by

$$\theta \mathrm{d}z_i = e^{2\pi \mathrm{i} v_i} \mathrm{d}z_i \,, \qquad \theta \mathrm{d}\bar{z}_i = e^{-2\pi \mathrm{i} v_i} \mathrm{d}\bar{z}_i \,, \tag{2.31}$$

i.e. any of the basic forms transform with a phase. The forms which are invariant under the point group survive the orbifolding procedure and build up the untwisted cohomology group. Note that:

- There are at least the three $(1,1)$-forms $\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, $i = 1 \dots 3$. Their corresponding Kähler moduli describe the sizes of the three 2-tori.

[1] Actually due to other twisting elements the fixed set is not always a torus but nevertheless will be called fixed torus.

[2] As in this case such a basis can always be chosen to be harmonic.

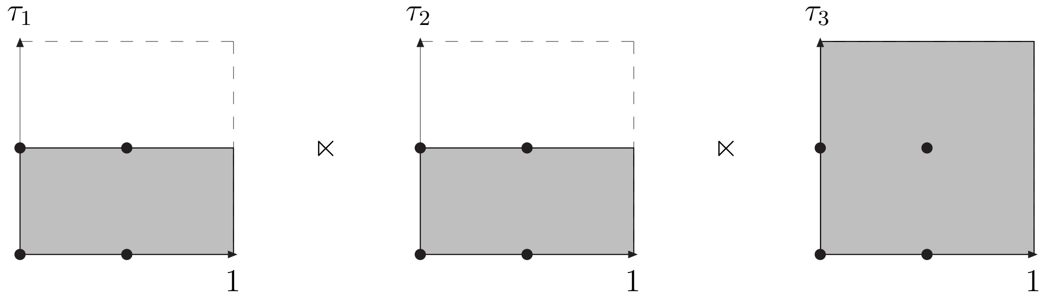


Figure 2.1: Setup of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold as a fibre product, $\mathcal{O} = T^2/\mathbb{Z}_2 \ltimes (T^2/\mathbb{Z}_2 \ltimes T^2)$. The grey region shows the fundamental domain. The dots show where the fixed lines are located in each torus. The product of two points corresponds to a fixed torus. The product of three dots is a point in which fixed tori intersect.

- The holomorphic 3-form $\Omega := \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$ is always invariant due to the CY condition (2.30).

- In order to have exactly $N = 1$ SUSY in $d = 4$ and not more, the holonomy must be a subgroup of SU(3) but not of SU(2). This means that no coordinate is allowed to stay untouched by the point group and in particular none of the $(1, 0)$ and $(0, 1)$-forms is invariant.

- From the previous fact and the existence of the holomorphic three form it follows that the $(2, 0)$ and $(0, 2)$-forms also do not survive.

The resulting Hodge diamond is exactly as required for a CY manifold, with $h^{0,0} = h^{3,0} = 1$, $h^{1,0} = h^{2,0} = 0$ and with only $h^{2,1}$ and $h^{1,1}$ being undetermined. In order to keep the CY condition they are the only ones who could be affected by the twisted sectors. In [13] one can find a detailed description how to get the twisted Hodge numbers.

For the singularities considered here there is one additional Kähler modulus and hence $(1, 1)$-form for each fixed point which acts as a blow-up mode. For other types of singularities there can be more that one $(1, 1)$-from and even further $(2, 1)$-forms if the topology of the fixed points allows more freedom in the choice of the complex structure.

Example: The $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

This thesis will deal with model building on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and its resolution. One strong reason for this explicit choice is that it has a $\mathbb{Z}_{2,\text{free}}$ freely acting symmetry which allows for new methods of symmetry breaking (see sec. 2.4).

The point group[3] contains three non-trivial elements, each of order two,

$$\theta_1 = \mathrm{diag}(1,-1,-1)\,, \quad \theta_2 = \mathrm{diag}(-1,1,-1)\,, \quad \theta_3 = \mathrm{diag}(-1,-1,1)\,, \quad (2.32)$$

which correspond to the twist vectors,

$$v_1 = (0, 1/2, -1/2)\,, \qquad v_2 = (-1/2, 0, 1/2)\,, \qquad v_3 = (1/2, -1/2, 0)\,. \quad (2.33)$$

We chose the convention that the i-th twist leaves the i-th torus invariant. Any two of them can be chosen as generating elements and the third is the product of them. Due to the fact that every $\mathbb{Z}_2$ twist acts on only two coordinates, the sets fixed under one twist are fixed tori. They are in fact not tori because the other $\mathbb{Z}_2$ folds them to a pillow $T^2/\mathbb{Z}_2 \cong \mathbb{CP}^1$. Let us take a closer look at e.g. the θ_1 sector where the second and third coordinates get inverted while the first is unaffected. On each affected torus there are four points which are mapped on themselves, either only by rotation or together with an appropriate lattice shift. These points are

$$z_i^1 = 0\,, \qquad z_i^2 = \frac{\tau_i}{2}\,, \qquad z_i^3 = \frac{1}{2}\,, \qquad z_i^4 = \frac{1+\tau_i}{2}\,, \qquad i = 1 \ldots 3\,. \quad (2.34)$$

The θ_1 sector has 16 fixed tori namely the sets $F_{1,\beta,\gamma} = \{z_2 = z_2^\beta,\ z_3 = z_3^\gamma\}$, similarly there are 16 fixed tori in the θ_2 sector, $F_{2,\alpha\gamma} = \{z_1 = z_1^\alpha,\ z_3 = z_3^\gamma\}$ and 16 in the θ_3 sector, $F_{3,\alpha\beta} = \{z_1 = z_1^\alpha,\ z_2 = z_2^\beta\}$. Here we introduced the convention to label the fixed loci in the first, second and third torus by α, β and γ respectively which all run from 1 to 4. Note that the points $(z_1^\alpha, z_2^\beta, z_3^\gamma)$ are the places where the fixed tori from different sectors intersect.

Next we want to explore the topology of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The only invariant toroidal $(1,1)$-forms are the $\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, $i = 1 \ldots 3$. Their Poincaré duals are the $(2,2)$-cycles which are obtained by fixing one coordinate on the bulk and have the topology of a $T^4/\mathbb{Z}_2 \cong \mathrm{K3}$. Further we find three invariant $(2,1)$-forms namely $\mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$, $\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_3$ and $\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_3$. The respective complex structure moduli are the torus parameters τ_i. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case they are all not restricted by the point group[4]. We find the untwisted Hodge diamond,

$$h^{p,q}_{\text{untwisted}} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,. \quad (2.35)$$

[3]In fact there is only one possibility for a factorized $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold which is CY.

[4]This freedom only exists if the twist is of order $N = 2$ in the respective torus. For $N = 3$ or $N = 6$ we need $\tau = e^{2\pi \mathrm{i}/3}$ and $N = 4$ requires $\tau = \mathrm{i}$, each up to $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ transformations.

Now let us look at the contributions from the twisted sectors. We have 48 fixed lines with topology $\mathbb{CP}^1$. This leads to one additional Kähler modulus per fixed torus which we call blow-up mode, $h^{1,1}_{\text{twisted}} = 48$. Since on an $\mathbb{CP}^1$ the complex structure is unique, there are no additional complex structure moduli, $h^{2,1}_{\text{twisted}} = 0$. Adding up the two sectors results in the total Hodge diamond which we should recover after blow-up,

$$h^{p,q} = h^{p,q}_{\text{untwisted}} + h^{p,q}_{\text{twisted}} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 51 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 51 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{2.36}$$

The Hodge diamond allows us to compute the Euler number

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q} = 96 . \tag{2.37}$$

2.3 Heterotic Compactification on Orbifolds

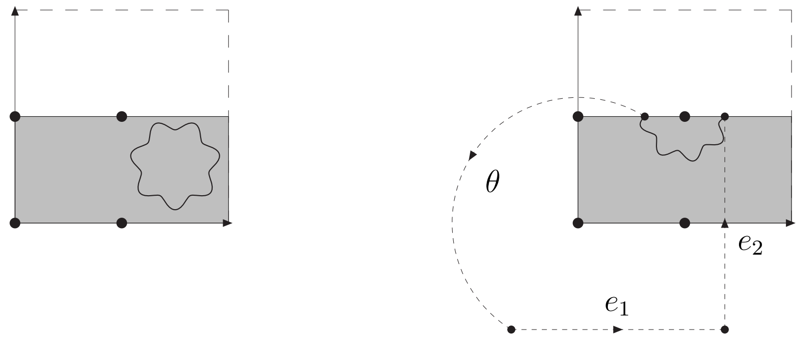
At this point we are ready to perform the compactification. This means that six of the bosonic coordinates (X^μ , $\mu = 4 \ldots 9$) take values on the orbifold $\mathcal{O}$. It is convenient to pair up the real coordinates to complex ones $Z^a = X^{2a+2} + \mathrm{i}X^{2a+3}$, $a = 1 \ldots 3$. The right-movers are now compactified on a 22 dimensional space. The simplest choice would be just to take the direct product of the orbifold and the 16-torus. In this case there would be no gauge group breaking and no $d = 4$ chiral matter which contradicts our intention to find the MSSM. A more general possibility is to take the T^{16} as a fiber over the orbifold which can be realized by embedding the space group into the automorphisms of the torus,

$$S \to \mathrm{Aut}(T^{16}) . \tag{2.38}$$

Then modding out the space group results in the fiber structure,

$$\left(\mathbb{C}^3 \times T^{16}\right) / S = \mathcal{O} \ltimes T^{16} , \tag{2.39}$$

where in general the fiber degenerates at the fixed lines. The only thing that has to be specified is the embedding (2.38). In the types of models considered here this is done by shift vectors, $\theta_i \to V_i^I$, and Wilson lines, $e_i \to W_i^I$, which act by translations in the gauge dimensions. In the low energy effective SYM theory they can be interpreted as a non-trivial 1-form field $\mathcal{A}$ wrapping a cycle. The shift vectors and Wilson lines are not completely arbitrary. If e.g. the twist θ is



(a) Untwisted String (b) Twisted String

Figure 2.2: Strings sectors visualized on a $T^2/\mathbb{Z}_2$ orbifold. The grey region is the fundamental region. The twisted string is closed by a spacegroup element $(\theta, e_1 + e_2)$.

of order N then N times the corresponding shift V_θ must also be the identity map on the T^{16}, i.e. a lattice vector, $NV \in \Lambda_{16}$. Similarly one must restrict the Wilson lines to take only discrete values in form of fractions of lattice vectors.

For strings living on the orbifold, new types of boundary conditions have to be imposed. For a space group element $g = (\theta, n_i e_i)$ we find for the orbifold coordinates Z^a and for the gauge coordinates X^I,

$$Z^a(\tau, \sigma + \pi) = gZ^a(\tau, \sigma) = \theta Z^a(\tau, \sigma) + n_i e_i \,, \tag{2.40a}$$

$$X^I(\sigma_+ + \pi) = gX^I(\sigma_+) = X^I(\sigma_+) + V^I + L \,, \tag{2.40b}$$

where

$$V^I = V^I_\theta + n_i W^I_i \,, \tag{2.41}$$

is the local shift and $L \in \Lambda$ a lattice vector. At this point we distinguish between the untwisted $\theta = 1$ and the twisted $\theta \neq 1$ sectors.

Untwisted Sector

The mode expansion in the untwisted sector is just the mode expansion on the underlying torus. Since the radii are undetermined we do not know the masses of the states with winding and momentum, so in general they will be massive. Hence we only look at the unwound string and its Hilbert space. In order to find the Hilbert space we first have to look at the transformation behavior of the operators and states under the space group. Under a twist θ we find

$$|q\rangle \to e^{-2\pi \mathrm{i} q\cdot v}|q\rangle \,, \tag{2.42a}$$

$$\tilde{\alpha}^a_n \to e^{2\pi \mathrm{i} v_a}\tilde{\alpha}^a_n \,, \tag{2.42b}$$

$$|P\rangle \to e^{2\pi \mathrm{i} P\cdot V_\theta}|P\rangle \,. \tag{2.42c}$$

The twist vector $v = (0, v_1, v_2, v_3)$ has been extended by a zero component to be able to act on the SO(8) weights q. With this convention the zero component corresponds to the spacetime helicity. For a state to survive the orbifolding means that its total transformation phase must be trivial. From the $d = 10$ SUGRA multiplet (2.24) there remains a $N = 1$, $d = 4$ SUGRA multiplet together with a tensor multiplet containing the model-independent axion,

$$\left.\begin{array}{l} |\pm(1,0,0,0)\rangle \\ |\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle \end{array}\right\} \otimes \alpha^{\mu}_{-1}|P=0\rangle\,, \tag{2.43}$$

and some moduli[5] multiplets which depend on the choice of the orbifold

$$|q\rangle \otimes \alpha^{a}_{-1}|P=0\rangle\,. \tag{2.44}$$

In the gauge multiplet (2.26) we see that all Cartan generators

$$\left.\begin{array}{l} |\pm(1,0,0,0)\rangle \\ |\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle \end{array}\right\} \otimes \alpha^{I}_{-1}|P=0\rangle\,, \tag{2.45}$$

still exist so we have not reduced the rank of the gauge group, which is always the case for Abelian embeddings. The non-Abelian part of the $d = 4$ gauge group G_4 comes from the roots $P \neq 0$ which satisfy

$$P \cdot V = 0 \bmod 1\,, \qquad P \cdot W = 0 \bmod 1\,, \tag{2.46}$$

for all shifts V and Wilson lines W:

$$\left.\begin{array}{l} |\pm(1,0,0,0)\rangle \\ |\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle \end{array}\right\} \otimes |P\rangle\,. \tag{2.47}$$

We see that the states (2.45) and (2.47) form complete vector multiplets as required for a SYM theory. But there can be more states of the form $|q\rangle \otimes |P\rangle$ which do not satisfy (2.46) but

$$q \cdot v - P \cdot V = 0 \bmod 1\,, \tag{2.48}$$

for all local shifts V with the associated twist vectors v. These states are called *untwisted matter* since they build $d = 4$ chiral multiplets and are charged under the gauge group. It turns out that these multiplets appear in C-conjugate pairs and hence loose their property of being chiral.

[5]These are actually the untwisted moduli. The twisted moduli or blow-up modes become visible in blow-up.

Twisted Sector

In the twisted sector the string is closed by a space group element which has a non-trivial twist, $\theta \neq 1$. On the one hand there are space group elements which act freely[6]. The corresponding states will again in general be massive (as for the untwisted wound string) so they will not be explored. The other elements $g = (\theta, n_i e_i)$ have fixed points or fixed tori. To satisfy (2.40a), the mode expansion around the fixed point must consist of fractional modes,

$$Z^a(\tau,\sigma) = z^a_{\text{fix}} + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}}\left(\frac{\alpha^\mu_{n-v^a}}{n-v^a}e^{-2\mathrm{i}(n-v^a)(\tau-\sigma)} + \frac{\tilde{\alpha}^\mu_{n+v^a}}{n+v^a}e^{-2\mathrm{i}(n+v^a)(\tau+\sigma)}\right) . \quad (2.49)$$

As a consequence the right-moving SO(8) weights q get shifted

$$q^a_{\text{sh}} = q^a + v^a . \quad (2.50)$$

At the same time (2.40b) leads to a shift of the momentum around the 16-torus

$$P_{\text{sh}} = P + V , \quad (2.51)$$

where V is the local shift (2.41). Also the normal ordering constant in the zero mode of the energy momentum tensor will be shifted by

$$\delta c = \sum_{a=1}^{3} v_a\left(1 - v_a\right) , \quad (2.52)$$

which modifies the mass equations

$$\frac{M_R^2}{8} = \frac{q_{\text{sh}}^2}{2} + \delta c - \frac{1}{2} , \quad (2.53a)$$

$$\frac{M_L^2}{8} = \frac{P_{\text{sh}}^2}{2} + \tilde{N} + \delta c - 1 . \quad (2.53b)$$

The twisted massless Hilbert space is again defined by a projection. The states and operators in the twisted sector transform under the space group the same way as (2.42), but replacing $q \to q_{\text{sh}}$, $P \to P_{\text{sh}}$ and the integer oscillators by fractional ones. Of all massless states only those which transform trivially under the space group appear in the on-shell Hilbert space. When we work on the covering space $\mathbb{C}^3$, these states are usually superpositions of states on all fixed points which are mapped onto each other by S. It turns out that all states in the twisted sector appear in complete representations of the $d = 4$ gauge group G_4 in form of $N = 1$ chiral multiplets. That is why they are called *twisted matter*. Since the states are trapped in the fixed points it seems reasonable to assume that the blow-up mode is among them. We will find out that some twisted states can indeed act as a blow-up modes.

[6]Such elements have a shift in a torus which they do not twist.

Modular Invariance

The choice of shift vectors V and Wilson lines W_i is reduced by some restrictions. We already know that they must be integer multiples of some fraction of the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice, e.g. for a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold

$$V \in \frac{1}{N}\Lambda_{16}\,, \tag{2.54a}$$

$$V' \in \frac{1}{M}\Lambda_{16}\,, \tag{2.54b}$$

$$W_i \in \frac{1}{N_i}\Lambda_{16}\,. \tag{2.54c}$$

N_i is the order of the i-th Wilson line and is determined by the point group. But there are further restrictions which come from one-loop modular invariance. A one-loop diagram in closed string theory is a torus and depends on the modular parameter τ. It is well known that a $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{2.55}$$

maps a torus onto itself so the amplitude should be invariant under such transformations. From this it follows that shift vectors and Wilson lines must satisfy a set of conditions. For a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold they read

$$N\left(V^2 - v^2\right) \equiv 0 \bmod 2\,, \tag{2.56a}$$

$$M\left(V'^2 - v'^2\right) \equiv 0 \bmod 2\,, \tag{2.56b}$$

$$\gcd(N,M)\,(V\cdot V' - v\cdot v') \equiv 0 \bmod 2\,, \tag{2.56c}$$

$$N_i\,(W_i\cdot V) \equiv 0 \bmod 2\,, \tag{2.56d}$$

$$N_i\,(W_i\cdot V') \equiv 0 \bmod 2\,, \tag{2.56e}$$

$$N_i\left(W_i^2\right) \equiv 0 \bmod 2\,, \tag{2.56f}$$

$$\gcd(N_i,N_j)\,(W_i\cdot W_j) \equiv 0 \bmod 2\,. \tag{2.56g}$$

Here N_i is the order of the i-th Wilson line, i.e. $N_i \cdot W_i \in \Lambda_{16}$ (no sum over i).

