Anomalies on Orbifolds

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Diplomarbeit in Physik
angefertigt im
Physikalischen Institut

vorgelegt der
Mathematisch–Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich–Wilhelms–Universität
Bonn

im Oktober 2002

Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Referent: Prof. Dr. Hans-Peter Nilles
Koreferent: Priv. Doz. Dr. Stefan Förste

Zuerst möchte ich Herrn Prof. Hans-Peter Nilles danken, der es mir ermöglicht hat, mich mit einem sehr interessanten Thema der theoretischen Physik zu beschäftigen. Mein Dank gilt der gesamten von ihm geleiteten Arbeitsgruppe für die stetige Hilfsbereitschaft und sehr angenehme Atmosphäre, namentlich Kang Sin Choi, Athanasios Dedes, Stefan Förste, Mark Hillenbach, Gabriele Honecker, Stefan Groot Nibbelink, Jihn E. Kim, Marek Olechowski, Marco Peloso, Jasmin Pielorz, Martin Walter und Akın Wingerter.
Insbesondere danke ich Stefan Groot Nibbelink und Martin Walter für die tolle Zusammenarbeit und die Beantwortung meiner vielen Fragen. Diesen beiden, sowie Mark Hillenbach möchte ich außerdem für das Korrekturlesen dieser Arbeit danken.
Nicht zuletzt bedanke ich mich bei meinen Eltern, die mir das Studium der Physik ermöglicht haben, sowie bei Robin Gareus, ohne den mir dieses Studium nur halb so viel Spaß gemacht hätte. Mein besonderer Dank gilt auch meiner Freundin Petra Gutjahr für Ihre Unterstützung und Geduld.

All science is either physics or stamp collecting.
(E. Rutherford)

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Chapter 1

Introduction

The standard model of particle physics (SM) (see e.g. [17]) describes the fundamental interactions, with the exception of gravity, in terms of (nonabelian) gauge field theories. Its predictions are tested experimentally to a very high precision, but the strong and electroweak interactions are treated as separate theories. Grand unified theories (GUTs) (see e.g. [65]) are attempts to unify these two into one gauge theory. The most promising gauge groups, containing the SM $SU(3) \times SU(2) \times U(1)$ group as a subgroup, are SU(5) and SO(10). To achieve a unification of the strong and electroweak forces, the corresponding gauge couplings should meet at the unification scale, which is not the case in the standard model. A very attractive approach to solve this problem is to consider supersymmetric models, relating bosonic and fermionic degrees of freedom. Supersymmetry is appealing also for other reasons, for example it accounts for the cancellation of quadratic divergences. Supersymmetric grand unified theories are very successful in predicting the quantum numbers of the standard model particles, but introduce new problems, including the details of gauge and supersymmetry breaking and the origin of doublet-triplet splitting. The latter problem occurs in SU(5) GUTs, where we get a colored triplet of Higgs bosons in addition to the standard model SU(2) doublet. The SM Higgses have to develop a vacuum expectation value to break electroweak symmetry, but the masses of the colored partners have to be much heavier, because they can mediate baryon number violating processes, like proton decay, that have to be suppressed.

Recently there has been growing interest in addressing these issues using extra spacelike dimensions, a concept already considered by Kaluza and Klein [44, 49]. The new ingredient is to use orbifold spaces for the compactification to (3+1) dimensions. These orbifold GUTs (see e.g. [3, 5, 35]) are able to explain the doublet-triplet splitting, it is possible to break supersymmetry and get chiral fermions in the effective theory. These very desirable properties rely on discrete symmetries. In orbifold models these discrete symmetries are projected out. This mechanism can be used to eliminate unwanted states in the effective four dimensional theory.

Although these models seem to be very attractive, they are quite unconstrained, in the sense that they cannot make any predictions for fields living at the fixed points.

In principle one is free to put arbitrary fields on these points, the only restriction is given by demanding the cancellation of chiral anomalies. This cancellation, which is needed for the consistency of the theories, is a nontrivial issue. Problems arise, because the orbifold projections do not necessarily preserve the consistency at the quantum level [37, 38]. A consistent orbifold field theory should be locally free of gauge and gravitational anomalies, which means that the corresponding symmetries are respected at each fixed point. Calculations of local anomalies for five- and six dimensional orbifolds have been carried out in [4, 8, 62, 67] and [6]. It has been shown, that theories which are anomaly free at the zero mode level could be made locally consistent by the introduction of appropriate Chern-Simons terms. In [1, 30, 31] the implications of localized U(1) anomalies, which lead to quadratically divergent Fayet-Iliopoulos terms, were addressed and a relation to localization effects of bulk fields has been established.