Brother Models

Now we know that an orbifold model is determined by a set of shift vectors and Wilson lines which both are rational multiples of $\mathrm{E}_8 \times \mathrm{E}_8$ lattice vectors. At this point one can ask the question what happens if one adds a lattice vector to one of them while respecting (2.56). This leads to the concept of *brother models* and was studied in [21] for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds. First one finds that the untwisted sector and in particular the gauge group stays the same. Also the choice of six-dimensional twisted states remains but the transformation phase (2.42) changes

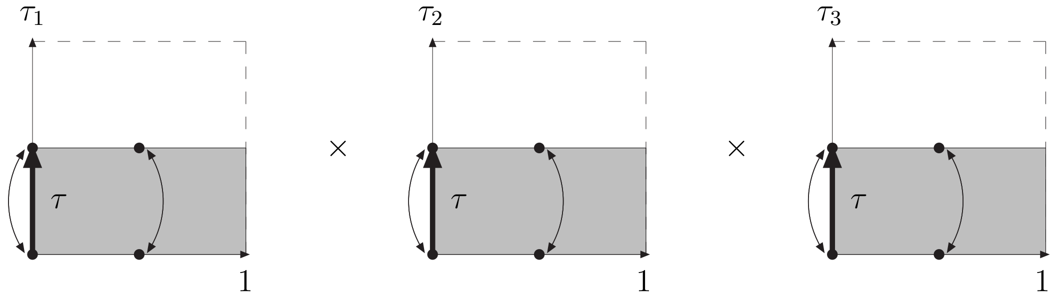


Figure 2.3: $\mathbb{Z}_{2,\text{free}}$ shift on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The fundamental (grey) region is now one eighth of the torus. The upper and lower fixed loci are mapped onto each other simultaneously.

and hence the set of states which survive the projection. Note that the addition of double lattice vectors does not change the transformation phase. Hence the number of brother models is finite. One can go one step further and try to add lattice vectors to each local twist separately such that it satisfies modular invariance. We call such models *grandchildren* or local orbifolds. It is clear that one has many more possibilities for grandchildren than for brothers. It seems that they are needed for an identification between orbifold and resolved models.

2.4 $\mathbb{Z}_{2,\text{free}}$ Shift on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

As already mentioned, this thesis will deal with model building on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold as well as its blown up version. Let us therefore shortly apply the analysis of sec. 2.3 to this particular case and fix the conventions. We need two independent shift vectors, each of order two, which we call V_1 and V_2. Their sum is called $V_3 = V_1 + V_2$ in correspondence with (2.32). The identities like $(\theta_2, e_1)^2 = 1$ imply that all Wilson lines W_i belonging to the Λ_6 lattice base vector e_i are also of order $N_i = 2$, and it turns out that they are all independent[7]. So we need in total eight half $\mathrm{E}_8 \times \mathrm{E}_8$ lattice vectors to specify one model.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold possesses a lot of symmetries. First all twists are equivalent and permuting them corresponds to permutations of the three T^2's. Further one can see by modular transformations that all fixed lines for one twist are of the same type so the following analysis holds for all 48 twisted sectors. In each of them the vacuum energy shift (2.52) has the value $\delta c = 1/4$. Now (2.53a) for massless states reads $q_{\text{sh}}^2 = 1/2$, and together with (2.50) implies that q_{sh} is of the form $\pm(\underline{\tfrac{1}{2}, \tfrac{1}{2}, 0, 0})$. A closer analysis reveals that each state appears as a chiral multiplet and has four degrees of freedom. The masslessness of the left-mover (2.53b) can be ensured in two ways. Either we choose $V_{\text{sh}}^2 = 3/2$ and $\tilde{N} = 0$, or we excite one $\tilde{\alpha}^a_{-1/2}$ oscillator and have $V_{\text{sh}}^2 = 1/2$. In sec. 4.1 we will

[7]For higher order twist the Wilson lines must necessarily be equal or must even vanish.

find out that only the states without oscillator are blow-up mode candidates.

$\mathbb{Z}_{2,\text{free}}$ Symmetry

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold possesses an additional $\mathbb{Z}_{2,\text{free}}$ symmetry. Free means that it has no new fixed points. It acts on the orbifold coordinates by a shift about a half Λ_6 lattice vector

$$\tau : z^a \longrightarrow z^a + \frac{\tau^a}{2}\,, \qquad \tau = \frac{1}{2}\left(e_2 + e_4 + e_6\right)\,. \tag{2.57}$$

$\mathbb{Z}_{2,\text{free}}$ and other $\mathbb{Z}_2$ shift symmetries are studied in [29], but $\mathbb{Z}_{2,\text{free}}$ is the only one which does not introduce new fixed points. As a consequence, a curve which closes by the action of τ cannot be shrunk to a point and the manifold obtained by modding out $\mathbb{Z}_{2,\text{free}}$ has a non-trivial fundamental group, $\pi_1(\mathcal{O}/\mathbb{Z}_{2,\text{free}}) = \mathbb{Z}_{2,\text{free}}$. This allows us to wrap a Wilson line W around this cycle so that we have a new possibility for gauge symmetry breaking. But in contrast to the local shifts which in blow-up turn out to be caused by localized fluxes at the resolved fixed points, W is a pure and topologically stable Wilson line. The symmetry breaking by such Wilson lines does not lead to new massive U(1) factors and hence it is a good candidate for breaking the SU(5) GUT to the standard model while preserving the hypercharge, see sec. 4.2.

The new Wilson line can again be implemented by embedding $\mathbb{Z}_{2,\text{free}}$ into the gauge transformations and modding it out. Now τ can be expressed in terms of half lattice vectors so the requirement to preserve the symmetry in the gauge embedding poses relations on the Wilson lines W_2, W_4, W_6 and W. The identity $\tau \circ \theta_i \circ \tau \circ \theta_i = e_{2i}$ implies

$$2W \equiv W_{2i} \qquad i = 1, 2, 3\,, \tag{2.58}$$

which leads to

$$W_2 \equiv W_4 \equiv W_6\,, \tag{2.59}$$

and which says that W is of order 4. In this context "$\equiv$" denotes equality up to an $\mathrm{E}_8 \times \mathrm{E}_8$ lattice vector. As for the shift vectors and the other Wilson lines, there are constraints on W imposed by modular invariance of the one-loop string amplitude,

$$2W^2 = 0 \text{ mod } 1\,, \tag{2.60a}$$
$$4W \cdot V_i = 0 \text{ mod } 1\,, \tag{2.60b}$$
$$4W \cdot W_i = 0 \text{ mod } 1\,. \tag{2.60c}$$

Their explicit derivation will be presented in [32]. $\mathbb{Z}_{2,\text{free}}$ maps two fixed points onto each other. Using the notation introduced in sec. 2.2 the mapping is

$$F_{1,\beta\gamma} \leftrightarrow F_{1,\beta'\gamma'}\,, \tag{2.61a}$$
$$F_{2,\alpha\gamma} \leftrightarrow F_{2,\alpha'\gamma'}\,, \tag{2.61b}$$
$$F_{3,\alpha\beta} \leftrightarrow F_{3,\alpha'\beta'}\,, \tag{2.61c}$$

with

$$\begin{array}{c|c} \alpha,\beta,\gamma & \alpha',\beta',\gamma' \\ \hline 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{array} \quad . \tag{2.62}$$

The symmetry requires that the twisted spectra localized at these points must be identified which can be ensured by condition (2.59) when we require equality without addition of lattice vectors. As a result the multiplicities in the twisted spectrum will be divided by two when modding out $\mathbb{Z}_{2,\text{free}}$.

Let us take a look at the Hodge diamond. All untwisted forms are invariant unter $\mathbb{Z}_{2,\text{free}}$ so the untwisted Hodge diamond (2.35) stays the same. In particular the holomorphic 3-form Ω survives which tells us that the manifold is still CY after modding out. In the twisted sectors the number of fixed lines is reduced by half, i.e. we are left with 24 fixed lines, 8 for each of the three twists. Since the geometry of the fixed tori is untouched, the twisted Hodge diamond will also be divided by two. We get

$$h^{p,q} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 27 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 27 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,, \tag{2.63}$$

and as a direct consequence

$$\chi = 48\,. \tag{2.64}$$

2.5 A 6-Generation GUT on $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

Our intention is of course to find a heterotic model which comes as close as possible to the MSSM. The highest priority is to reproduce the gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and the chiral spectrum with three families of quarks

and leptons and a pair of Higgses. One promising possibility is to have a SU(5) GUT with matter in the appropriate representations which is broken at a scale of $M_{\mathrm{GUT}} \sim 10^{16}$ GeV. We will follow this path and first try to find a GUT model on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and then break it down to the SM via the Wilson line that comes up by modding out $\mathbb{Z}_{2,\mathrm{free}}$. But doing so, the multiplicity of twisted matter will be halved, so we need a GUT model with six twisted generations since the untwisted sector is not chiral. $\mathbb{Z}_{2,\mathrm{free}}$ also requires (2.59) to be satisfied. Using [33] we find that there are only few inequivalent models satisfying these constraints. One of them will be presented here. Its phenomenological details are about to be published in [31].
The shift vectors and Wilson lines of this model are

$$V_1 = \left(\frac{1}{2},\frac{1}{2},2,0,0,0,1,-1,\quad 0,1,1,0,1,0,0,-1\right)\,, \tag{2.65a}$$

$$V_2 = \left(\frac{5}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\quad \frac{1}{2},\frac{1}{2},0,0,0,0,0,4\right)\,, \tag{2.65b}$$

$$W_1 = \left(0^{16}\right) \tag{2.65c}$$

$$W_3 = \left(1,-1,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\quad \frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{5}{4},-\frac{3}{4},\frac{1}{4},-\frac{3}{4},\frac{1}{4}\right)\,, \tag{2.65d}$$

$$W_5 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\quad \frac{1}{2},\frac{1}{2},0,0,0,0,-\frac{1}{2},-\frac{1}{2}\right)\,, \tag{2.65e}$$

$$\begin{aligned} W_2 &= W_4 = W_6 \\ &= \left(-1,-1,0-1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\quad -\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\,. \end{aligned} \tag{2.65f}$$

The non-Abelian part of the gauge group of this model is SU(5) × SU(4) × SU(4) where the SU(5) arises from the first E_8. Since we have no rank reduction, there are six U(1) factors, one of which is anomalous,

$$t_{\mathrm{anom}} = \left(-1,-1,-2,-1,1,1,1,2,\quad -\frac{1}{2},-\frac{1}{2},0,0,0,0,-\frac{1}{2},-\frac{1}{2}\right)\,. \tag{2.66}$$

A summary of the chiral spectrum is shown in tab. 2.1. For more details see appendix B. We observe that the multiplicities in the twisted sectors are all even. The multiplicities in the θ_2 and θ_3 sectors are actually multiples of four which follows from the vanishing of the first Wilson line. In total we have six **10**'s, nine **5**'s and fifteen $\overline{\mathbf{5}}$'s which guarantees non-Abelian anomaly freedom. The forth-order Wilson line W is chosen to be

$$W = \left(-\frac{1}{2},-\frac{1}{2},0-\frac{1}{2},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\quad -\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)\,, \tag{2.67}$$

(a) Untwisted Sector

#	irrep
6	$(\mathbf{1},\mathbf{1},\mathbf{1})$
3	$(\mathbf{5},\mathbf{1},\mathbf{1})$
3	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$

(b) θ_1 Twisted Sector

#	irrep
2	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$
18	$(\mathbf{1},\mathbf{1},\mathbf{1})$
2	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$
2	$(\mathbf{1},\mathbf{6},\mathbf{1})$
4	$(\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$
4	$(\mathbf{1},\mathbf{4},\mathbf{1})$
2	$(\mathbf{1},\mathbf{1},\mathbf{4})$
4	$(\mathbf{5},\mathbf{1},\mathbf{1})$
2	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}})$

(c) θ_2 Twisted Sector

#	irrep
4	$(\mathbf{5},\mathbf{1},\mathbf{1})$
12	$(\mathbf{1},\mathbf{1},\mathbf{1})$
4	$(\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$
4	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}})$
4	$(\mathbf{1},\mathbf{1},\mathbf{4})$

(d) θ_3 Twisted Sector

#	irrep
4	$(\mathbf{5},\mathbf{1},\mathbf{1})$
16	$(\mathbf{1},\mathbf{1},\mathbf{1})$
4	$(\mathbf{1},\mathbf{4},\mathbf{1})$
4	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$
4	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$

Table 2.1: Summary of the orbifold model matter spectrum.

i.e. $W = W_2/2$. It breaks $\mathrm{SU}(5) \to \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ with the hypercharge generator

$$t_{\mathrm{Y}} = \left(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \quad 0^8\right) . \tag{2.68}$$

The $\mathbf{10}$'s and six of the $\bar{\mathbf{5}}$'s will then become three families of quarks, leptons and their superpartners. Further two of the $\mathbf{5}$'s and $\bar{\mathbf{5}}$'s will result in a pair of up- and down-Higgses. The remaining ones should decouple from the low energy spectrum by getting high mass terms. The hidden sector gauge group is broken to $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)^4$ which leaves the possibility for SUSY breaking at low scales by gaugino condensation. For more details see [31].

Chapter 3

Blow-Up of Orbifold Singularities

In the last chapter we have seen how the spectra of the low energy effective field theories are obtained from the heterotic orbifold construction. In this theory one has to construct the superpotential which in general will give vacuum expectation values (vevs) to some fields. Due to the appearance of the anomalous $\mathrm{U}(1)_{\mathrm{anom}}$, the induced *Fayet–Iliopoulos* (FI) term requires some charged fields to attain a vev in order to preserve $N = 1$ supersymmetry. These fields must be spacetime scalars and charged under $\mathrm{U}(1)_{\mathrm{anom}}$, so most of them are found in the twisted sectors. If they get a non-trivial vev, this has a backreaction on the geometry and results in a resolution of the fixed point. It follows that the curvature which was localized in the fixed point is now smoothed out and we obtain a differentiable CY manifold. We want do describe this manifold and explore its geometrical and topological properties. The mathematical tool which we use to describe the blown up singularity is called toric geometry and belongs to the field of algebraic geometry. Toric means that it deals with spaces, so-called toric varieties, which contain an algebraic torus $(\mathbb{C}^*)^n$ as a dense open subset, i.e. (partial) compactifications of them. A mathematical introduction is given in [15]. Toric geometry will allow us to take a singularity of the type $\mathbb{C}^n/P$ where $P = \mathbb{Z}_N$ or $P = \mathbb{Z}_N \times \mathbb{Z}_M$, remove the fixed points or lines and replace them with smooth spaces of codimension two. The orbifold coordinates are not able to parameterize these smooth spaces, so the idea of toric geometry is to construct new coordinates and mod out a continuous group action to keep the dimension. This way a topological structure becomes visible to which we had no access in the singular limit. A complete orbifold can then be blown up by cutting out each singularity, blowing it up and gluing the resolved spaces together, see fig. 3.1.

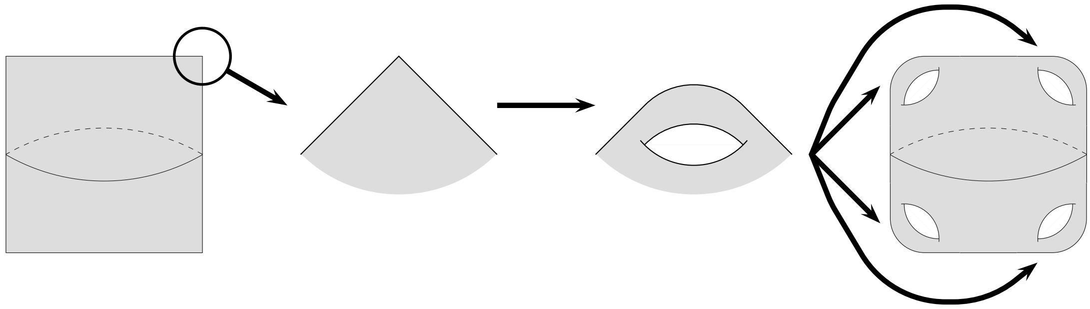


Figure 3.1: To blow up an orbifold we first cut out neighborhoods around the singularities then replace the singular point by a smooth space and finally glue the resolved spaces together.

3.1 Toric Geometry

We want to describe the concepts and methods of toric geometry by the illustrative example of complex projective space $\mathbb{CP}^n$. $\mathbb{CP}^n$ is a compactification of $\mathbb{C}^n$ and it can be obtained by adding a lower dimensional space, a $\mathbb{CP}^{n-1}$, which glues the boundary of $\mathbb{C}^n$ together. But the compactification can be done in many other ways or one can perform just a partial compactification. Such possibilities can be described by toric geometry. Here we will present the homogeneous coordinate construction of the toric variety since it is more illustrative.

The starting point of such a description is the toric diagram. It consists of a set of points in an n dimensional lattice isomorphic to $\mathbb{Z}^n$, where n is the complex dimension of the variety. For $\mathbb{CP}^n$ (fig. 3.2) we choose the set $V = \{v_i\}$, $i = 1 \dots n+1$ given by

$$v_i = (0, \dots, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, \dots, 0)\,, \qquad i = 1 \dots n \tag{3.1a}$$

$$v_{n+1} = (-1, \dots, -1)\,. \tag{3.1b}$$

To each of these vectors we associate a homogeneous coordinate z_i so we start with a $\mathbb{C}^{n+1}$.