In this thesis which is based on two publications [26, 27] (in collaboration with S. Groot Nibbelink, J. E. Kim, H. M. Lee, H. P. Nilles, M. Olechowski and M. Walter) the problem of local anomaly cancellation has been considered from a string theoretical perspective. Orbifolds have been indeed used first in heterotic string theory compactifications [18, 19]. This approach has two major advantages. Firstly string theory can be consistently defined on the whole orbifold, which leads to the prediction of additional chiral matter states, located at the fixed points. Therefore the contribution to local anomalies is completely specified. Secondly ten-dimensional string theory models are anomaly free due to the Green-Schwarz mechanism [29].

Thus an investigation of local anomaly cancellation in the effective field theories of heterotic string theory models on orbifolds is very instructive to gain insight into problems that arise in the pure field theoretical approach. In [27] the question was raised, how localized nonabelian anomalies can be canceled. As it turned out [26], this type of anomaly is always canceled locally by means of contributions from heavy Kaluza-Klein modes. In the case of the abelian anomalies no such general statement can be made, but their structure can be determined and their relation to U(1) anomalies at the zero mode level can be specified.

This work is organized as follows. In the next chapter we give an introduction to the geometrical construction of orbifolds in general and the $T^6/\mathbb{Z}_3$ case in particular, which we will consider throughout this work. The chapters 3 and 4 deal with the formulation of field and string theory on these spaces, respectively. Having introduced the important concepts, we turn to the anomalies in chapter 5 and apply the results to some illuminating examples in chapter 6. In the appendices some technical issues of spinors and Lie algebras are addressed and properties of special orbifold models are listed.

Chapter 2

Orbifold construction

In this chapter we will discuss the geometrical issues of orbifold compactifications, starting with the simplest examples in one and two dimensions. In the second part we will analyze the properties of the $\mathbb{Z}_3$ orbifold, the construction that has received a lot of attention in string theory compactifications and which we will use later to construct some explicit models.

2.1 One- and two-dimensional orbifolds

Orbifolds are quotient spaces of a compact manifold with a non freely acting group. The resulting space is not a manifold, because it contains singular points, the fixed points of the quotient group. Nevertheless, this space has, away from the fixed points, the geometry of a manifold and it is possible to formulate consistent string theories on these spaces, as we will see later.

In general, an orbifold can be described by

$$O := M/P, \tag{2.1}$$

where the discrete group P is called the *point group* of the orbifold.

We are interested in toroidal base manifolds, $M = T^n$, which can always be constructed as a coset space from $\mathbb{R}^n$ via

$$T^n = \mathbb{R}^n/\Gamma, \tag{2.2}$$

where Γ is an integral lattice. Therefore we can describe the orbifold as the coset

$$O = \mathbb{R}^n/S, \tag{2.3}$$

where S is called the *space group*, which is the semi-direct product of Γ and P. The product is only semi-direct, because in general the orbifold twist and the lattice translations do not commute.

2.1.1 The orbifold $\mathbf{S^1/\mathbb{Z}_2}$

The simplest possible orbifold can be constructed from a circle, $M = S^1$, with point group $P = \mathbb{Z}_2$.

$$O := S^1/\mathbb{Z}_2 \tag{2.4}$$

The physical space is described by the set of equivalence classes of points of S^1 under the equivalence relation induced by P. We implement the action of $\mathbb{Z}_2$ as a reflection

$$y \simeq -y. \tag{2.5}$$

As mentioned in the previous section, we can describe our model by the action of a space group according to (2.3), if we construct $T^1 = S^1$ as a quotient space

$$S^1 = \mathbb{R}/\Gamma, \tag{2.6}$$

where Γ is a lattice defined by

$$\Gamma = \{\ell = 2\pi R\,\mathbb{Z}\} \tag{2.7}$$

with $R =$ const. being the radius of the circle. Therefore points on the circle are given by the equivalence classes defined by

$$y \simeq y + \ell, \qquad \ell \in \Gamma. \tag{2.8}$$

Now we can combine the action of $p = (1, -1) \in P$ and $\ell \in \Gamma$ into the action of the space group S, which gives the equivalence relation

$$y \simeq p\,y + \ell. \tag{2.9}$$

This identification reduces $\mathbb{R}$ to an interval $y \in [0, \pi]$. One can think of this as a two-step operation, first compactifying $\mathbb{R}$ to S^1 and reducing it then to the interval via P, or considering directly the action of S (see figure 2.1).

There are special points ζ on the orbifold, called *fixed points*, which are left *invariant* under the action of S. They are defined by

$$p\,\zeta + \ell = \zeta, \tag{2.10}$$

and in our case given by 0 and πR, where the origin is trivially invariant under the reflection and πR fulfills

$$-\pi R + 2\pi R = \pi R. \tag{2.11}$$

2.1.2 $\mathbb{Z}_n$ point groups and complex coordinates

In higher dimensional orbifolds it is possible to consider also higher dimensional cyclic groups. The finite, cyclic groups of order n, denoted by $\mathbb{Z}_n$, are generated by one element, which we will call θ, in the following way

$$\mathbb{Z}_n = \left\{\theta^k;\ k = 0, \ldots, n-1\right\}. \tag{2.12}$$

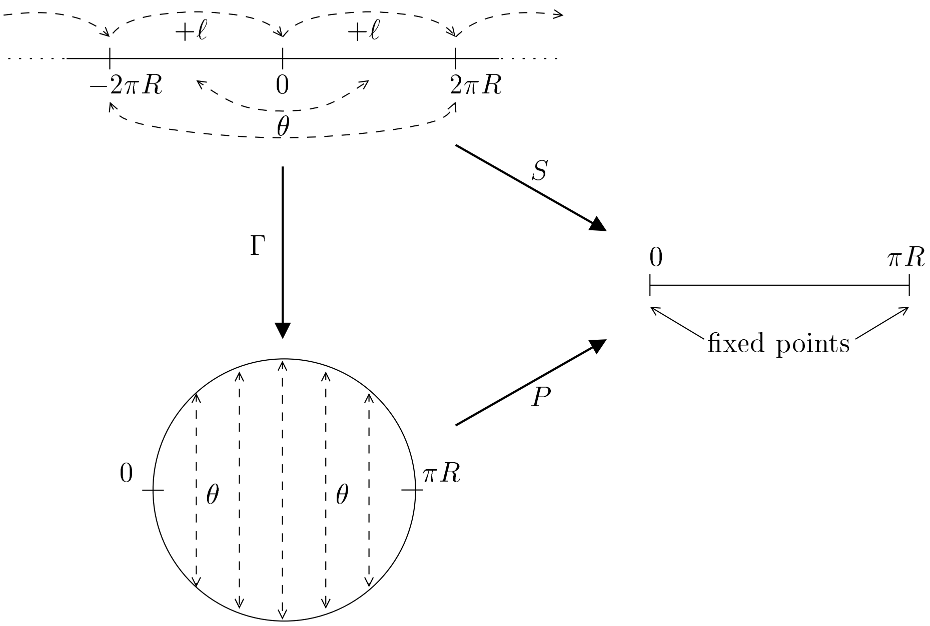


Figure 2.1: *The actions of the space and point group reduces $\mathbb{R}$ resp. S^1 to an interval with two fixed points. The dashed lines connect points that are identified under group actions.*

The cyclicity of $\mathbb{Z}_n$ is reflected by the condition, that

$$\theta^n = 1, \tag{2.13}$$

where 1 is the identity or neutral element of the group.

To describe two dimensional toroidal orbifolds it is convenient to introduce a complex coordinate z. The action of the generator of $\mathbb{Z}_n$ is implemented as a rotation in the complex plane, given by

$$\theta = e^{2\pi i/n}, \tag{2.14}$$

acting on z or $\bar{z}$ as

$$z \simeq \theta z = e^{2\pi i/n} z \tag{2.15a}$$
$$\bar{z} \simeq \bar{\theta}\bar{z} = e^{-2\pi i/n} \bar{z} \tag{2.15b}$$

In addition to 2.13 we find the following relation for the twist

$$\sum_{k=0}^{n-1} \theta^k = 0. \tag{2.16}$$

The action of the space group S is given now as $(\theta, \ell)\, z = \theta z + \ell$ and the fixed points ζ are in general defined by

$$(\theta, \ell)\, \zeta = \theta\zeta + \ell = \zeta, \tag{2.17}$$

with solution

$$\zeta = (1-\theta)^{-1}\ell. \tag{2.18}$$

2.1.3 The orbifold $\mathbf{T^2/\mathbb{Z}_3}$

We want to apply this now to the case of $T^2/\mathbb{Z}_3$, the two-dimensional torus modded out by the cyclic group with three elements. We use the complex notation introduced in the last section and define the torus as a quotient manifold of $\mathbb{C}$ and an appropriate lattice, given in analogy to (2.7) as

$$\Gamma = \{\ell = n\,\hat{\imath} + m\,\theta\hat{\imath};\ n,m \in \mathbb{Z}\}, \tag{2.19}$$

where $\hat{\imath}$ and $\theta\hat{\imath}$ are the two basis vectors of the lattice. They define the *fundamental domain* of the torus (see fig. 2.2).