The next step is to triangulate the toric diagram. This means we have to divide the diagram into a set of convex cones[1] spanned by subsets of V such that the intersection of two cones of same dimension is a lower dimensional cone. This set of cones is called *fan*. The triangulation is not always unique and we will see that different triangulations lead to different topologies since the cones in the fan correspond to non-trivial cycles on the variety.

Now we choose the *exclusion set* $Z \in \mathbb{C}^{n+1}$ which contains all simultaneous zero loci of all combinations of coordinates z_i such that the associated vectors v_i

[1] A cone spanned by a set of vectors $\{\phi_i\}$ is the set $\sigma = \left\{\sum \alpha_i \phi_i \middle| \alpha_i \geq 0\right\}$. It is convex if $\omega \in \sigma - \{0\}$ implies $-\omega \notin \sigma$.

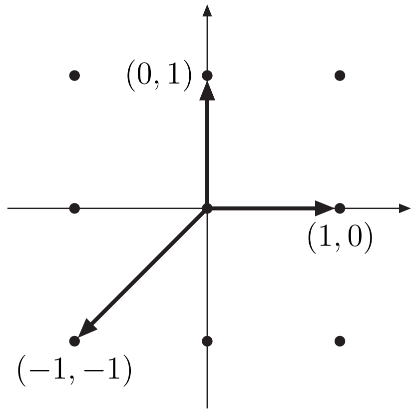


Figure 3.2: Toric Diagram of a $\mathbb{CP}^2$

do not span a cone in the fan. For the $\mathbb{CP}^n$ one can easily see that there exists only one triangulation in which any proper subset $V' \subsetneq V$ spans a convex cone whereas the cone spanned by V is not convex. Hence the exclusion set is

$$Z = \{z_1 = \ldots = z_{n+1} = 0\} \ . \tag{3.2}$$

This set will be subtracted in order to avoid singularities.

At this point we still have more complex coordinates than the dimension of the variety we wish to describe. Therefore the final step is to find a $(\mathbb{C}^*)^r$ action and reduce the dimension by modding it out. Here r is the difference between the number of homogeneous coordinates and the desired dimension, $r = \#V - n$. In the toric diagram we have $n + r$ vectors in a n dimensional lattice so there are r independent linear equivalences of the form

$$\sum_i \alpha_i^k v_i = 0\,, \qquad k = 1 \ldots r\,, \tag{3.3}$$

where the vectors α^k are linearly independent. This equivalences can now be translated into a $(\mathbb{C}^*)^r$ action on the homogeneous coordinates

$$z_i \xrightarrow{\lambda} \prod_{k=1}^{r} \left(\lambda^k\right)^{\alpha_i^k} \cdot z_i\,. \tag{3.4}$$

One can easily see that the $(\mathbb{C}^*)^r$ action only depends on the space spanned by the α^k and not on the particular choice. Hence it is uniquely determined by the toric diagram. For the $\mathbb{CP}^n$ we have $r = 1$, the only linear equivalence reads $\sum_i v_i = 0$ and thus the $\mathbb{C}^*$ acts as $z_i \to \lambda z_i$. This way we found our standard recipe to construct a $\mathbb{CP}^n$

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}}{\mathbb{C}^*}\,. \tag{3.5}$$

For a general toric variety it reads

$$\mathcal{V} = \frac{C^{n+r} - Z}{(\mathbb{C}^*)^r}\,. \tag{3.6}$$

One can construct inhomogeneous coordinates which are invariant under (3.4) out of the homogeneous ones. One quickly finds the following inhomogeneous coordinates

$$Z_k = \prod_i z_i^{(v_i)_k}\,, \qquad k = 1\ldots n\,. \tag{3.7}$$

Furthermore, all monomials in the Z_k are allowed as inhomogeneous coordinates. A set of n inhomogeneous coordinates can only describe a subset of the variety and we need several such sets to cover the whole variety just in the sense of a manifold.

3.1.1 Divisors and Intersections

Now that we have created the variety we can start studying its topology. First one defines the divisors[2] as the zero loci of the homogeneous coordinates,

$$D_i = \{z_i = 0\}\ . \tag{3.8}$$

They are of complex codimension one and they are non-trivial cycles in the $H_{n-1,n-1}$ homology group. By Poincaré duality we can associate a $(1,1)$-form[3] to each of the divisors which we will by abuse of notation also call D_i. It should always be clear from the context whether we use the divisor as a cycle or as a form.

There is a naturally defined product on the space of divisors: For k divisors $D_{i_1}, \ldots, D_{i_k}$ we define

$$D_{i_1} \ldots D_{i_k} = D_{i_1} \cap \ldots \cap D_{i_k} \in H_{n-k,n-k}\,. \tag{3.9}$$

In form language this product turns out to be the wedge product

$$D_{i_1} \ldots D_{i_k} = D_{i_1} \wedge \ldots \wedge D_{i_k} \in H^{k,k}\,. \tag{3.10}$$

The product is not always well-defined by (3.9) for cycles since due to the orientation a sign can appear and since it does not define self-intersections. But one can always turn to forms to avoid these problems. A product of n divisors is a finite set of points or as its Poincaré dual a multiple of the volume form. In abuse

[2]Generally, zero loci of polynomials in homogeneous coordinates are called divisors.

[3]These forms can also be defined as the first Chern class of the line bundle defined by the transition functions of the polynomial of which D_i is the zero locus.

of notation we will denote the number of these points with the sign encoding relative orientations by such products. This number, which is to the integral of the dual (n, n)-form over the whole variety, is called intersection number,

$$D_{i_1} \dots D_{i_n} = \int_{\mathcal{V}} D_{i_1} \wedge \dots \wedge D_{i_n} = \# D_{i_1} \cap \dots \cap D_{i_n} \,. \tag{3.11}$$

Let us consider the case when all divisors in such a product are non-compact which can happen on a non-compact variety. If the cycles have an intersection point it can be pushed to infinity and away off the variety. In such a case the intersection number will turn out to be a rational number. In the other case, when at least one divisor is compact, the intersection numbers are all integers.

The compactness of a divisor can be easily checked from the toric diagram: If and only if the corresponding vector is surrounded by cones, i.e. if it is not at the boundary of the diagram, the divisor is compact. Hence a variety whose cones cover the whole lattice is compact, and otherwise it is not. Now all compact intersection numbers of distinct divisors can be read off from the triangulated toric diagram. If the associated vectors span a cone in the fan the divisors intersect in exactly one point. This is no surprise since the exceptional set contains exactly those points in which divisors not spanning a cone in the fan could have intersected. If the cone is not divisible into two cones but does not belong to the fan or if the n vectors are linearly dependent the divisors do not intersect.

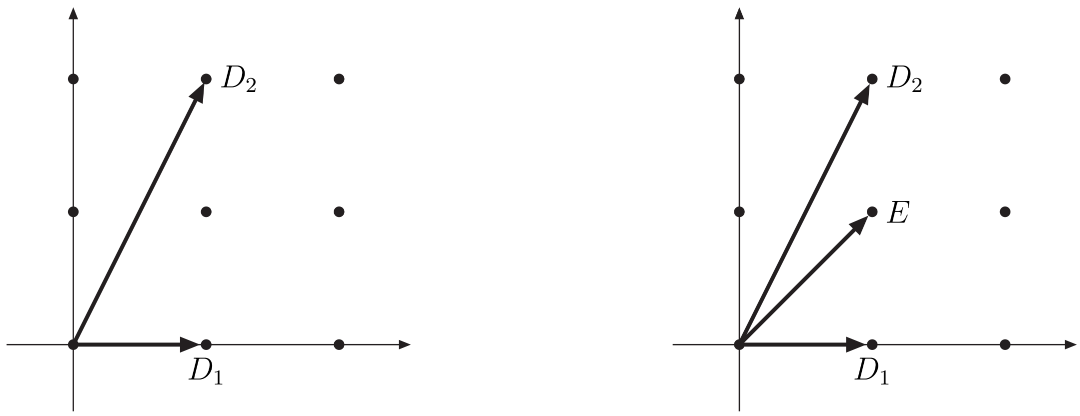
The next step is to obtain all possible intersection numbers, in particular those including self intersection. To this end, we can make use of n linear equivalences between the divisors,

$$\sum_i (v_i)_k \, D_i \sim 0 \,, \qquad k = 1 \dots n \,. \tag{3.12}$$

Here "$\sim$" means equality up to a boundary or an exact form, respectively. These linear equivalences together with the basic intersection numbers allow us to compute the remaining ones. How this works will be described in more detail for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold in sec. 3.3. To sum up, we have found $\# V = n + r$ non-trivial $(1, 1)$-forms and n linear equivalence relations so only r of them are independent and we find $h^{1,1} = r$.

Chern Class

Much topological information is encoded in the Chern class of a complex manifold. The CY condition can be formulated as the vanishing of the first Chern class, $c_1(\mathrm{CY}) = 0$. Furthermore, the Euler characteristic of a complex n-fold can



(a) $\mathbb{C}^2/\mathbb{Z}_2$ (b) $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$

Figure 3.3: Toric diagram of $\mathbb{C}^2/\mathbb{Z}_2$ and its resolution. It is convenient to label the vectors with the divisors.

be obtained by

$$\chi = \int_{X_n} c_n \,. \tag{3.13}$$

For a toric variety the total Chern class is expressed by the divisors (as $(1,1)$-forms)

$$c(\mathcal{V}) = \prod_i (1 + D_i) \ . \tag{3.14}$$

This can now be used to extract[4] the single Chern classes, e.g. the first one is $c_1(\mathcal{V}) = \sum_i D_i$.

3.2 Resolution of non-Compact Singularities

We want to use toric geometry to blow up the singularities of an orbifold. In a few particular cases [23, 26, 27] the blow-up can be done more explicitly by constructing a Kähler metric on the resolved space. This supports non-Abelian bundles and hence more possibilities for gauge symmetry breaking. For the remaining cases like the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we must arrange the model building by the topological data. For isolated singularities this is done in [24].

3.2.1 $\mathbb{C}^2/\mathbb{Z}_2$

We start with the blow up of a single $\mathbb{C}^2/\mathbb{Z}_2$ singularity. It is the simplest type of singularity with $\mathrm{SU}(N)$ holonomy and hence very illustrative. Furthermore, all

[4]The n-th Chern class is the (n,n)-form part of the total Chern class.

singularities of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold are of this type once we fix the coordinate on the fixed torus. The general procedure how to blow up any orbifold singularity will be depicted in sec. 3.2.2.

The toric diagram for $\mathbb{C}^2/\mathbb{Z}_2$, fig. 3.3(a), consists of the vectors

$$u_1 = (1, 0)\,, \qquad u_2 = (1, 2)\,. \tag{3.15}$$

There are only two homogeneous coordinates and hence no $\mathbb{C}^*$ action to mod out so one would naively say that we describe a $\mathbb{C}^2$. But a look at the inhomogeneous coordinates $Z_1 = z_1^2$ and $Z_2 = z_1 z_2$ reveals that we are describing a $\mathbb{C}^2/\mathbb{Z}_2$ with $\mathbb{Z}_2$ action $(z_1, z_2) \to (-z_1, -z_2)$ since Z_1 and Z_2 describe a basis for invariant monomials on it. Generally a toric variety is non-singular if and only if the vectors in the toric diagram span the whole lattice by integer linear combinations which is clearly not the case here.

Resolution

Hence we need another divisor to make the variety smooth. This divisor is called exceptional divisor E and the corresponding vector is $w = (1, 1)$, see fig. 3.3(b). The homogeneous coordinate is denoted by x. In two dimensions the triangulation is always unique. Here the sets of vectors that do not span cones in the fan are $\{u_1, u_2\}$ and $\{u_1, u_2, w\}$ thus we find the exclusion set $Z = \{z_1 = z_2 = 0\}$.

The toric vectors are linearly dependent,

$$u_1 + u_2 - 2w = 0\,, \tag{3.16}$$

so we find the $\mathbb{C}^*$ action

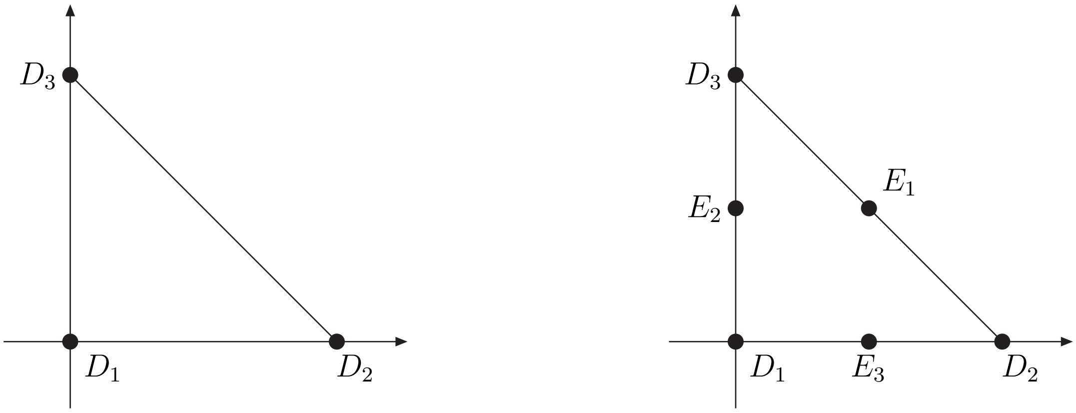
$$(z_1, z_2, x) \to \left(\lambda z_1, \lambda z_2, \lambda^{-2} x\right)\,, \tag{3.17}$$

which leads us to the definition of the variety

$$\mathcal{V} = \frac{\mathbb{C}^3 - Z}{\mathbb{C}^*}\,. \tag{3.18}$$

Let us explore the geometry of the variety. First we look at the coordinate patch defined by $U = \{x \neq 0\}$. Here we can use the $\mathbb{C}^*$ action (3.17) to fix the value of x to $x = 1$ by choosing $\lambda = \pm\sqrt{x}$. We see that there is a remaining $\mathbb{Z}_2$ ambiguity which further identifies the points (z_1, z_2) and $(-z_1, -z_2)$. Taking into account the exceptional set we find that U is a $\mathbb{C}^2/\mathbb{Z}_2$ where the singularity $\{z_1 = z_2 = 0\}$ has been removed. The other part of the variety is the exceptional divisor $E = \{x = 0\}$. The remaining $\mathbb{C}^*$ action on z_1 and z_2 shows that E is a $\mathbb{CP}^1$. This $\mathbb{CP}^1$ now replaces the singularity which can be seen by constructing inhomogeneous coordinates $Z_i = x z_i^2$ and letting $Z_i \to 0$ which corresponds to $x \to 0$. Hence the singularity has been resolved,

$$\mathcal{V} = U \cup E = \mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)\,. \tag{3.19}$$



(a) $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. (b) $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$, untriagulated

Figure 3.4: Projected toric diagrams. The figure shows the $x = 1$ plane of the unprojected diagram.

The linear equivalences (3.12) can be written as $2D_i + E \sim 0$, $i = 1, 2$ and we read off the basic intersection numbers $D_i E = 1$, $i = 1, 2$. This immediately leads to the remaining intersection numbers $E^2 = -2$, $D_i D_j = -1/2$. Note that the D_i are not compact and hence their intersection number is fractional. Furthermore, the intersection $D_1 D_2$ should vanish due to the toric diagram but it does not because of non-compactness. In the compact case this issue is resolved by the presence of additional divisors.

The first Chern class (3.14) is $c_1 = E + D_1 + D_2$ and the linear equivalences show that it vanishes. Thus our resolved space is still a CY. Using the intersection numbers we can integrate the second Chern class $\chi = \int c_2 = D_1 D_2 + D_1 E + D_2 E = 3/2$. Gluing 16 such singularities together results in a compact $T^4/\mathbb{Z}_2$ with Euler number $\chi = 16 \cdot 3/2 = 24$ which is the Euler number of a K3 as expected.

3.2.2 $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

In the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we deal with 48 fixed lines which meet in the 64 points $(z_1^\alpha, z_2^\beta, z_3^\gamma)$, $\alpha, \beta, \gamma = 1 \ldots 4$. A neighborhood of such a point looks like a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ so these are the singular spaces which we want to resolve. The blow-up procedure of such a singularity always starts with the construction of the toric diagram. We first choose homogeneous coordinates z_i, $i = 1 \ldots 3$ on the covering space $\mathbb{C}^3$ and associate a vector u_i to each of them. The vectors must be chosen such that the monomials $\prod_i z_i^{(u_i)_k}$ are invariant under the point group and are an integer basis of all invariant monomials. For a singularity with SU(3) holonomy the monomial $z_1 z_2 z_3$ is always invariant so we can choose $(u_i)_1 \equiv 1$. This means the toric diagram lies in a plane and leads us to the projected toric diagram which is easier to visualize. For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity with point group

(2.32) we take the vectors

$$u_1 = (1,0,0)\,, \qquad u_2 = (1,2,0)\,, \qquad u_3 = (1,0,2)\,. \tag{3.20}$$

The respective divisors, called ordinary divisors, are labeled by D_i, see fig. 3.4(a). The next step is to resolve the orbifold. We add one exceptional divisor E_i for each point group element $\theta_i = \exp\{2\pi\mathrm{i}\,\mathrm{diag}\,(v_i^1, v_i^2, v_i^3)\}$ with $0 \leq v_i^a < 1$ [5] and $\sum_a v_i^a = 1$. The associated toric vector is

$$w_i = \sum_a v_i^a u_a\,. \tag{3.21}$$

For the $\mathbb{Z}_2\times\mathbb{Z}_2$ twist all non-trivial point group elements satisfy these conditions. Thus we add the three vectors

$$w_1 = (1,1,1)\,, \qquad w_2 = (1,0,1)\,, \qquad w_3 = (1,1,0)\,. \tag{3.22}$$

These vectors lie in the same plane as the u_i, so we can draw them into the projected toric diagram (fig. 3.4(b)). We recover three times the structure of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ (fig. 3.3(b)) which is one exceptional divisor between two ordinary divisors. Note that the exceptional divisors are not compact. The respective homogeneous coordinates are denoted by x_i.

Now we must triangulate the toric diagram which is not unique in this case. There are four possible triangulations denoted by “symm” (for symmetric), “E_1”, “E_2” and “E_3”, see fig. 3.5. From the triangulation we can read of the exclusion set

$$\begin{aligned} Z_{\text{“symm”}} =& \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\} \\ & \cup \{z_1 = x_1 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_3 = x_3 = 0\}\,, \end{aligned} \tag{3.23a}$$

$$\begin{aligned} Z_{\text{“}E_1\text{”}} =& \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\} \\ & \cup \{x_2 = x_3 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_3 = x_3 = 0\}\,, \end{aligned} \tag{3.23b}$$

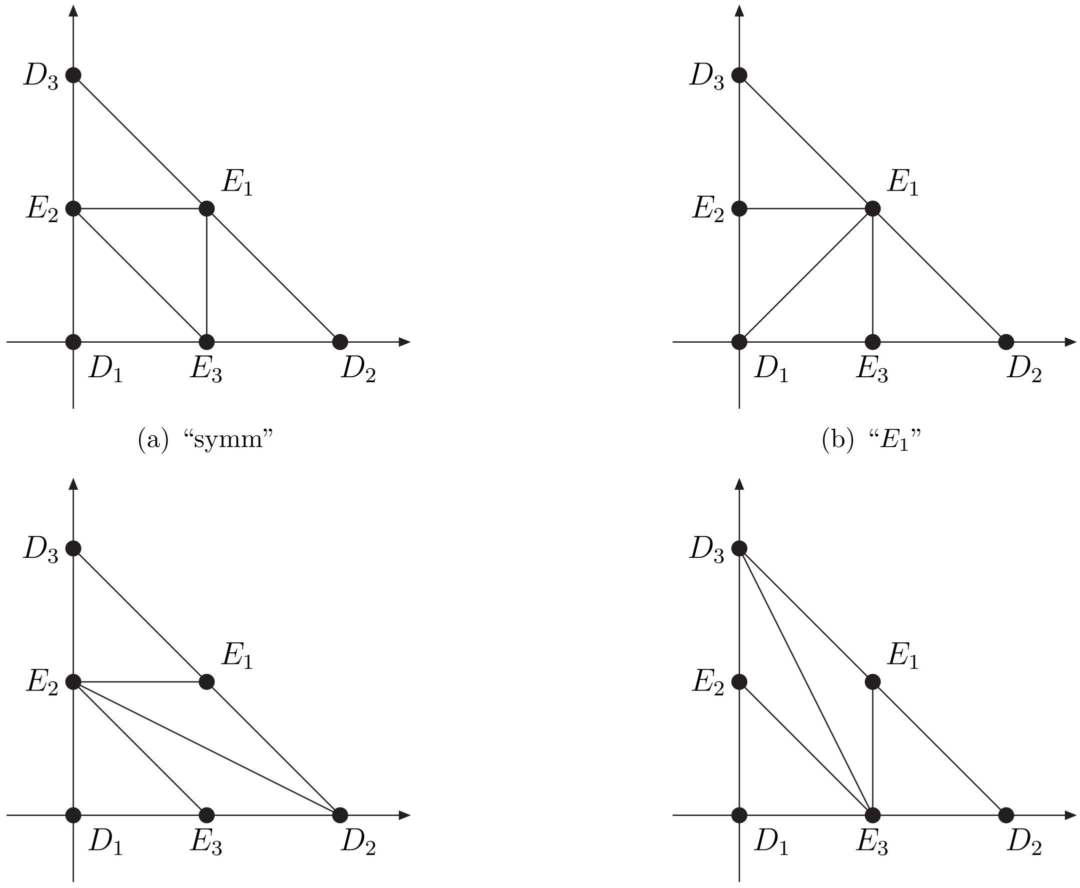
$$\begin{aligned} Z_{\text{“}E_2\text{”}} =& \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\} \\ & \cup \{z_1 = x_1 = 0\} \cup \{x_1 = x_3 = 0\} \cup \{z_3 = x_3 = 0\}\,, \end{aligned} \tag{3.23c}$$

$$\begin{aligned} Z_{\text{“}E_3\text{”}} =& \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\} \\ & \cup \{z_1 = x_1 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{x_1 = x_2 = 0\}\,. \end{aligned} \tag{3.23d}$$

The six vectors u_i, w_i obey three linear dependences which are exactly the definitions of the exceptional vectors (3.21). They read

$$u_2 + u_3 - 2w_1 = 0\,, \tag{3.24a}$$
$$u_3 + u_1 - 2w_2 = 0\,, \tag{3.24b}$$
$$u_1 + u_2 - 2w_3 = 0\,, \tag{3.24c}$$

[5] This can always be achieved by adding integers.



(a) "symm"

(b) "E_1"

(c) "E_2"

(d) "E_3"

Figure 3.5: Different triangulations of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$

triangulation	"symm"	"E_1"	"E_2"	"E_3"
$E_1E_2E_3$	1	0	0	0
$E_1E_2^2$	-1	-2	0	0
$E_1E_3^2$	-1	-2	0	0
$E_1^2E_2$	-1	0	-2	0
$E_2E_3^2$	-1	0	-2	0
$E_1^2E_3$	-1	0	0	-2
$E_2^2E_3$	-1	0	0	-2
E_1^3	1	0	2	2
E_2^3	1	2	0	2
E_3^3	1	2	2	0

Table 3.1: Intersection numbers of exceptional divisors for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

and lead to the $(\mathbb{C}^*)^3$ action

$$(z_1, z_2, z_3, x_1, x_2, x_3) \xrightarrow{(\lambda_1,\lambda_2,\lambda_3)} \left(\lambda_2\lambda_3 z_1, \lambda_1\lambda_3 z_2, \lambda_1\lambda_2 z_3, \frac{x_1}{\lambda_1^2}, \frac{x_2}{\lambda_2^2}, \frac{x_3}{\lambda_3^2}\right) . \tag{3.25}$$

Now we can define the resolved space

$$\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)_{\mathrm{triang}} = \frac{\mathbb{C}^6 - Z_{\mathrm{triang}}}{(\mathbb{C}^*)^3} , \tag{3.26}$$

with triang $\in$ {"symm","E_1","E_2","E_3"}. There are three linear dependence relations (3.12),

$$D_1 + D_2 + D_3 + E_1 + E_2 + E_3 \sim 0 , \tag{3.27a}$$

$$2D_2 + E_1 + E_3 \sim 0 , \tag{3.27b}$$

$$2D_3 + E_1 + E_2 \sim 0 . \tag{3.27c}$$

It is convenient to replace the first one to have them in a more symmetric manner,

$$2D_1 + E_2 + E_3 \sim 0 . \tag{3.27d}$$

They can be used to compute all intersection numbers from the basic ones. The results for the exceptional divisors are listed in table 3.1. Intersection numbers containing ordinary divisors can be calculated using (3.27). We observe:

- The different triangulations will result in different intersection numbers and hence in different topologies, which in blow-down become the same singularity. At this point it is not clear which triangulation we should choose in blow-up. In sec. 3.4 we will see that the choice of triangulation depends on the Kähler moduli and that flop transitions between them are possible.

- The permutation symmetry between the complex coordinates $z_i \to z_{\sigma(i)}$ is still there when we also permute the exceptional divisors $E_i \to E_{\sigma(i)}$ and the triangulation "E_i" $\to$ "$E_{\sigma(i)}$". The symmetric triangulation is mapped to itself.

- The lines between the divisors correspond to the simultaneous zero loci of coordinates which are not excluded by the exceptional set (3.23). On the variety this means that these divisors intersect in a $(1,1)$-cycle. Thus varieties of different triangulations contain different complex curves. We also know that triangles in the toric diagram are points ($(0,0)$-cycles) on the variety in which the divisors at the corners intersect. Generally the real dimension of the cones is the complex codimension of the associated cycles.

- The first Chern class $c_1 = \sum_i D_i + \sum_i E_i$ vanishes due to (3.27a). Once again we see that the CY condition is equivalent to the toric vectors lying in a plane.

- Using (3.27) we can rewrite the third Chern class in terms of the exceptional divisors only,

$$\begin{aligned} c_3 = &\frac{1}{8}(2E_1^3 + 2E_2^3 + 2E_3^3 - E_1^2 E_2 \\ &- E_1^2 E_3 - E_2^2 E_1 - E_2^2 E_3 - E_3^2 E_1 - E_3^2 E_2)\,. \end{aligned} \tag{3.28}$$

  Although the single terms are triangulation-dependent the value of the integral of c_3 over the variety turns out to be universal. We find $\int c_3 = 3/2$ which fits exactly with the fact that 64 such varieties can be glued to a resolution of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with $\chi = 96$ (2.37).

3.3 Resolution of Compact $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

In the last section we saw how to blow up singularities that appear in toroidal orbifolds. Our intention is of course to have a description of a resolved version of the whole compact orbifold. Although this can not be done by a single toric description we can nevertheless use the terminology and the results from the non compact resolved singularity. The blow-up of many compact orbifolds is described in [22]. In [28] the resolution of the $\mathbb{Z}_{6-\mathrm{II}}$ orbifold and heterotic model building are studied.

Divisors

In the local case we had three ordinary divisors $D_i = \{z_i = 0\}$. In blow-down a product of two such divisors was a fixed line and a product of three was the point in which they intersect. In the global orbifold case we have now in each torus four loci in which the fixed lines sit, see fig. 2.1. Hence we define four ordinary divisors on each torus, i.e. 12 in total, in the following way:

$$D_{1,\alpha} = \{z_1 = z_1^\alpha\}\ , \qquad D_{2,\beta} = \left\{z_2 = z_2^\beta\right\}\ , \qquad D_{3,\gamma} = \{z_3 = z_3^\gamma\}\ . \tag{3.29}$$

The labels α, β, γ are used in the same manner as in (2.34). As in the non-compact case we can recover a fixed torus by taking a product of two of them.

$$F_{1,\beta\gamma} = D_{2,\beta}D_{3,\gamma}\ , \qquad F_{2,\alpha\gamma} = D_{1,\alpha}D_{3,\gamma}\ , \qquad F_{3,\alpha\beta} = D_{1,\alpha}D_{2,\beta}\ . \tag{3.30}$$

Each of these 48 fixed lines is a $\mathbb{C}^2/\mathbb{Z}_2$ singularity times a $T^2/\mathbb{Z}_2$ so to blow them up we replace them by 48 exceptional divisors which lie "between" the ordinary divisors, 16 in the θ_1 sector, $\underset{D_{2,\beta}}{\bullet}$ — $\underset{E_{1,\beta\gamma}}{\bullet}$ — $\underset{D_{3,\gamma}}{\bullet}$, 16 in the θ_2 sector, $\underset{D_{1,\alpha}}{\bullet}$ — $\underset{E_{2,\alpha\gamma}}{\bullet}$ — $\underset{D_{3,\gamma}}{\bullet}$, and 16 in the θ_3 sector, $\underset{D_{1,\alpha}}{\bullet}$ — $\underset{E_{3,\alpha\beta}}{\bullet}$ — $\underset{D_{2,\beta}}{\bullet}$. These divisors are all non-trivial $(2,2)$-cycles or closed $(1,1)$-forms.

On the orbifold there are three more $(2,2)$-cycles which are obtained by fixing one coordinate on the bulk. On the covering torus we define these so called inherited divisors in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant way

$$R_i = \{z_i = z_{\text{const}}\} \cup \{z_i = -z_{\text{const}}\}\ , \qquad i = 1\ldots 3\,. \tag{3.31}$$

In blow-down the inherited divisors are $T^4/\mathbb{Z}_2$ orbifolds so in blow-up we expect them to be smooth K3's. The Poincaré duals are the forms $\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$. When we move z_{const} towards a fix locus, $z_{\text{const}} \to z_i^\delta$, we find the linear equivalence relation for all ordinary divisors valid at the orbifold point,

$$2D_{i,\delta} \sim R_i\ , \qquad i = 1,2,3\ , \qquad \delta = 1\ldots 4\,. \tag{3.32}$$

In blow-up we have to take into account that each ordinary divisor is locally equivalent to exceptional divisors (3.27) so we find the blow-up linear equivalence relations,

$$2D_{1,\alpha} + \sum_\beta E_{3,\alpha\beta} + \sum_\gamma E_{2,\alpha\gamma} \sim R_1\ , \tag{3.33a}$$

$$2D_{2,\beta} + \sum_\alpha E_{3,\alpha\beta} + \sum_\gamma E_{1,\beta\gamma} \sim R_2\ , \tag{3.33b}$$

$$2D_{3,\gamma} + \sum_\alpha E_{2,\alpha\gamma} + \sum_\beta E_{1,\beta\gamma} \sim R_1\,. \tag{3.33c}$$

They show that all ordinary divisors are linear combinations of exceptional and inherited ones. Furthermore, the exceptional and inherited divisors are linearly independent. Their number is $48 + 3 = 51 = h^{1,1}$ so we have found a basis of the $H_{2,2}$ homology or the $H^{1,1}$ cohomology, respectively. Note that we profit a lot from the various permutation symmetries of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. For other orbifolds the construction of these divisors turns out to be more complicated since every twisted sector has to be analyzed separately. In general even the different fixed loci in the same torus behave different, see e.g. [28].

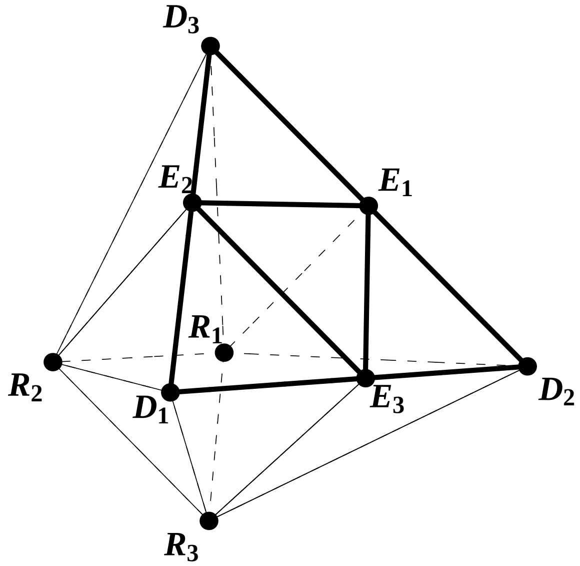


Figure 3.6: The topology of the resolved $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is described by 64 auxiliary polyhedra, one for each combination of fixed point labels α,β,γ, which here are dropped for convenience.

Auxiliary Polyhedra

To be able to compute integrals of forms over various cycles we are interested in the intersection numbers of these 51 divisors. In blow-down we had 64 points $(z_1^\alpha, z_2^\beta, z_3^\gamma)$ in which the fixed lines $F_{1,\beta\gamma}$, $F_{2,\alpha\gamma}$ and $F_{3,\alpha\beta}$ intersect. The neighborhood of these points topologically look like $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, so for each of them we can draw a toric diagram to determine the basic intersection numbers. But there are also the inherited divisors which extent the toric diagrams to 64 *auxiliary polyhedra*, one for each combination of fixed point labels α,β,γ. To display them more symmetrically, we choose other vectors for the ordinary divisors[6] $D_{1,\alpha}$, $D_{2,\beta}$, $D_{3,\gamma}$,

$$u_1 = (2, 0, 0)\,, \qquad u_2 = (0, 2, 0)\,, \qquad u_3 = (0, 0, 2)\,. \tag{3.34}$$

Again the vectors representing the exceptional divisors $E_{1,\beta\gamma}$, $E_{2,\alpha\gamma}$, $E_{3,\alpha\beta}$ are given by (3.21),

$$w_1 = (0, 1, 1)\,, \qquad w_2 = (1, 0, 1)\,, \qquad w_3 = (1, 1, 0)\,. \tag{3.35}$$

[6]This choice violates the requirement that the inhomogeneous coordinates $Z_k = \prod_i z_i^{(u_i)_k}$ are an integer basis of invariant monomials.

The vectors t_i for the inherited divisors R_i, which are the same in all polyhedra, are uniquely chosen such that they reproduce the linear equivalences (3.33),

$$t_1 = (-1, 0, 0)\,, \qquad t_2 = (0, -1, 0)\,, \qquad t_3 = (0, 0, -1)\,. \tag{3.36}$$

Now we draw the auxiliary polyhedra (see fig. 3.6) which have to be triangulated. Its front faces are distorted projected toric diagrams for which we have the four possibilities shown in fig. 3.5. These 64 polyhedra can now be used to determine all intersection numbers between three divisors denoted by S_i with toric vectors s_i in the following way:

- If there is no auxiliary polyhedron in which all three divisors appear their intersection number is zero. This is clear since at least two of these divisors must differ in at least one label α, β or γ which means they live at different fixed loci and hence do not intersect.

- The inherited divisors do not intersect with themselves as can easily be seen when writing them as forms, $R_i \sim \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$.

- If the three divisors are different and do not span a cone in the triangulation then they also do not intersect. More generally one can say that the intersection $S_iS_jS_k$ vanishes if at least one of the curves S_iS_j, S_iS_k or S_jS_k does not exist, i.e. if the corresponding vectors are not connected by a line[7].

- The only non-vanishing intersection numbers of three different divisors are the ones whose vectors span a cone in at least one auxiliary polyhedron. Its value is given by

$$S_iS_jS_k = \left| \frac{\mathscr{N}}{\det(s_is_js_k)} \right| \,, \tag{3.37}$$

  where $\mathscr{N}$ is a normalization constant which can be determined by one known intersection number. For the polyhedra considered here we find $\mathscr{N} = 2$.