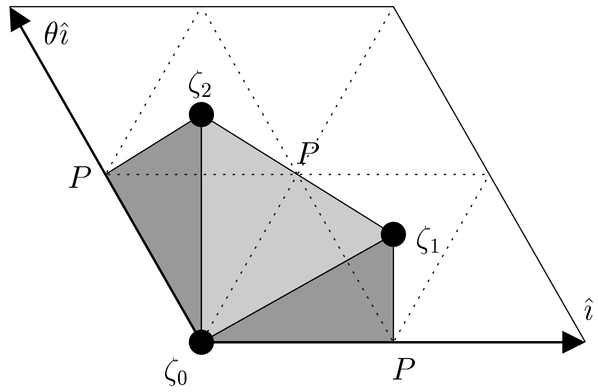


Figure 2.2: *The $T^2/\mathbb{Z}_3$ orbifold. The figure shows the fundamental domain of the torus, spanned by the two lattice vectors, $\hat{\imath}$ and $\theta\hat{\imath}$. The area shaded in light grey is the fundamental domain of the orbifold, with the additional pieces shaded in dark grey flipped into this area and glued at the edges (such that the points marked with P are identified). The black dots mark the three fixed points of $\mathbb{Z}_3$.*

The twist generator θ is defined according to (2.14) as $e^{2\pi i/3}$, implementing the action of $\mathbb{Z}_3$ as a rotation about 120 degrees in the complex plane. The relations (2.13) and (2.16) read now

$$\theta^3 = 1, \tag{2.20}$$
$$1+\theta+\theta^2 = 0. \tag{2.21}$$

The fixed points can be determined using (2.18), which can be rewritten using (2.21) as

$$\zeta = (1-\theta)^{-1}\ell = \tfrac{1}{3}(2+\theta)\ell, \tag{2.22}$$

leading to three fixed points (see fig. 2.2)

$$\zeta_0 = 0, \qquad \zeta_1 = \tfrac{1}{3}(2+\theta) \quad \text{and} \quad \zeta_2 = \tfrac{1}{3}(1+2\theta). \tag{2.23}$$

Their transformation under twists is related to shifts in the torus lattice according to

$$\theta\zeta_0 = \zeta_0, \qquad \theta\zeta_1 = \zeta_1 - 1, \qquad \theta\zeta_2 = \zeta_2 - \theta - 1. \tag{2.24}$$

2.2 The $\mathbb{Z}_3$ orbifold

In this section we want to generalize the torus construction to six dimensions. We are especially interested in the case of six dimensions, because we want to deal with compactifications of heterotic string theory, which lives in ten dimensions and has to be compactified on a six dimensional manifold to arrive at a four dimensional description.

The $\mathbb{Z}_3$ orbifold is a six dimensional torus T^6, modded out by $\mathbb{Z}_3$.

$$O = T^6/\mathbb{Z}_3 \tag{2.25}$$

The six dimensional torus can be described by a lattice as a generalization of the two-torus lattice (2.19) used in the previous section. A coordinate on T^6 can be described by three complex coordinates z^i and their complex conjugates $\bar{z}^i = z^{\underline{i}}$ (each parameterizing a T^2 sub-torus), that we will unify in a vector $z = (z^1, z^2, z^3) \in \mathbb{C}^3$, with the equivalence relations

$$z \simeq z + \ell, \qquad \ell \in \Gamma \tag{2.26}$$

and the definition of the lattice as

$$\Gamma = \left\{ \ell = n_i\,\hat{\imath} + m_i\,\theta_i\hat{\imath};\ n_i, m_i \in \mathbb{Z}; \quad \theta_i = e^{2\pi i\,\phi_i} \right\}, \tag{2.27}$$

using unit vectors $\hat{\imath}$ and $\theta_i\hat{\imath}$ for one two-torus as in the previous section. The generator of $\mathbb{Z}_3$ for each two-torus is given by

$$\theta_i := e^{2\pi i\,\phi_i}, \tag{2.28}$$

with $\phi_i = const.$ and satisfying $3\phi_i \equiv 0$, where we introduced the notation $a \equiv b$ for $a = b \mod \mathbb{Z}$, which we will use throughout this work.