- The remaining intersection numbers, i.e. those containing self-intersections, can be computed using the linear equivalence relations (3.33). The most general procedure is to multiply the 12 relations with all $\frac{51\cdot 52}{2}$ products of two divisors and insert all known intersection numbers to obtain 15912 equations for the remaining ones. But many of these equations contain only products of divisors which do not intersect, so we are left with 3672 equations. If we now count equal equations only once we obtain a system of 1172 linear equations with 708 unknowns which can be solved. The fact that there are more equations than unknowns can be seen as a cross-check.

[7]Such a line may not go through another divisor.

triang(α,β,γ)	"symm"	"E_1"	"E_2"	"E_3"
$E_{1,\beta\gamma}E_{2,\alpha\gamma}E_{3,\alpha\beta}$	1	0	0	0
$E_{1,\beta\gamma}E_{2,\alpha\gamma}^2$	-1	-2	0	0
$E_{1,\beta\gamma}E_{3,\alpha\beta}^2$	-1	-2	0	0
$E_{1,\beta\gamma}^2E_{2,\alpha\gamma}$	-1	0	-2	0
$E_{2,\alpha\gamma}E_{3,\alpha\beta}^2$	-1	0	-2	0
$E_{1,\beta\gamma}^2E_{3,\alpha\beta}$	-1	0	0	-2
$E_{2,\alpha\gamma}^2E_{3,\alpha\beta}$	-1	0	0	-2

Table 3.2: Intersection numbers of exceptional divisors for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

As result we present all non vanishing intersection numbers between exceptional and inherited divisors. The ones with ordinary divisors can be obtained using (3.33). First we observe that the intersection numbers containing an inherited divisor are triangulation-independent. This was to be expected since the triangulation only affects the ordinary and exceptional divisors. We find

$$R_1R_2R_3 = 2\,, \tag{3.38}$$

$$R_1E_{1,\beta\gamma}^2 = R_2E_{2,\alpha\gamma}^2 = R_3E_{3,\alpha\beta}^2 = -2\,. \tag{3.39}$$

The intersection numbers containing at least two different exceptional divisors only depend on the triangulation of the auxiliary polyhedron in which they meet. Note that having two different divisors completely fixes the labels α, β, γ and hence there is only one polyhedron in which they meet, see tab. 3.2. Here we exactly recover the results of the non-compact case (see tab. 3.1) when we drop the fixed point labels. Finally there are the triple self-intersection numbers of exceptional divisors. Since one such divisor only fixes two fixed point labels, their value will depend on four auxiliary polyhedra where the remaining fixed point label runs from 1 to 4. When we define

- $N^{\text{symm}}_{\beta\gamma}$:= number of polyhedra (α, β, γ) with symmetric triangulation,
- $N^{E_i}_{\beta\gamma}$:= number of polyhedra (α, β, γ) with "E_i" triangulation,

we find for the triple intersection:

$$E_{1,\beta\gamma}^3 = N^{\text{symm}}_{\beta\gamma} + 2N^{E_2}_{\beta\gamma} + 2N^{E_3}_{\beta\gamma}\,, \tag{3.40a}$$

$$E_{2,\alpha\gamma}^3 = N^{\text{symm}}_{\alpha\gamma} + 2N^{E_1}_{\alpha\gamma} + 2N^{E_3}_{\alpha\gamma}\,, \tag{3.40b}$$

$$E_{3,\alpha\beta}^3 = N^{\text{symm}}_{\alpha\beta} + 2N^{E_1}_{\alpha\beta} + 2N^{E_2}_{\alpha\beta}\,. \tag{3.40c}$$

The interpretation of this is that the intersection numbers are the sums of the local ones from the four polyhedra.

We saw that when we want to resolve the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold completely we have to specify 64 triangulations which determine the intersections of the divisors and hence the topology. For each of them there are four choices so we totally gain $4^{64} \approx 3 \cdot 10^{38}$ possibilities to triangulate the whole resolution. But not all of these possibilities result in inequivalent spaces. By permuting the three T^2's and the four fixed point labels inside each torus we can map one triangulation to another which shows that the described spaces are homeomorphic. Since few of the resolved spaces are symmetric under such permutations, this gives rise to a lower bound on the number of inequivalent CY manifolds which have the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold as singular limes,

$$N_{\text{diffeomorphism classes}} \gtrsim \frac{4^{64}}{3! \cdot 4!^3} \approx 4 \cdot 10^{33} \,. \tag{3.41}$$

Chern class

The total Chern class is of the form

$$c = \prod_S (1+S)^{n_S} \;, \tag{3.42}$$

where the sum runs over all divisors. From the local resolutions we know that the coefficients of the ordinary and exceptional divisors are $n_D = n_E = 1$. To determine n_R we do the following. The total Chern class can be expressed by the curvature 2-form $\mathcal{R}$

$$c = \det\left(1 - \frac{\mathcal{R}}{2\pi \mathrm{i}}\right) \,. \tag{3.43}$$

We know that on the orbifold the curvature vanishes on the bulk and develops a singularity on the fixed lines. In blow-up the fixed lines are replaced by the exceptional divisors and hence the Chern class must be a function only of them and not of the inherited divisors. We decompose the Chern class as $c = C_E C_1 C_2 C_3$ with $C_E = \prod_E (1+E)$ and $C_i = (1+R_i)^{n_R} \prod_\delta (1+D_{i,\delta})$ such that the whole R_i dependence is contained in C_i. Now we replace the D's according to (3.33) and find e.g. for $i=1$

$$\begin{aligned} C_1 &= (1+R_1)^{n_R} \prod_\alpha \left(1 + \frac{1}{2}R_1 - \frac{1}{2}\sum_\beta E_{3,\alpha\beta} - \frac{1}{2}\sum_\gamma E_{2,\alpha\gamma}\right) \\ &= 1 + (n_R+2)R_1 - \frac{1}{2}\sum_{\alpha\beta} E_{3,\alpha\beta} - \frac{1}{2}\sum_{\alpha\gamma} E_{2,\alpha\gamma} \,. \end{aligned} \tag{3.44}$$

Here we have used the fact that the curves R_1^2, $R_1 E_{2,\alpha\gamma}$, $R_1 E_{3,\alpha\beta}$, $E_{2,\alpha\gamma}E_{2,\alpha'\gamma}$, $E_{3,\alpha\beta}E_{3,\alpha'\beta}$ and $E_{2,\alpha\gamma}E_{3,\alpha'\beta}$ do not exist. The requirement of flatness on the bulk in blow-down implies $n_R = -2$. Thus we can write down the total Chern class

$$c = \prod_{i,\delta}(1+D_{i,\delta}) \prod_{i,\delta,\epsilon}(1+E_{i,\delta\epsilon}) \prod_i (1-2R_i) \;. \tag{3.45}$$

We observe:

- The first Chern class vanishes, $c_1 = 0$ due to the linear equivalence relations (3.33). Thus the resolved spaces we create are all CY.

- The integrated third Chern class is always equal to the expected Euler number (2.37) independent of the vast choice of triangulations, $\int c_3 = 96$.

- The Euler number of the divisors can be calculated using the adjunction formula

$$\int_S c_2 = S c_2 = \chi(S) - S^3 \,. \tag{3.46}$$

  For the inherited divisors we find $\chi(R_i) = R_i c_2 = 24 = \chi(\mathrm{K3})$.

- For the exceptional divisors, e.g. $E_{1,\beta\gamma}$ we find

$$\int_{E_{1,\beta\gamma}} c_2 = E_{1,\beta\gamma} c_2 = -4 + 2N_{\beta\gamma}^{\text{symm}} + 4N_{\beta\gamma}^{E_1} \,, \tag{3.47}$$

  so together with (3.40) and (3.46) we obtain

$$\chi\left(E_{1,\beta\gamma}\right) = 4 + N_{\beta\gamma}^{\text{symm}} + 2N_{\beta\gamma}^{E_1} \,, \tag{3.48}$$

  i.e. the topology of the exceptional divisors is triangulation-dependent.

3.4 Kähler Moduli and Triangulation

A CY manifold is by definition a Kähler manifold. A manifold is Kähler if the *Kähler form* $\mathcal{J}$ is closed. On the resolution of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we have found a basis of closed forms, namely the divisors R_i and E_r where r is a multi-index that runs over all exceptional divisor labels. Thus we can expand the Kähler form in these divisors

$$\mathcal{J} = \sum_i a_i R_i - \sum_r b_r E_r \,. \tag{3.49}$$

In the $d = 4$ low energy effective theory the coefficients a_i and b_r are scalar fields which are called Kähler moduli. Note that $\mathbb{Z}_{2,\text{free}}$ will require moduli b_r from the exceptional divisors, that are mapped onto each other, to be equal and will effectively reduce their number from 48 to 24.

The Kähler form can be used to compute the volumes of complex curves C ((1,1)-cycles), divisors S ((2,2)-cycles) and of the whole CY manifold X (a

Cycle C	Vol(C)	triang(α,β,γ)
$E_{2,\alpha\gamma}E_{3,\alpha\beta}$	$-b_{1,\beta\gamma}+b_{2,\alpha\gamma}+b_{3,\alpha\beta}$	“symm”
	0	“E_1”
	$2b_{3,\alpha\beta}$	“E_2”
	$2b_{2,\alpha\gamma}$	“E_3”
$D_{1,\alpha}E_{1,\beta\gamma}$	$b_{1,\beta\gamma}-b_{2,\alpha\gamma}-b_{3,\alpha\beta}$	“E_1”
	0	else
$D_{1,\alpha}E_{2,\alpha\gamma}$	$a_2+\sum\limits_{\beta}\left\{\begin{matrix} b_{2,\alpha\gamma}-b_{1,\beta\gamma} & \text{if triang}(\alpha,\beta,\gamma) = \text{“}E_1\text{”} \\ -b_{3,\alpha\beta} & \text{else}\end{matrix}\right\}$	
$R_1E_{1,\beta\gamma}$	$2b_{1,\beta\gamma}$	
$R_1D_{2,\beta}$	$a_3-\sum\limits_{\gamma}b_{1,\beta\gamma}$	
R_1R_2	$2a_3$	

Table 3.3: Volumes of $(1,1)$-cycles in terms of the Kähler moduli.

$(3,3)$-cycle).

$$\mathrm{Vol}(C) = \int_C \mathcal{J} = \mathcal{J}C\,, \tag{3.50a}$$

$$\mathrm{Vol}(S) = \frac{1}{2!}\int_S \mathcal{J}\wedge\mathcal{J} = \frac{1}{2}\mathcal{J}\mathcal{J}S\,, \tag{3.50b}$$

$$\mathrm{Vol}(X) = \frac{1}{3!}\int_X \mathcal{J}\wedge\mathcal{J}\wedge\mathcal{J} = \frac{1}{6}\mathcal{J}\mathcal{J}\mathcal{J}\,. \tag{3.50c}$$

Using these formulæ and the intersection numbers, we can express the volumes of the cycles on our manifold by the Kähler moduli values. The permutation symmetry will make the following listings complete.

In table 3.3 we list the volume of the $(1,1)$-cycles which we express as products of two divisors. The requirement of the volumes to be positive puts restrictions on the Kähler moduli. We will impose this requirement on the various cycles and discuss its implications.

- We first note that volumes of cycles which are products with inherited divisors are triangulation-independent. Cycles like $E_{2,\alpha\gamma}E_{3,\alpha\beta}$ and $D_{1,\alpha}E_{1,\beta\gamma}$ correspond to interior lines in the toric diagram and hence are compact in the local resolutions. Therefore their volume formula only depends on the particular triangulation. By contrast the cycles of type $D_{1,\alpha}E_{2,\alpha\gamma}$ are only compact in the global description and their volume depends on the triangulations of the 4 auxiliary polyhedra in which they appear.

- From $\text{Vol}(R_i E_{i,\delta\epsilon}) \geq 0$ and $\text{Vol}(R_i R_j) \geq 0$ it follows that the values of all 51 moduli a_i and $b_{i,\delta\epsilon}$ must be positive. Here we also see the reason for the minus sign in front of the $b_{i,\delta\epsilon}$ in the Kähler form (3.49). Further due to $h^{1,1} = 51$ theses cycles are a basis of the homology class $H_{1,1}$. But because of the prefactor 2 in their volumes which does not appear in the other ones they can not be an integral basis.

- The cycle $D_{1,\alpha} E_{1,\beta\gamma}$ exists in the "E_1" triangulation but not in the symmetric one whereas the opposite is true for $E_{2,\alpha\gamma} E_{3,\alpha\beta}$. Positivity of the volumes implies that $b_{1,\beta\gamma} > b_{2,\alpha\gamma} + b_{3,\alpha\beta}$ is equivalent to having the "E_1" triangulation[8]. The point in moduli space where $b_{1,\beta\gamma} = b_{2,\alpha\gamma} + b_{3,\alpha\beta}$ is a transition between these two triangulations. At this point both cycles are shrunk to zero and we have found a smooth transition between these triangulations through a singular point[9].

- At the transition from the symmetric to e.g. the "E_2" triangulation the volume of $E_{2,\alpha\gamma} E_{3,\alpha\beta}$ changes smoothly since at this point $b_{2,\alpha\gamma} = b_{1,\beta\gamma} + b_{3,\alpha\beta}$.

- The volumes of $R_i D_{j,\delta}$ are of the form "$a_i - b_j$" hence the values of the b moduli must be small compared to the a moduli. This fits nicely to them being blow-up modes which locally resolve the singularity but not having big influence on the global geometry.

As a strong result one can say that we have divided the moduli space of our CY manifold into disjoint regions which have the form of cones and which correspond to the different triangulations. The choice of triangulation is no longer arbitrary but follows immediately from the ratios of the vevs of the moduli fields. The boundaries of these regions correspond to singular points in which the topology of the manifold changes. For completeness we also give the volumes of the exceptional and inherited divisors,

$$\text{Vol}(R_1) = 2a_2 a_3 - \sum_{\beta,\gamma} b_{1,\beta\gamma}^2 \,, \tag{3.51}$$

$$\text{Vol}(E_{1,\beta\gamma}) = a_1 b_{1,\beta\gamma} + f_{1,\beta\gamma}^{(2)}(b_r) \,, \tag{3.52}$$

and of the whole manifold,

$$\text{Vol}(X) = 2a_1 a_2 a_3 - \sum_{\beta,\gamma} a_1 b_{1,\beta\gamma}^2 - \sum_{\alpha,\gamma} a_2 b_{2,\alpha\gamma}^2 - \sum_{\alpha,\beta} a_3 b_{3,\alpha\beta}^2 + f^{(3)}(b_r) \,. \tag{3.53}$$

[8] In the same way $b_{2,\alpha\gamma} > b_{1,\beta\gamma} + b_{3,\alpha\beta}$ is equivalent to triang(α,β,γ) = "E_2" and $b_{3,\alpha\beta} > b_{1,\beta\gamma} + b_{2,\alpha\gamma}$ to triang(α,β,γ) = "E_3" .

[9] This singularity is less singular than the orbifold point in which all internal cycles have zero volume.

Here $f_r^{(2)}(b_r)$ and $f^{(3)}(b_r)$ are homogeneous second or third order polynomials in the twisted moduli which are highly triangulation-dependent. From (3.52) we read off that $b_r \to 0$ corresponds to the blow down limes. Increasing b_r slowly, i.e. blowing up, gives volume to the exceptional divisors while it eats away the volume of the inherited divisors and of the whole manifold. Again we see that b_r cannot grow arbitrarily but must stay small compared to a_i.

Chapter 4

Model Building on Resolved Orbifolds

Now that we have a topological description of the resolved $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, we can use it as a compactification space for the heterotic string. Unfortunately, we are not able to give a CFT description of this compactification, i.e. to explicitly construct the strings on the manifold and to quantize them. Therefore we take another approach. First we take the low energy limit of the heterotic string which is a 10 dimensional $N = 1$ SUGRA with an $\mathrm{E}_8 \times \mathrm{E}_8$ SYM theory called heterotic SUGRA. For this theory we know all massless modes (2.24), (2.26) and we can write down an action. Then we compactify on our CY space and explore the four-dimensional massless spectrum.

4.1 Gauge Flux Embedding

The task at this point is to match the orbifold models to models on the resolved space. The input data of an orbifold model are the shift vectors and the Wilson lines. Let us concentrate on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case where there are 2 independent shift vectors V_1 and V_2 and 6 Wilson lines W_i. They correspond to local shifts around each fixed line. If a fixed point is fixed by the space group element $g = (\theta_1^{m_1}\theta_2^{m_2}, n_i e_i)$ then the local shift vector is $V_g = m_1 V_1 + m_2 V_2 + n_i W_i$. This local shift corresponds to an internal gauge field $\mathcal{A}$ that wraps a 1-cycle around the singularity, i.e. a cycle that is closed by g. Such a cycle can be shrunk to zero length by contracting it to the fixed point which means that it is the boundary of a 2-surface that contains the fixed point. By applying Stokes' theorem this means that there is an Abelian 2-form flux $\mathcal{F} = \mathrm{d}\mathcal{A}$ in form of a delta-peak localized in the singularity. Now in blow-up, the singularity is smoothed out and so should be the flux.

A flux on a CY manifold must satisfy the *Hermitian Yang–Mills* (HYM) equations,

$$\mathcal{F}_{ab} = \mathcal{F}_{\bar{a}\bar{b}} = 0\,, \tag{4.1}$$

$$\mathcal{F}_{a\bar{b}}\mathcal{G}^{a\bar{b}} = 0\,, \tag{4.2}$$

in order to preserve $N = 1$ SUSY. $\mathcal{G}$ is the CY metric. To be more concrete, (4.1) is equivalent to F-flatness and (4.2) to D-flatness.

Equation (4.1) further implies that $\mathcal{F}$ is a $(1,1)$-form. Thus, to obtain a stable flux that is able to match the orbifold description we need to expand $\mathcal{F}$ in closed but not exact $(1,1)$-forms that are localized in the fixed lines in blow-down. Such forms are exactly the exceptional divisors E so we write

$$\mathcal{F} = E_r V_r^I H^I\,. \tag{4.3}$$

The H^I are the Cartan generators of $\mathrm{E}_8 \times \mathrm{E}_8$ which ensures the flux to be Abelian. The V_r^I are 48 vectors with 16 components called *bundle vectors*. Matching them with the orbifold can be made with Stokes' theorem by focusing on the local blow-up picture. To get e.g. the shift vector $V_{1,\text{local shift}}$ we integrate over the curve $\mathcal{C} : c(\phi) = (z_1, 0, r\,e^{\mathrm{i}\phi})$. For $r \to \infty$, $\mathcal{C}$ is the boundary of the cycle R_1D_2 and we find,

$$V^I_{1,\text{local shift}} H^I = \int\limits_{\mathcal{C}} \mathcal{A} = \int\limits_{R_1D_2} \mathcal{F}_{\text{local}} = V^I_{1,\text{bundle}} H^I\,, \tag{4.4}$$

which explains the usage of the same symbol for the shift and bundle vector. The other bundle vectors are identified with the local shifts similarly and we find in particular that they all must be quantized just as the shift vectors and Wilson lines. It turns out that demanding strictly the identification (4.4) it is not possible to solve the Bianchi Identities (see sec. 4.1.1). Therefore we require (4.4) to be valid only up to addition of lattice vectors. Otherwise the underlying orbifold model would necessarily have to be a grandchild model. Due to the ambiguity of adding lattice vectors it seems that we have infinitely many possibilities for gauge fluxes for a given set of shift vectors and Wilson lines. But soon we will see that this choice is indeed limited and finite. Note that this choice of bundle vectors is exactly the same for brother and grandchildren sec. 2.3 so it is not yet clear whether we blow up one of them.

4.1.1 Bianchi Identities

In the low energy effective theory we have to deal with various anomalies. Fortunately, in heterotic SUGRA all anomalies can be cancelled by the Green–Schwarz mechanism [11]. This mechanism requires the 2-form field B_2 to be charged under gauge and coordinate transformations. This implies that its 3-form field strength must be modified to be invariant,

$$H_3 = \mathrm{d}B_2 + \omega^{\mathrm{L}}_{3,\mathrm{CS}} - \omega^{\mathrm{YM}}_{3,\mathrm{CS}}\,. \tag{4.5}$$

Here we introduce the *Lorentz Chern–Simons 3-form*

$$\omega^{\mathrm{L}}_{3,\mathrm{CS}} = \mathrm{Tr}\left(\omega \wedge \mathcal{R} - \frac{1}{3}\omega \wedge \omega \wedge \omega\right)\,, \tag{4.6}$$

with ω being the spin connection, and the *Yang–Mills Chern–Simons 3-form*

$$\omega^{\mathrm{YM}}_{3,\mathrm{CS}} = \mathrm{Tr}\left(\mathcal{A} \wedge \mathcal{F} - \frac{1}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)\,. \tag{4.7}$$

In both cases "Tr" denotes an appropriately normalized trace. Since H_3 appears in the action it must be globally defined. The same is true for its exterior derivative

$$\mathrm{d}H_3 = \mathrm{Tr}\,\mathcal{R} \wedge \mathcal{R} - \mathrm{Tr}\,\mathcal{F} \wedge \mathcal{F}\,. \tag{4.8}$$

which is an exact form. Equation (4.8) is called *Bianchi Identity* (BI). One way to solve it, called standard embedding, is to embed the spin connection, which is of SU(3) type, into the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group. But we are interested in Abelian fluxes and hence we make the ansatz (4.3). Due to Stokes' theorem its integral over all closed 4-cycles must vanish,

$$0 = \int_S \mathrm{d}H_3 = \int_S \left(\mathrm{Tr}\,\mathcal{R} \wedge \mathcal{R} - \mathrm{Tr}\,\mathcal{F} \wedge \mathcal{F}\right)\,. \tag{4.9}$$

On the resolved space a basis of 4-cycles is given by the exceptional and inherited divisors $\{E_r, R_i\}$ and therefore we obtain 51 independent equations from (4.9) which we will refer to when speaking about BI's in the following. Inserting $\mathrm{Tr}\,\mathcal{R}^2 = -2c_2(X)$ and (4.3) and using the intersection numbers we find that the BI's are Diophantine[1] equations for the bundle vectors V_r^I.

Bianchi Identities on Inherited Divisors

Let us first have a look at the BI's integrated over the inherited divisors R_i. Since intersection numbers containing R_i are triangulation-independent these BI's are as well. They read

$$\sum_{\beta,\gamma} V^2_{1,\beta\gamma} = 24\,, \tag{4.10a}$$

$$\sum_{\alpha,\gamma} V^2_{2,\alpha\gamma} = 24\,, \tag{4.10b}$$

$$\sum_{\alpha,\beta} V^2_{3,\alpha\beta} = 24\,, \tag{4.10c}$$

[1] A Diophantine equation is a polynomial equation for which one is interested in integer solutions.

on R_1, R_2 and R_3 respectively. They imply that the choice of bundle vectors is finite. In sec. 4.2 we will see that the bundle vectors are the weights of the blow-up modes. Obviously the choice for bundle vectors V_r and shifted weights V_{sh} in each twisted sector are the same. In sec. 2.4 we saw that for massless modes we either have $V_{\text{sh}}^2 = 1/2$ or $V_{\text{sh}}^2 = 3/2$. Hence the only way to solve (4.10) with only massless blow-up modes is to choose

$$V_r^2 = 3/2 \tag{4.11}$$

for all bundle vectors. Otherwise, choosing $V_r^2 = 1/2$ for one bundle vector would imply that there must be some other bundle vector with $V_r^2 > 3/2$ which would correspond to a massive blow-up mode.

Bianchi Identities on Exceptional Divisors

The BI's integrated over the exceptional divisors are, in contrast to (4.10), highly triangulation-dependent. Thus, when trying to solve them, it is necessary to select one out of the approximately $4 \cdot 10^{33}$ inequivalent triangulations. Here we only present the BI's for the two most symmetric cases. The most symmetric one is to choose the symmetric triangulation for all 64 polyhedra. With the results from sec. 3.4 this is the case when the values of all twisted moduli are approximately of the same size. In this triangulation, all allowed intersection numbers between exceptional divisors, i.e. all which can be drawn in one auxiliary polyhedron, do not vanish and hence the BI's are highly coupled. On the divisors $E_{1,\beta\gamma}$, $E_{2,\alpha\gamma}$ and $E_{3,\alpha\beta}$ they read

$$4V_{1,\beta\gamma}^2+\sum_\alpha \left(V_{2,\alpha\gamma}\cdot V_{3,\alpha\beta} - 2V_{1,\beta\gamma}\cdot(V_{2,\alpha\gamma}+V_{3,\alpha\beta}) - V_{2,\alpha\gamma}^2 - V_{3,\alpha\beta}^2\right) = -8\,, \tag{4.12a}$$

$$4V_{2,\alpha\gamma}^2+\sum_\beta \left(V_{1,\beta\gamma}\cdot V_{3,\alpha\beta} - 2V_{2,\alpha\gamma}\cdot(V_{1,\beta\gamma}+V_{3,\alpha\beta}) - V_{1,\beta\gamma}^2 - V_{3,\alpha\beta}^2\right) = -8\,, \tag{4.12b}$$

$$4V_{3,\alpha\beta}^2+\sum_\gamma \left(V_{1,\beta\gamma}\cdot V_{2,\alpha\gamma} - 2V_{3,\alpha\beta}\cdot(V_{1,\beta\gamma}+V_{2,\alpha\gamma}) - V_{1,\beta\gamma}^2 - V_{2,\alpha\gamma}^2\right) = -8\,. \tag{4.12c}$$

The next-highest symmetric possibility would be to have e.g. the "E_1" triangulation in all polyhedra. This corresponds to the case in which the b_1 moduli are significantly larger than the others, i.e. the singularities from the θ_1 sector are "more resolved". Now the BI's look much simpler

$$\sum_\alpha \left(V_{2,\alpha\gamma}^2 + V_{3,\alpha\beta}^2\right) = 12\,, \tag{4.13a}$$

$$2V_{2,\alpha\gamma}^2 - \sum_\beta V_{2,\alpha\gamma}\cdot V_{1,\beta\gamma} = 2\,, \tag{4.13b}$$

$$2V_{3,\alpha\beta}^2 - \sum_\gamma V_{3,\alpha\beta}\cdot V_{1,\beta\gamma} = 2\,. \tag{4.13c}$$

First one finds that the assumption (4.11) already solves (4.13a). But also the others are less coupled, such that this triangulation seems more promising for finding solutions. The methods and results of searching for solutions will be presented in sec. 4.3.

4.2 Low Energy Effective Action

The starting point for describing a physical theory is to write down an action. From it all properties of the theory can be deduced. In the process of CY compactification, which we follow here, we start with $d = 10$, $N = 1$ heterotic SUGRA containing an $\mathrm{E}_8 \times \mathrm{E}_8$ SYM theory. Its massless field content is shown in (2.24) and (2.26). For us, the bosonic part of the action is of particular interest. It reads

$$\begin{aligned} S_{\mathrm{b}} =& \frac{1}{2\kappa^2} \int e^{-2\phi} \left(R \wedge *1 + 4\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{2} H_3 \wedge *H_3 \right) \\ & - \frac{1}{2g^2} \int \mathrm{Tr}\, F \wedge *F + S_{\mathrm{GS}}\,, \end{aligned} \tag{4.14}$$

with S_{GS} being a topological term postulated by the Green–Schwarz mechanism and $*1 = \mathrm{d}^{10}x\sqrt{-g}$ being the volume form. For the compactification we factorize the fields in a four dimensional spacetime part and a six dimensional CY part which is an eigenfunction of the derivative operator appearing in the respective e.o.m. The zero modes, i.e. those with zero eigenvalue will be the massless modes in the $d = 4$ theory while the others build a generalized Kaluza–Klein tower. For bosonic fields this operator is the Laplace operator and hence the zero modes are the harmonic functions. Since in each cohomology class there is exactly one harmonic form the number of zero modes is given by the Hodge numbers.

Identification of Blow-up Modes

Let us first expand the two-form B_2. Since a CY space has $h^{1,0} = 0$, its indices must be either both spacetime indices or both internal indices. For the first case there is only one zero mode which is the constant function. The corresponding $d = 4$ field is called *model-independent axion* b_2 since it is dual to an pseudoscalar. For the second case there are 51 independent harmonic 2-forms on the resolved orbifold which are the divisors. The fields wrapping them are the *model-dependent axions* α_i and β_r. We find

$$B_2 = b_2 + \alpha_i R_i + \beta_r E_r\,. \tag{4.15}$$

A similar expansion has been done for the Kähler form in (3.49). On the CY the Kähler form and the 2-form can be joined to the complexified Kähler form whose expansion coefficients are complex scalars, i.e. the bosonic components of the moduli multiplets

$$\mathcal{J} - \mathrm{i}B_2 = (a_i - \mathrm{i}\alpha_i)\, R_i - (b_r + \mathrm{i}\beta_r)\, E_r\,. \tag{4.16}$$

From the neutralness of H_3 we can read off the behavior of the model-dependent axions under Abelian gauge transformations Λ^I,

$$\beta_r \longrightarrow \beta_r - V_r^I \Lambda^I\,. \tag{4.17}$$

To obtain a linearly transforming field we make a field redefinition by exponentiating

$$\Psi_r = e^{2\pi(b_r - \mathrm{i}\beta_r)}\,, \qquad \Psi_r \longrightarrow e^{2\pi\mathrm{i}V_r^I\Lambda^I}\,\Psi_r\,. \tag{4.18}$$

In blow-down these fields are localized at the fixed lines F_r and hence are twisted orbifold states with charges V_r^I. Since we only have fixed lines, these states are six-dimensional states which may be projected out when going to four dimensions. These states contain further the Kähler moduli b_r which are responsible for the blow-up. Hence we have identified the blow-up modes on the orbifold. But there is still one inconsistency in this picture. In the geometrical picture in sec. 3.4 we saw that $b_r \rightarrow 0$ corresponds to the blow-down limes. In this limes the blow-up modes Ψ_r should have a vanishing vev which is equivalent to $b_r \rightarrow -\infty$. The solution is that if the length and curvature scales of the CY approach the string scale, the string corrections become relevant and modify the classical geometrical picture which we used before. For a detailed discussion see [14].

Anomalous U(1)'s

The four-dimensional gauge group of heterotic orbifold models with Abelian gauge embedding may contain non-Abelian factors, together with U(1) factors to keep the rank. In turns out that the U(1)'s can be rotated such that only one of them is anomalous and hence is called $\mathrm{U}(1)_{\mathrm{anom}}$. Here the model-independent axion can be used to cancel this anomaly. In blow-up there are many anomalous U(1) factors [25] which will be cancelled by the model-dependent axions. To see this we look closer at the 3-form flux (4.5) term in the bosonic action (4.14). Inserting the expansions of the 2-form (4.15), the gauge flux (4.3) and the Abelian part of the 1-form potential, $\mathcal{A} = A^I H^I$, we find among others the term

$$H_3 = E_r(\mathrm{d}\beta_r + V_r^I A^I) + \dots\,, \tag{4.19}$$

which shows that (4.17) is the right transformation behavior to make H_3 gauge invariant. We further see that certain linear combinations of axions are eaten by the gauge bosons and act now as their longitudinal dof while the linear combinations orthogonal to them remain untouched. We can perform a gauge transformation to gauge the eaten axions away and obtain in this way a mass term for the four dimensional gauge bosons by inserting (4.19) into (4.14)

$$\int_X H_3 \wedge *H_3 = A^I_\mu A^{J,\mu} M^{2\,IJ} + \dots\,, \tag{4.20}$$

with the mass matrix

$$M^{2\,IJ} = V^I_r V^J_s \cdot \int_X E_r \wedge *_6 E_s\,. \tag{4.21}$$

Here $*_6$ denotes the Hodge star on the internal space X. The scalar product on divisor space, given by

$$\langle S_1, S_2 \rangle = \int_X S_1 \wedge *_6 S_2\,, \tag{4.22}$$

is positive definite in complete blow-up which implies that the rank of the mass matrix is equal to the rank of V^I_r as a matrix. From this it follows that the U(1)'s which are orthogonal to all bundle vectors remain massless while the others get a mass in the process of anomaly cancellation. This mass matrix depends on the moduli since the metric appears in the Hodge star and we strongly assume this dependence to be such that the mass matrix vanishes in an appropriate blow-down limit[2] since at the orbifold point there is at best one anomalous U(1).

Chiral Spectrum

Now we know that the Abelian part of the $d = 4$ gauge group G_4 is given by the U(1)'s which are orthogonal to the bundle vectors. But in $d = 10$ we started with a $E_8 \times E_8$ SYM theory, i.e. we had 480 gauge multiplets with non-trivial root vectors, see (A.1). It turns out that roots which are orthogonal to the bundle vectors remain massless after the compactification and can enhance the gauge group to a non-Abelian one. The other roots do not survive the compactification as gauge bosons but they can still appear in the massless spectrum in form of chiral multiplets. Group-theoretically this corresponds to a decomposition of the adjoint of $E_8 \times E_8$ into the adjoint of G_4 plus a couple of other representations.

[2] Using the formula $*S = 3/4\mathcal{J}^2 \cdot \int_X \mathcal{J}^2 S / \int_X \mathcal{J}^3 - S\mathcal{J}/2$ from [12] which is valid for harmonic $(1,1)$-forms S in classical geometry one can show that in the limes $b_r \to 0$ the mass matrix does not vanish and even keeps its rank. For a proper blow-down limit we again have to enter a "stringy" geometry regime.

The multiplicity in which these chiral multiplets appear can be seen by integrating the gaugino anomaly polynomial in $d = 10$ over the manifold which results in the $d = 4$ anomaly polynomial in which the *multiplicity operator*

$$\mathcal{N} = \int_X \left(\frac{1}{6}\mathcal{F}^3 - \frac{1}{24} \operatorname{tr} \mathcal{R}^2 \cdot \mathcal{F} \right) \tag{4.23}$$

appears. The powers are understood in terms of wedge products. For details see e.g. [23]. The Cartan generators H^I which appear in $\mathcal{N}$ act on the $\mathrm{E}_8 \times \mathrm{E}_8$ roots $|P\rangle$ as

$$H^I |P\rangle = P^I |P\rangle \,, \tag{4.24}$$

such that $|P\rangle$ are all eigenvectors of $\mathcal{N}$ with the eigenvalue being the multiplicity. The multiplicity operator is an odd polynomial in the 2-form flux which implies that the sign of the multiplicity flips when the weight vector does. We see that the chiral dofs appear in CPT-conjugate pairs and one only has to count the positive multiplicities. Note that computing the integral in (4.23) is performed with the help of intersection numbers and is, in particular, triangulation-dependent. The multiplicity operator can also be obtained by the *Dirac index theorem* which in this context counts the number of spinorial zero modes. One important result of the spectra obtained from $\mathcal{N}$ is that they are free of pure non-Abelian anomalies.