We can combine the action of the twist into one operator Θ, acting on the coordinates as

$$\Theta\,z = (\theta_1\,z^1, \theta_2\,z^2, \theta_3\,z^3). \tag{2.29}$$

The vector ϕ can be defined arbitrarily in principle, but we want the additional condition

$$\sum_i \phi_i = 0 \tag{2.30}$$

to be fulfilled, in order to get a SU(3) holonomy group under which the coordinates transform as a triplet. This will be needed for the breaking of supersymmetry as explained in the next chapter.

A consistent choice for ϕ that satisfies the requirements (2.30) is

$$\phi = \frac{1}{3}(1, 1, -2), \tag{2.31}$$

which we will use from now on.

The lattice of one of the three two-tori (see fig. 2.2) is isomorphic to the SU(3) root lattice, denoted by $\Gamma_{\mathrm{SU}(3)}$, therefore the whole lattice is isomorphic to the root lattice of $\mathrm{SU}(3)^3$.

The fixed points can be constructed from the fixed points of $T^2/\mathbb{Z}_3$, given by (2.23). Since we have three fixed points on each of the three T^2 sub-tori we have in total $3^3 = 27$ fixed points on the orbifold [57]. We will denote these fixed points by

$$\mathfrak{z}_s = \mathfrak{z}_{s_1,s_2,s_3} = (R^{(1)}\zeta^{(1)}_{s_1}, R^{(2)}\zeta^{(2)}_{s_2}, R^3\zeta^{(3)}_{s_3}), \quad s_i = 0, 1, 2. \tag{2.32}$$

The volume of the orbifold is given by one third of the torus volume. This can be seen directly in (fig. 2.2), but one has to take into account, that we are not reducing the fundamental domains of all three T^2 tori, but the fundamental domain of the six torus by one third. Reducing all two-tori would correspond to the case of taking the direct product of three $T^2/\mathbb{Z}_3$ orbifolds.

Chapter 3

Higher dimensional field theory

In this chapter we want to investigate how orbifolds can be used in field theoretical models. After an introduction to higher dimensional theories, we will consider a simple five dimensional model, compactified to Minkowski space. In this example we will use the orbifold $S^1/\mathbb{Z}_2$, described in (2.1.1). In a second step we will consider Yang-Mills theory and supergravity in ten dimensions, compactified on the $\mathbb{Z}_3$ orbifold. The results will be used in chapter 5 to investigate global and local anomalies of these models.

3.1 Kaluza-Klein theories

The original approach of Kaluza and Klein [44, 49] in the 1920's was to consider Einstein gravity in five dimensions on a manifold $\mathbb{R}^{3,1} \times S^1$, where $\mathbb{R}^{3,1}$ is the usual Minkowski space-time, interpreting the $g_{\mu 5}$ components of the metric as the electromagnetical field. In the following we will take the metric to be diagonal (see app. A) and consider only higher dimensional fields in this background.

As an example let us take a five dimensional theory of a massless complex scalar field ϕ and compactify the fifth dimension on a circle.

The action on $\mathbb{R}^{4,1}$ is given by

$$S = \int \mathrm{d}^5 x \, \partial_M \phi^*(x) \partial^M \phi(x), \qquad M = 0, \ldots, 4 \tag{3.1}$$

and the field equation is of standard form, using the five dimensional Laplacian $\Delta^{(4,1)}$, defined as

$$\Delta^{(4,1)} \phi = \partial_M \partial^M \phi = 0. \tag{3.2}$$

Now we want to compactify the fifth dimension on a circle of radius R and then integrate over this compact dimension to get an effective four dimensional action. We will denote the four dimensional Minkowski coordinates with x^μ and the coordinate on the circle with y.