D-Flatness

We have not yet discussed the second HYM equation (4.2). It can be multiplied with the six dimensional volume element and integrated over the whole manifold,

$$\int_X \mathcal{F}_{a\bar{b}} \mathcal{G}^{a\bar{b}} * 1 \sim \int_X \mathcal{J}^2 \mathcal{F} = 0 \,. \tag{4.25}$$

Inserting (4.3) and using (3.50b) shows that this is a set of equations for the exceptional divisor volumes,

$$\int_X \mathcal{J}^2 \mathcal{F} = \sum_r \mathrm{Vol}(E_r) V_r^I = 0 \,, \qquad I = 1 \ldots 16 \,. \tag{4.26}$$

The number of independent equations is the rank of V_r^I. We find that in most cases this equation would force many or even all exceptional divisor volumes to be zero which would mean we go to (complete) blow-down. But this statement gets modified by two corrections. First, when the divisor volumes approach zero the classical geometrical description is replaced by a "stringy" geometrical one with a different measure, i.e. the volume formula (3.50b) is no longer valid.

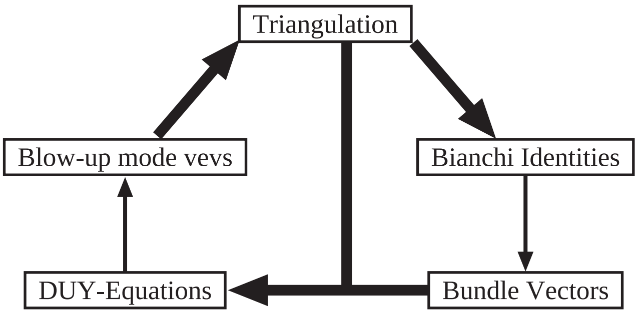


Figure 4.1: Ring of strict implications (bold arrows) and restrictions (thin arrows).

Second, equation (4.25) is valid only at tree level. Taking into account one-loop corrections changes it to the *Donaldson–Uhlenbeck–Yau* equation (DUY),

$$\int\limits_X \mathcal{J}^2 \mathcal{F} = \frac{e^{2\phi}}{8\pi} \int\limits_X \left[\left(\operatorname{Tr} \mathcal{F}'^2 - \frac{1}{2} \operatorname{Tr} \mathcal{R}^2 \right) \mathcal{F}' + (\mathcal{F}' \to \mathcal{F}'') \right] . \tag{4.27}$$

The primed and double-primed field strengths come from the first and second E_8 factor, respectively. Once we fix the triangulation and the gauge flux, these are inhomogeneous equations for the ratios of the moduli to the string coupling $e^{2\phi}$.

Furthermore, in sec. 3.4 we have seen that the ratios of the moduli determine the triangulation uniquely. This closes a ring of equations and properties which affect one another circularly, see fig. 4.1.

In a $d = 4$ SUSY theory there are two requirements for unbroken $N = 1$ SUSY which are F-flatness and D-flatness. The D-term of a SYM theory with further charged states is

$$D^a = -g \sum_i \phi_i^* T^a \phi_i \,, \tag{4.28}$$

where the sum runs over all chiral multiplets with ϕ_i being their scalar component and T^a are the generators. For an Abelian part of the gauge group we can replace the generators by the charges Q_i^a of the multiplet. If such an Abelian factor is anomalous, the GS mechanism induces a FI D-term ξ^a and the D flatness condition reads

$$g \sum_i Q_i^a \, |\phi_i|^2 = \xi^a \,. \tag{4.29}$$

This equation is very similar to the DUY equations (4.27) when writing it in terms of the divisor volumes, cf. (4.26), since the bundle vectors are the weight vectors of the blow-up modes (4.18) attaining a vev and their U(1) charges are just linear combinations of them. Further the DUY one-loop correction corresponds to the FI term and affects only the anomalous U(1)'s since the space spanned by the bundle vectors is the space of anomalous U(1)'s. One only has to replace the blow-up mode vev squares by the divisor volumes which are obviously not equal. But this discrepancy is not yet completely understood.

4.3 Blow-Up of the 6-Generation $\mathbb{Z}_2 \times \mathbb{Z}_2$ Model

Finally, we present a model on the resolution of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold which could result from blowing up the singularities in the orbifold model. The blow-up modes attain vevs which induce mass terms for some of the chiral multiplets and project them out of the low energy spectrum. Yet it is not clear whether we really blow up this model or one of its brothers or even grandchildren.

The challenge is to find a solution to the 51 integrated BI's (4.9) for which one first has to choose a triangulation. Such a solution in form of 48 bundle vectors must be of the form,

$$V_{1,\beta\gamma} \equiv V_1 + \sum_{i=3,4,5,6} n_i W_i \,, \tag{4.30a}$$

$$V_{2,\alpha\gamma} \equiv V_2 + \sum_{i=1,2,5,6} n_i W_i \,, \tag{4.30b}$$

$$V_{3,\alpha\beta} \equiv V_1 + V_2 + \sum_{i=1,2,3,4} n_i W_i \,. \tag{4.30c}$$

where the matching of indices α, β, γ with the n_i can be read off from the following tables,

α	n_1	n_2
1	0	0
2	0	1
3	1	0
4	1	1

,

β	n_3	n_4
1	0	0
2	0	1
3	1	0
4	1	1

,

γ	n_5	n_6
1	0	0
2	0	1
3	1	0
4	1	1

.

In sec. 4.1.1 it was shown that the set of potential solutions is finite but from first principles it is not clear how many solutions exist.

$\mathbb{Z}_{2,\text{free}}$ Symmetric Model Building

We want to blow up a model with six generations of SU(5) GUT and then mod out the $\mathbb{Z}_{2,\text{free}}$ to break it down to a SM with three generations. Alternatively we could first mod out $\mathbb{Z}_{2,\text{free}}$ to obtain a three-generation MSSM on a

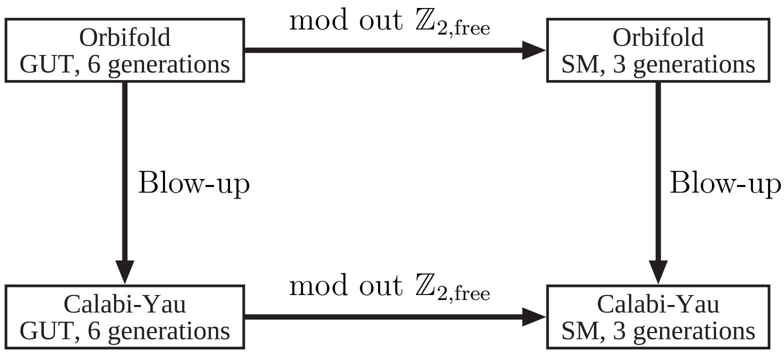


Figure 4.2: Two ways to obtain the MSSM on a CY from a 6 generation GUT on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

non-factorizable orbifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2,\text{free}})$, before blowing it up, see fig. 4.2. We assume these two procedures to commute. The blow-up of the GUT model must satisfy some requirements in order to preserve the $\mathbb{Z}_{2,\text{free}}$ symmetry. Since it maps pairs of fixed points onto each other, their blow-ups must be done in a symmetric way. Thus we must choose the same bundle vectors at these resolved points,

$$V_{1,\beta\gamma} = V_{1,\beta'\gamma'}\,, \qquad V_{2,\alpha\gamma} = V_{2,\alpha'\gamma'}\,, \qquad V_{2,\alpha\beta} = V_{2,\alpha'\beta'}\,. \tag{4.31}$$

For the index matching, see (2.62). In the same way the blow-up modes at the divisors which replace the fixed points, must be equal to obtain a symmetric geometry. This reduces the number of Kähler moduli from 51 to 27 as required by the Hodge numbers (2.63). On the orbifold, $\mathbb{Z}_{2,\text{free}}$ maps the points in which fixed tori meet onto each other,

$$\left(z_1^\alpha, z_2^\beta, z_3^\gamma\right) \leftrightarrow \left(z_1^{\alpha'}, z_2^{\beta'}, z_3^{\gamma'}\right). \tag{4.32}$$

On the resolution this corresponds to a mapping of the loci in which exceptional divisors intersect, i.e. which are described by one of the 64 auxiliary polyhedra. Here $\mathbb{Z}_{2,\text{free}}$ dictates that the triangulations of the polyhedra must be equal, but this is already implied by the equality of the blow-up modes. The number of inequivalent triangulations fulfilling this can be estimated in the following way: We must choose 32 independent triangulation so we start with 4^{32} possibilities. The permutations which we divide out in (3.41) are slightly modified since all triangulations are symmetric unter the simultaneous fixed permutation of the fixed loci which is given by $\mathbb{Z}_{2,\text{free}}$. Thus we obtain,

$$N^{\mathbb{Z}_{2,\text{free}}\text{ symmetric}}_{\text{diffeomorphism classes}} \gtrsim \frac{4^{32}}{3!\cdot 4!^3/2} \approx 4\cdot 10^{14}\,. \tag{4.33}$$

A result of sec. 3.3 was that intersection numbers of only exceptional divisors are either determined by just one triangulation or can be split into contributions from four polyhedra in the case of triple self-intersections. This implies that integrals of products of exceptional divisors only can be rewritten into sums over the 64 polyhedra and we see that modding out $\mathbb{Z}_{2,\text{free}}$ divides such integrals by two if they are $\mathbb{Z}_{2,\text{free}}$-symmetric. One such integral is the third Chern class integrated over the whole manifold, which implies that the Euler number is halved just as expected by (2.37) and (2.64). Another one is the multiplicity operator (4.23), so the amount of chiral matter will be halved apart from effects of the new Wilson line. As one consequence, we expect multiplicities of the resolved GUT model to be even.

Search Procedure

The explicit model search and the computation of the properties of the models are done by extensive use of computers. To this end a `c++` code has been developed. It first creates a choice of all bundle vector candidates of the form (4.30) with length postulated by (4.11). This way the BI's on the inherited divisors are fulfilled. This further corresponds to the blow-up mode being a massless twisted orbifold state, at least in $d = 6$. Since we do not want to break the GUT SU(5) we remove all vectors from this choice which are not orthogonal to the SU(5) simple roots.

To keep the $\mathbb{Z}_{2,\text{free}}$, we restrict the search to bundle vectors satisfying (4.31). Another simplification can be done due to the vanishing of the first Wilson line (2.65c). It motivates the restriction to choose the same bundle vector at exceptional divisors which differ by W_1, i.e. whose associated fixed points differ by $e_1/2$ in blow-down. The number of independent bundle vectors and of remaining BI's is reduced to 16.

The next step is to fix the triangulation. A search over all triangulations on an ordinary computer would probably last longer that a human lifetime and is, in particular, not possible within this thesis.

We performed the search mainly for the overall symmetric and the overall E_1" triangulation. In the symmetric case, 16 independent and highly coupled equations (4.12) must be solved and we find that there exist no solutions. In the E_1" triangulation the BI's on the $E_{1,\beta\gamma}$ (4.13a) are already fulfilled, thus we only have to solve 8 independent and much simpler equations, (4.13b) and (4.13c). By a systematic search we can find roughly one hundred solutions per second. These models differ by the $d = 4$ hidden sector gauge group and by the multiplicities of the chiral matter which could be explained by brother or grandchildren models.

During the search among other 6 generation GUT models we haven't found any model whose bundle vectors all correspond to $d = 4$ massless states. One always has to include the weights of states which are projected out by the second $\mathbb{Z}_2$. In this model it is even necessary from the beginning since in some fixed sectors all chiral states are projected out on the orbifold.

$V_{1,11}, V_{1,22}$	$\left(0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$	✓
$V_{1,12}, V_{1,21}$	$\left(0,0,\frac{1}{2},-\frac{1}{2},0,0,0,0,\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{3}{4},-\frac{1}{4},-\frac{1}{4}\right)$	e
$V_{1,13}, V_{1,24}$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\right)$	x
$V_{1,14}, V_{1,23}$	$\left(0,0,-\frac{1}{2},\frac{1}{2},0,0,0,0,-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$	✓
$V_{1,31}, V_{1,42}$	$\left(0,0,-\frac{1}{2},0,0,0,0,\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4}\right)$	✓
$V_{1,32}, V_{1,41}$	$\left(-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2},0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0\right)$	x
$V_{1,33}, V_{1,44}$	$\left(0,0,-\frac{1}{2},0,0,0,0,\frac{1}{2},-\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$	x
$V_{1,34}, V_{1,43}$	$\left(-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2},0,0,0,-\frac{1}{2},0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2}\right)$	✓
$V_{2,11}, V_{2,22}, V_{2,31}, V_{2,42}$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0\right)$	x
$V_{2,12}, V_{2,21}, V_{2,32}, V_{2,41}$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	✓
$V_{2,13}, V_{2,24}, V_{2,33}, V_{2,44}$	$\left(-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{3}{4},0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2}\right)$	✓
$V_{2,14}, V_{2,23}, V_{2,34}, V_{2,43}$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$	x
$V_{3,11}, V_{3,22}, V_{3,31}, V_{3,42}$	$\left(\frac{1}{4},\frac{1}{4},-\frac{1}{4},\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\right)$	✓
$V_{3,12}, V_{3,21}, V_{3,32}, V_{3,41}$	$\left(-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	x
$V_{3,13}, V_{3,24}, V_{3,33}, V_{3,44}$	$\left(\frac{1}{4},\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	e
$V_{3,14}, V_{3,23}, V_{3,34}, V_{3,43}$	$\left(-\frac{1}{4},-\frac{1}{4},-\frac{3}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},0,0,0,0,0,0,0,0\right)$	✓

Table 4.1: Bundle Vectors of one particular resolution model with six GUT generations. The last column denotes if the blow-up mode is a $d = 4$ state (✓), a projected out state (x) or if the twisted sector is empty in $d = 4$ (e).

One Particular Solution

As a final result we present one particular solution which comes very close to the MSSM after modding out $\mathbb{Z}_{2,\text{free}}$. The corresponding bundle vectors are listed in tab. 4.1. By construction they fulfill the assumptions we made to simplify the search. The non-Abelian gauge group in $d = 4$ can be obtained by looking at the unbroken simple roots. One possibility to choose them are the vectors,

$$\alpha_1 = \left(0,0,0,0,1,-1,0,0, \quad 0^8\right) , \tag{4.34a}$$

$$\alpha_2 = \left(0,0,0,0,0,1,-1,0, \quad 0^8\right) , \tag{4.34b}$$

$$\alpha_3 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}, \quad 0^8\right) , \tag{4.34c}$$

$$\alpha_4 = \left(1,-1,0,0,0,0,0,0, \quad 0^8\right) , \tag{4.34d}$$

$$\alpha_5 = \left(0^8 \quad -\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) . \tag{4.34e}$$

A look at the Cartan matrix reveals that the α_i, $i = 1 \ldots 4$ span an SU(5) which is by construction the SU(5) from the orbifold model. In the hidden sector an SU(2) spanned by α_5 remains. The summary of the massless chiral spectrum is given by

#	irrep
202	$(\mathbf{1},\mathbf{1})$
8	$(\bar{\mathbf{5}},\mathbf{1})$
2	$(\mathbf{5},\mathbf{1})$
6	$(\mathbf{10},\mathbf{1})$
20	$(\mathbf{1},\mathbf{2})$

.

For more details see appendix C. Now we mod out $\mathbb{Z}_{2,\text{free}}$ and switch on the Wilson line W. Even though on a CY the first Betti number vanishes, we have created a non-contractible cycle, i.e. the one which is closed by $\mathbb{Z}_{2,\text{free}}$, around which the Wilson line can wrap. Since this resolution model is related to the orbifold model, we still require W to be of the form (2.58), i.e. it is half the second Wilson line plus an arbitrary half lattice vector. Let us first try $W = W_2/2$. This results in a breaking of the roots α_3 and α_5. In the visible sector we obtain the desired GUT $\rightarrow$ SM breaking where α_1 and α_2 span $\text{SU}(3)_C$, α_4 spans $\text{SU}(2)_L$ and the standardly embedded hypercharge generator $t_Y = 2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4$ is the same as in (2.68). Unfortunately, W also breaks the remaining non-Abelian part of the hidden sector, but this can be avoided by adding a half lattice vector, e.g.,

$$\begin{aligned} W' &= \frac{W_2}{2} + \left(0^8, 0, 0, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0\right) \\ &= \left(-\frac{1}{2}, -\frac{1}{2}, 0 - \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \quad -\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right) . \end{aligned} \tag{4.35}$$

W' does the same as W in the visible sector but also leaves the hidden SU(2) alive.

Finally we obtain three families of $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$ from the $\mathbf{10}$'s and six of the $\bar{\mathbf{5}}$'s. The $\mathbf{5}$'s and the remaining two $\bar{\mathbf{5}}$'s will become a pair of up- and down-Higgses plus the triplets which yet have to be decoupled. So up to the doublet-triplet splitting we found the exact MSSM spectrum. We further have ten hidden SU(2) doublets and one hundred and one non-Abelian singlets which can act e.g. as right-handed neutrinos.

Chapter 5

Conclusions

In this thesis we have studied heterotic model building on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds and on its resolved CY manifold. The $\mathbb{Z}_{2,\text{free}}$ symmetry allows us to build new kinds of models with methods of gauge symmetry breaking that have not yet been investigated in orbifold constructions. The probably most important result is that we have obtained a way of breaking an SU(5) GUT to the SM which does not lead to an anomalous hypercharge in blow-up. We first found that on the orbifold there exist models which contain the MSSM in its massless spectrum. Motivated by this we could construct models on the resolution which come even closer to the MSSM. The number of these resolution models is enormous and they differ by their amount of chiral matter (i.e. SM families) and we need brother and perhaps even grandchildren models to explain this. Nevertheless the model search was not complete due to our limited technical possibilities. Perhaps one day a quantum computer can give us the possibility to perform a complete search and to figure out an MSSM model with best properties.

In the identification of the blow-up modes we saw that for a complete blow-up it is necessary to allow six-dimensional fields, which have no massless four-dimensional mode, to act as blow-up mode and obtain a vev. However, it is not completely clear if this is allowed and how such a vev should look like.

Another result of this thesis is that the Kähler moduli dictate the topology of the resolved manifold. For a realistic model it should be possible to stabilize the moduli such that the Bianchi identities of the corresponding topology are solved by the gauge flux. But this issue is yet far from being realized. Probably to this aim we need “stringy geometry” which can describe the regime between the stringy construction at the orbifold point and the SUGRA approach for small curvatures.