Define the lattice of the circle as in (2.7), then we have the dual lattice

$$\Gamma^* = \left\{ q = \frac{n}{2\pi R};\ n \in \mathbb{Z} \right\}. \tag{3.3}$$

Choose a complete set of eigenfunctions of ∂_y on S^1 as

$$\eta_q(y) = \frac{1}{\sqrt{2\pi R}} e^{2\pi i\, qy}, \qquad q \in \Gamma^*. \tag{3.4}$$

The functions η_q have been chosen to form an orthonormal basis on S^1, their orthogonality and completeness relations are given by

$$\left\langle \eta_p^*(y), \eta_q(y) \right\rangle = \int_{S^1} \mathrm{d}y\, \eta_p^*(y)\, \eta_q(y) = \frac{1}{2\pi R} \int_{S^1} \mathrm{d}y\, e^{2\pi i\,(q-p)y} = \delta_{q,p} \tag{3.5}$$

$$\sum_{q\in\Gamma^*} \eta_q^*(y)\, \eta_q(z) = \frac{1}{2\pi R} \sum_q e^{2\pi i\, q(z-y)} = \delta(z - y - \Gamma), \tag{3.6}$$

where $\delta(y - \Gamma) = \sum_{\ell\in\Gamma} \delta(z - \ell)$ is the delta function on the circle.

The derivative of $\eta_q(y)$ with respect to y is given by

$$\partial_y \eta_q(y) = 2\pi i\, q\, \eta_q(y). \tag{3.7}$$

Now we can expand the field $\phi(x)$ in terms of these functions with some coefficients $\widetilde{\phi}(x_\mu)$ that depend on the uncompactified coordinates only.

$$\phi(x) = \sum_{q\in\Gamma^*} \widetilde{\phi}_q(x_\mu)\, \eta_q(y) = \frac{1}{\sqrt{2\pi R}} \widetilde{\phi}_q(x_\mu)\, e^{2\pi i\, qy}. \tag{3.8}$$

Using this decomposition for $\phi(x)$ and the property (3.7) of $\eta_q(y)$, we can rewrite the action (3.1) as

$$S = \int \mathrm{d}^4x \int_{S^1} \mathrm{d}y \sum_{q,p\in\Gamma^*} \left(\partial_\mu \widetilde{\phi}_q^*(x_\mu)\, \partial^\mu \widetilde{\phi}_p(x_\mu) + 4\pi^2 q^2 \right) \eta_q^*(y) \eta_p(y). \tag{3.9}$$

Integration over the fifth coordinate using (3.5) simplifies this to

$$S = \int \mathrm{d}^4x \sum_{q\in\Gamma^*} \left(\partial_\mu \widetilde{\phi}_q^*\, \partial^\mu \widetilde{\phi}_q + M_q^2 \widetilde{\phi}_q^*\, \widetilde{\phi}_q \right), \tag{3.10}$$

where

$$M_q := 2\pi q = \frac{n}{R}, \qquad n \in \mathbb{Z}, \tag{3.11}$$

is the coefficient of a mass term of the four dimensional field $\widetilde{\phi}_q$.

Looking at (3.10) we realize, that we have a four dimensional theory of infinitely many scalar fields. The spectrum contains one massless field ($q = 0$) and an infinite set of massive states, called the *Kaluza-Klein tower*.

In the *dimensionally reduced* theory, which is reached in the limit $R \to 0$, we are left with the massless state only. This limit is interesting in the context of small extra dimensions, because in models of this type the mass scale of the Kaluza-Klein excitations is in the range of the GUT scale. Hence interactions of the massive modes are heavily suppressed and the states decouple in the effective theory.

3.2 Quantum field theory on orbifolds

During the last years there has been growing interest in models of this type and various models on S^1 and T^2 with enhanced point groups have been constructed and their properties have been analyzed [3, 9, 10, 34, 35, 47].

We will first review some aspects of the original Kawamura model, which uses the simplest possible orbifold (see 2.1.1) to implement a supersymmetric SU(5) GUT. With this example at hand we explain the mechanisms of supersymmetry and gauge symmetry breaking, which will be used for the $\mathbb{Z}_3$ orbifold in the next section.

3.2.1 An orbifold model on $\mathbf{S^1}/\mathbb{Z}_2$

The same notation as in 3.1 is used to describe the five dimensional space-time, now given by

$$\mathbb{R}^{3,1} \times S^1/\mathbb{Z}_2. \tag{3.12}$$

The action of the $\mathbb{Z}_2$ transformation is implemented as a reflection on the internal coordinate (2.5).