Appendix A

Group Theory

We want to present some important properties of the Lie algebras which appear in the thesis. A complete analysis together with a huge collection of group tables can be found in [30].

$\mathbf{E_8}$

The E_8 root lattice is the only eight-dimensional even and self dual lattice. It is spanned by the roots

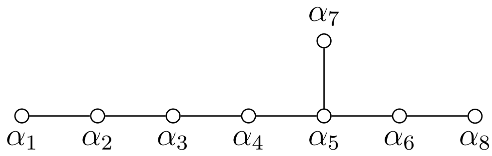
$$\Big(\underline{\pm 1, \pm 1, 0, 0, 0, 0, 0, 0}\Big)\,, \tag{A.1a}$$

$$\Big(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]\Big)\,. \tag{A.1b}$$

We again use the notations that the underline denotes permutations and the rectangular brackets denote even number of sign flips. Note that the roots all have length square equal to 2. A set of simple roots is given by,

$$\alpha_1 = \big(0, 1, -1, 0, 0, 0, 0, 0\big)\,, \tag{A.2a}$$
$$\alpha_2 = \big(0, 0, 1, -1, 0, 0, 0, 0\big)\,, \tag{A.2b}$$
$$\alpha_3 = \big(0, 0, 0, 1, -1, 0, 0, 0\big)\,, \tag{A.2c}$$
$$\alpha_4 = \big(0, 0, 0, 0, 1, -1, 0, 0\big)\,, \tag{A.2d}$$
$$\alpha_5 = \big(0, 0, 0, 0, 0, 1, -1, 0\big)\,, \tag{A.2e}$$
$$\alpha_6 = \big(0, 0, 0, 0, 0, 0, 1, -1\big)\,, \tag{A.2f}$$
$$\alpha_7 = \big(0, 0, 0, 0, 0, 0, 1, 1\big)\,, \tag{A.2g}$$
$$\alpha_8 = \big(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\big)\,. \tag{A.2h}$$

They lead to the Dynkin diagram,

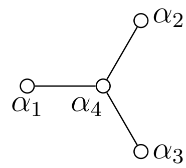


.

The root vectors are the weight vectors of the **248** which is at the same time the fundamental and the adjoint representation. It is convenient to characterize an irreducible representation (irrep) by the Dynkin label of its highest weight (DLHW). The DLHW of the **248** of E_8 is $[1, 0, 0, 0, 0, 0, 0, 0]$.

SO(8)

SO(8) is of particular interest since it the little group of ten-dimensional massless particles. The Dynkin diagram is



.

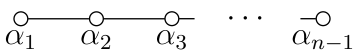
Its symmetry implies that the dimensions of many irreps are equal so in order to distinguish them we label them with indices, **V** for vectorial, **S** for spinorial and **C** for conjugate spinorial. The most important non-trivial irreps are listed below.

irrep	weights	DLHW
$\mathbf{8_V}$	$\underline{(\pm 1, 0, 0, 0)}$	$[1, 0, 0, 0]$
$\mathbf{8_S}$	$\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]\right)$	$[0, 1, 0, 0]$
$\mathbf{8_C}$	$\left(\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]\right)$	$[0, 0, 1, 0]$
28	$\underline{(\pm 1 \pm 1, 0, 0)}$	$[0, 0, 0, 1]$

Note that the **28** is the adjoint and hence its weights are the roots.

SU(N)

In the models considered in this thesis the non-Abelian part of the $d = 4$ gauge group G_4 was made out of SU(N) factors. Thus we also discuss them and their irreps which can appear here. Decomposing the **248** of E_8 results in the adjoint of G_4 and some irreps. It turns out that the DLHW of these irreps contain only zeros and one "1". The Dynkin diagram of SU(N) is



.

The relevant irreps are

irrep	DLHW
$\mathbf{N}$	$[1, 0, \ldots, 0]$
$\bar{\mathbf{N}}$	$[0, \ldots, 0, 1]$
$\mathbf{N^2 - 1}$	$[1, 0, \ldots, 0, 1]$
$\binom{\mathbf{N}}{\mathbf{i}}$	$[0, \ldots, 0, \underset{\substack{\uparrow \\ \text{i-th}}}{1}, 0, \ldots, 0]$

.

Note that the $\binom{\mathbf{N}}{\mathbf{i}}$ is the antisymmetrized i-fold product of the fundamental $\mathbf{N}$.

Appendix B

Details of the Orbifold Model

We give an overview of the complete massless chiral spectrum of the orbifold model presented in sec. 2.5. The first column shows the irrep under the non-Abelian part of the $d = 4$ gauge group, SU(5) × SU(4) × SU(4). The second column shows the U(1) charges of the states using the following basis of U(1) directions:

$$\mathrm{U}(1)_0 = (-2,-2,-4,-2,2,2,2,4,-1,-1,0,0,0,0,-1,-1)\,, \tag{B.1a}$$
$$\mathrm{U}(1)_1 = (-4,-4,-26,38,-2,-2,-2,-10,-2,-2,0,0,0,0,-2,-2)\,, \tag{B.1b}$$
$$\mathrm{U}(1)_2 = (-28,-28,-68,0,24,24,24,44,366,366,0,0,0,0,-14,-14)\,, \tag{B.1c}$$
$$\mathrm{U}(1)_3 = (-14,-14,-34,0,12,12,12,22,0,0,0,0,0,0,176,176)\,, \tag{B.1d}$$
$$\mathrm{U}(1)_4 = (74,74,-34,0,12,12,12,22,0,0,0,0,0,0,0,0)\,, \tag{B.1e}$$
$$\mathrm{U}(1)_5 = (0,0,-2,0,-8,-8,-8,10,0,0,0,0,0,0,0,0)\,. \tag{B.1f}$$

$\mathrm{U}(1)_0$ is the anomalous one.

Untwisted Sector	
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(4,8,56,28,-148,0)$
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$(4,-4,48,24,24,-16)$
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(-4,4,-48,-24,-24,16)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,-8,-56,-28,148,0)$
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$(6,30,96,48,-40,2)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(2,-26,8,4,-84,-18)$
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(-6,-30,-96,-48,40,-2)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-2,26,-8,-4,84,18)$
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(-2,-22,-40,-20,-108,-2)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(2,34,48,24,-64,18)$
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$(2,22,40,20,108,2)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-2,-34,-48,-24,64,-18)$

$F_{1,11}$,$F_{1,22}$ Sector	
$(\mathbf{10},\mathbf{1},\mathbf{1})$	$(-2,2,-24,-12,-12,8)$
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$(-4,10,-44,-22,-22,-10)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(8,-14,92,46,46,-6)$

$F_{1,13}$,$F_{1,24}$ Sector	
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(4,8,-324,-162,-74,0)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,380,190,-74,0)$
$(\mathbf{1},\mathbf{6},\mathbf{1})$	$(-2,-4,-28,-14,74,0)$

$F_{1,14}$,$F_{1,23}$ Sector	
$(\mathbf{1},\mathbf{4},\mathbf{1})$	$(0,30,210,105,17,1)$
$(\mathbf{1},\mathbf{4},\mathbf{1})$	$(-2,-34,142,71,-17,-1)$

$F_{1,31}$,$F_{1,42}$ Sector	
$(\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	$(-4,-8,-246,60,-28,-6)$
$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}})$	$(-3,-6,-232,-116,-28,-6)$

$F_{1,32}$,$F_{1,41}$ Sector	
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(0,12,388,11,11,5)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,-8,-436,-35,-35,11)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-30,360,-3,85,5)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,22,-416,-25,63,-5)$

$F_{1,33}$,$F_{1,44}$ Sector	
$(\mathbf{1},\mathbf{1},\mathbf{4})$	$(-3,-6,-232,-116,-28,-6)$
$(\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	$(-4,-8,134,-116,-28,-6)$

$F_{1,34}$,$F_{1,43}$ Sector	
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(0,12,8,187,11,5)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,-8,-56,-211,-35,11)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-30,-20,173,85,5)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,22,-36,-201,63,-5)$

$F_{2,11}$,$F_{2,22}$,$F_{2,31}$,$F_{2,42}$ Sector	
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(3,3,-340,13,-31,-4)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-5,-25,300,-33,-77,2)$

$F_{2,12}$,$F_{2,21}$,$F_{2,32}$,$F_{2,41}$ Sector	
$(\mathbf{1},\mathbf{4},\mathbf{1})$	$(-5,5,130,-118,14,3)$
$(\mathbf{1},\mathbf{1},\mathbf{4})$	$(-6,3,116,58,14,3)$

$F_{2,13}$,$F_{2,24}$,$F_{2,33}$,$F_{2,44}$ Sector	
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(3,-39,12,-177,43,-4)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-5,5,-60,153,21,-14)$

$F_{2,14}$,$F_{2,23}$,$F_{2,34}$,$F_{2,43}$ Sector	
$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}})$	$(-6,3,116,58,14,3)$

$F_{3,11}$,$F_{3,22}$,$F_{3,31}$,$F_{3,42}$ Sector	
$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(-1,-5,364,-1,43,-4)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-5,-25,-460,-47,-3,2)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(1,-13,384,9,53,14)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-3,39,-392,-13,31,4)$

$F_{3,12}$,$F_{3,21}$,$F_{3,32}$,$F_{3,41}$ Sector	
$(\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	$(-3,9,-222,72,-60,3)$

$F_{3,14}$,$F_{3,23}$,$F_{3,34}$,$F_{3,43}$ Sector	
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$(-3,27,-20,-10,-54,-1)$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(9,3,116,58,14,3)$
$(\mathbf{10},\mathbf{1},\mathbf{1})$	$(-3,-15,-48,-24,20,-1)$

Appendix C

Details of the Resolution Model

We present the details of the massless spectrum of the six-generation GUT model on the resolution, introduced in sec. 4.3. The simple roots of the gauge group SU(5) $\times$ SU(2) are given in (4.34). They can be used to obtain all states within the irreps by appropriate subtractions. States with zero multiplicity are not considered. States with negative multiplicity are the CPT conjugates of the states mentioned here. First we list the chiral visible sector:

highest weight	multiplicity	irrep
$\big(1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,-1,1,0,0,0,0,0,0,0,0,0,0,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0\big)$	6	$(\mathbf{10},\mathbf{1})$
$\big(0,0,-1,0,0,0,0,1,0,0,0,0,0,0,0,0\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,-1,0,0,0,1,0,0,0,0,0,0,0,0\big)$	16	$(\mathbf{1},\mathbf{1})$
$\big(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\big)$	2	$(\mathbf{5},\mathbf{1})$
$\big(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\big)$	4	$(\bar{\mathbf{5}},\mathbf{1})$
$\big(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\big)$	4	$(\bar{\mathbf{5}},\mathbf{1})$
$\big(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$

Finally, the chiral hidden sector:

highest weight	multiplicity	irrep
$\big(0,0,0,0,0,0,0,0,-1,-1,0,0,0,0,0,0\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,1,0,-1,0,0,0,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1\big)$	2	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,0,-1,-1,0,0,0,0,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,-1,0,-1,0,0,0,0\big)$	10	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,-1,0,0,-1,0,0,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,-1,0,0,0,-1,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,0\big)$	6	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,1\big)$	10	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,-1\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,-1\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,0,0,1,-1,0\big)$	2	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,-1\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	8	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	12	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$	8	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{2})$
$\big(0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\big)$	2	$(\mathbf{1},\mathbf{1})$
$\big(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\big)$	4	$(\mathbf{1},\mathbf{1})$

Bibliography

[1] H. Georgi, S. Glashow, *Unity of All Elementary-Particle Forces*, Phys. Rev. Lett. 32:438-441, 1974.

[2] K. Becker, M. Becker, J. H. Schwarz, *String Theory and M-Theory*, Cambridge University Press, 2007.

[3] David J. Gross, Jeffrey A. Harvey, Emil J. Martinec, and Ryan Rohm. *The Heterotic String*. Phys. Rev. Lett., 54:502-505, 1985.

[4] David J. Gross, Jeffrey A. Harvey, Emil J. Martinec, and Ryan Rohm. *Heterotic String Theory. 1. The Free Heterotic String*. Nucl. Phys., B256:253 , 1985.

[5] H. P. Nilles, *Supersymmetry, Supergravity and Particle Physics*, Phys. Rept. 110,1-162, 1984.

[6] L. Dixon, J. Harvey, C. Vafa, E. Witten, *Strings on Orbifolds*, Nucl. Phys., B261:678-686, 1985.

[7] L. Dixon, J. Harvey, C. Vafa, E. Witten, *Strings on Orbifolds (II)*, Nucl. Phys., B274:285-314, 1986.

[8] L. E. Ibáñez, *The Search for a Standard Model $SU(3)_C \times SU(2)_L \times U(1)_Y$ Superstring: An Introduction to Orbifold Constructions*, CERN-th.4769/87, 1987.

[9] L. E. Ibáñez, H. P. Nilles, F.Quevedo, *Orbifolds and Wilson Lines* Phys. Lett., B187:25-32, 1987.

[10] L. E. Ibáñez, J. E. Kim, H. P. Nilles, F.Quevedo, *Orbifold Compactifications with Three Families of $SU(3)_C \times SU(2)_L \times U(1)_Y$* Phys. Lett., B191:282-286, 1987.

[11] M. B. Green, J. H. Schwarz, *Anomaly Cancellations in Supersymmetric $D = 10$ Gauge Theory and Superstring Theory*, Phys. Lett. B 149:117-22, 1984.

[12] A. Strominger, *Yukawa Couplings in Superstring Compactification*, Phys. Rev. Lett. 55,2547-2550,1985.

[13] E. Zaslow, *Topological Orbifold Models and Quantum Cohomology Rings*, Commun. Math. Phys. 156, 301-331 (1993).

[14] P. S. Aspinwall, B. R. Greene, D. R. Morrison *Measuring small distances in N=2 sigma models*, Nucl. Phys. B420 (1994) 184242, hep-th/9311042, 1993.

[15] W. Fulton, *Introduction to Toric Varieties*, Princton University Press, 1993.

[16] O. Lebedev, H. P. Nilles, S. Raby, S. Ramos-Sánchez, M. Ratz, P. K. S. Vaudrevange, A. Wingerter, *A Mini-Landscape of exact MSSM Spectra in Heterotic Orbifolds*, hep-th/0611095v2, 2006.

[17] O. Lebedev, H. P. Nilles, S. Raby, S. Ramos-Sánchez, M. Ratz, P. K. S. Vaudrevange, A. Wingerter, *The Heterotic Road to the MSSM with R parity*, arXiv:0708.2691v1, 2007.

[18] O. Lebedev, H. P. Nilles, S. Ramos-Sánchez, M. Ratz, P. K. S. Vaudrevange, *Heterotic mini-landscape(II): completing the search for MSSM vacua in a* $\mathbb{Z}_6$ *orbifold*, arXiv:0807.4384v1, 2008.

[19] W. Buchmüller, K. Hamaguchi, O. Lebedev, M. Ratz, *Supersymmetric Standard Model from the Heterotic String.*, hep-ph/0511035, 2005.

[20] W. Buchmüller, K. Hamaguchi, O. Lebedev, M. Ratz, *Supersymmetric Standard Model from the Heterotic String (II).*, hep-th/0606187, 2006.

[21] F. Plöger, S. Ramos-Sánchez, M. Ratz, P. K. S. Vaudrevange, *Mirage torsion*, hep-th/0702176v2, 2007.

[22] D. Lüst, S. Reffert, E. Scheidegger, S. Stieberger, *Resolved Toroidal Orbifolds and their Orientifolds*, hep-th/0609014v2, 2006.

[23] S. Groot Nibbelink, M. Trapletti, M.G.A. Walter, *Resolutions of* $\mathbb{C}^n/\mathbb{Z}_n$ *Orbifolds, their* $U(1)$ *Bundles, and Applications to String Model Building*, hep-th/0701227v3, 2007.

[24] S. Groot Nibbelink, T.-W. Ha, M.Trapletti, *Toric Resolutions of Heterotic Orbifolds*, arXiv:0707.1597v2, 2007.

[25] S. Groot Nibbelink, H. P. Nilles, M. Trapletti, *Multiple anomalous U(1)'s in heterotic blow-ups*, hep-th/070321v2, 2007.

[26] S. Groot Nibbelink, D. Klevers, F. Plöger, M. Trapletti, P. K. S. Vaudrevange, *Compact heterotic orbifolds in blow-up*, arXiv:0802.2809v1, 2008.

[27] S. Groot Nibbelink, F. Paccetti Correia, M. Trapletti, *Non-Abelian bundles on heterotic non-compact K3 orbifold blowups*, arXiv:0809.4430v2, 2008.

[28] S. Groot Nibbelink, J. Held, F. Rühle, M. Trapletti, P. K. S. Vaudrevange, *Heterotic* $\mathbb{Z}_{6-II}$ *MSSM Orbifolds in Blowup*, arXiv:0901.3059v1, 2009.

[29] R. Donagi, K. Wendland, *On orbifolds and free fermion constructions*, arXiv:0809.0330v2, 2008.

[30] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rept. 79,1-128, 1981.

[31] M. Blaszczyk, S. Groot Nibbelink, M.Ratz, F. Rühle, M. Trapletti, P. K. S. Vaudrevange, *A* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *Standard Model*, arXiv:0909.xxxx, 2009.

[32] M. Blaszczyk, S. Groot Nibbelink, M.Ratz, F. Rühle, M. Trapletti, P. K. S. Vaudrevange, *Heterotic MSSM on a Resolved* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *Orbifold with a* $\mathbb{Z}_{2,free}$ *Involution*, arXiv:0910.xxxx, 2009.

[33] S. Ramos-Sánchez, P. K. S. Vaudrevange, A. Wingerter, *C++ Orbifolder*