Considering a five dimensional field (*bulk field*), this gives rise to an intrinsic $\mathbb{Z}_2$ parity P, defined by the transformation

$$\phi(x_\mu, y) \to \phi(x_\mu, -y) = P\phi(x_\mu, y), \qquad P = \pm 1. \tag{3.13}$$

We take the expansion (3.8) of $\phi(x)$ and denote fields with $P = \pm 1$ by $\phi^{(\pm)}$. Taking the parity into account, we see that for the expansion of $\phi^{(+/-)}$ the $\mathbb{Z}_2$ odd/even modes have to vanish, so we get different expansions

$$\phi^{(+)}(x_\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{q\in\Gamma*} \widetilde{\phi}_q^{(+)}(x_\mu) \cos(2\pi\, qy), \tag{3.14a}$$

$$\phi^{(-)}(x_\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{q\in\Gamma*} \widetilde{\phi}_q^{(-)}(x_\mu) \sin(2\pi\, qy). \tag{3.14b}$$

Obviously only the even four dimensional fields $\widetilde{\phi}^{(+)}$ have a massless mode (*zero mode*), because the term with $q = 0$ vanishes in the expansion of $\phi^{(-)}$. This offers the possibility of reducing local symmetries, if it is possible to assign different parities to subsets of the generators. Going to the effective four dimensional theory in the limit where the massive Kaluza-Klein modes decouple ($R \to 0$), only the subset containing generators which are invariant under the twist is present.

The assignment of parities for fields is equivalent to the specification of boundary conditions at the fixed points. The $\phi^{(+/-)}$ fields fulfill Dirichlet or Neumann boundary conditions respectively,

$$\begin{aligned} \partial_y \phi^{(+)}(x_\mu, 0) = \partial_y \phi^{(+)}(x_\mu, \pi R) &= 0, && (3.15a) \\ \phi^{(-)}(x_\mu, 0) = \phi^{(-)}(x_\mu, \pi R) &= 0. && (3.15b) \end{aligned}$$

3.2.2 Supersymmetry breaking

As we have seen in the previous section, we can assign different parities (or boundary conditions at the fixed points) to the fields of our theory, as long as the Lagrangian is left invariant under the point group transformation. This offers the possibility of supersymmetry and/or gauge symmetry breaking. We will explain supersymmetry breaking in the framework of the $S^1/\mathbb{Z}_2$ orbifold considered above, starting from supersymmetric Yang-Mills theory (SYM) in five dimensions.

Take a vector multiplet of minimal supersymmetry ($N = 1$) in five dimensions. By simple dimensional reduction to four dimensions one obtains an $N = 2$ supermultiplet, but with an additional orbifolding we will have reduced $N = 1$ supersymmetry at the fixed points.

To realize this, we have to ask first what kind of parity assignments are consistent with a five dimensional super-Yang-Mills action. A $5d$ vectormultiplet consists of a gauge field A_M, a gaugino λ^i, where i is an internal SU(2) spinor index (related to the additional generator Γ_4), and a real scalar ϕ. In $4d, N = 1$ language this is a vector multiplet $V = (A_\mu, \lambda^1)$ and a chiral multiplet $\Sigma = (\lambda^2, \phi + iA_5)$. We use an on-shell formalism here, for an off-shell treatment see [53]. In the following we write all fields contracted with the adjoint matrices of the gauge group T^a. As an example consider the gaugino, which reads in this notation $\lambda = \sum_a \lambda^a T^a$. The five dimensional SYM action is given by

$$S = \int \mathrm{d}^5 x \, \mathrm{Tr} \left(-\tfrac{1}{2} F^2 + (D_M \phi)^2 + \bar{\lambda} \not{D} \lambda + \bar{\lambda}[\phi, \lambda] \right), \tag{3.16}$$

with covariant derivative

$$D_M \phi = \partial_M \phi - i[A_M, \phi] \quad , \quad \not{D} = D_M \Gamma^M. \tag{3.17}$$

To keep (3.17) invariant under the twist we have to assign opposite parities for A_μ and A_5, but we are free to include an additional overall transformation U, which acts as a rotation in the gauge degrees of freedom. The transformation is given by

$$\begin{aligned} A_\mu(x_\mu, y) &\to A_\mu(x_\mu, -y) = U A_\mu(x_\mu, y) U^{-1} && (3.18a) \\ A_5(x_\mu, y) &\to A_5(x_\mu, -y) = -U A_5(x_\mu, y) U^{-1}, && (3.18b) \end{aligned}$$

which can also be justified by noticing that we can write the gauge field as a one-form $A = A_M \mathrm{d}x^M$, which should transform as UAU^{-1}.

Our theory contains spinors, therefore we must work out a representation R of the $\mathbb{Z}_2$ twisting. It can be found by the requirements that $R^2 = \mathbb{1}$ and that the five dimensional gamma matrices γ_M (see app. A.2) have to transform as

$$-\gamma_0 R^\dagger \gamma_0 \gamma_M R = \begin{cases} \gamma_\mu & : M = 0, \dots, 3 \\ -\gamma_4 & : M = 4 \end{cases}. \tag{3.19}$$

Obviously the generator we are looking for is γ_4 (in analogy to the usual parity operator γ_0). In the five dimensional case we see, that the two parts of λ^i have to transform with an opposite sign under the twist. This is also true for the supersymmetry generators $\xi^i = (\xi^1, \xi^2)$.

Collecting the results, we have to assign parities P as listed in tab. 3.1. In the effective four dimensional theory we will get only states that transform with an even parity (3.14). This implies, that one half of the supersymmetry is broken, because only one of the supersymmetry generators survives. The matter content turns out to be an $N = 1$ vector multiplet, made up from (A_μ, λ^1).

Field	A_μ	A_5	λ^1	λ^2	ϕ	ξ^1	ξ^2
Parity P	$+$	$-$	$+$	$-$	$-$	$+$	$-$

Table 3.1: $\mathbb{Z}_2$ *parity assignments for a* 5*d vector multiplet.*

3.2.3 Gauge symmetry breaking

From the discussion above it is clear, that we are also able to break the five dimensional gauge group by the orbifold procedure [36, 33]. We can implement this using the additional transformation U for the generators of the original gauge group. At the fixed points only the generators with an even transformation under the twist are present, and we are left with a reduced gauge group.

Take for example an SU(5) model and represent the gauge fields as matrices in the fundamental representation. We choose the transformation U of (3.18), which is a 5×5 matrix in this case, to be

$$U = \mathrm{diag}(1, 1, 1, -1, -1). \tag{3.20}$$

This breaks the SU(5) gauge group at one fixed point to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, projecting out all the other generators. To see this, consider $UA_\mu U^{-1}$, which can be written in matrix notation as

$$UA_\mu U^{-1} = \begin{pmatrix} A_\mu^{(3,3)} & -A_\mu^{(3,2)} \\ -A_\mu^{(2,3)} & A_\mu^{(2,2)} \end{pmatrix}. \tag{3.21}$$

Because only the generators with a plus sign have zero modes, we are left with $A_\mu^{(3,3)}$, $A_\mu^{(2,2)}$ and an additional diagonal generator. These three components represent the generators of SU(3), SU(2) and U(1), respectively.

More generally we can embed the full space group S nontrivially into the gauge group G. In the example given above we considered only gauge transformations corresponding to space-time twists, but we can also perform transformations corresponding to lattice shifts. This is done by defining an homomorphism between the space group and transformations in the group lattice[1]

$$S \quad \to \quad G \tag{3.22a}$$

$$(\Theta, \ell) \quad \mapsto \quad (\tilde{\Theta}, a^I), \qquad I = 1, \ldots, r;\ r = \mathrm{rank}\,(G), \tag{3.22b}$$

where $\tilde{\Theta}$ is an automorphism of the root lattice of G, i.e. an element of the Weyl group (see app. B.4), and a^I represents a shift in this lattice. The shift corresponds to a gauge transformation associated to a non-contractible loop in the compact space, defined by ℓ, and therefore corresponds to a *Wilson line*. Wilson lines can also be used to implement a gauge group breaking, which is known as the *Hosotani mechanism* [39]. In general the actions of $\tilde{\Theta}$ and a^I do not commute, which can be used to lower the rank of the gauge group (this has been shown in the context of string theory in [43]).

We will restrict ourselves in the following to the commuting case. This implies, that we can represent $\tilde{\Theta}$ by a shift vector v^I. The transformation matrix U for a specific choice of twist and lattice shift $(\Theta, \hat{\imath})$ is then given by $U = \exp(2\pi i(v + a_i) \cdot H)$, where H^I are the Cartan generators of G (see app. B.1).

3.3 Yang-Mills theory and supergravity on $\mathbf{T^6}/\mathbb{Z}_3$

In this section we want to provide the basis for the discussion of anomalies on the $\mathbb{Z}_3$ orbifold. Therefore we have to consider the field theory limit of heterotic string theory, which is known to be ten dimensional supergravity coupled to $\mathrm{E}_8 \times \mathrm{E}_8{}'$ super Yang-Mills gauge theory [28]. Here the problem arises that the orbifolding procedure and taking the field theory limit (which means to put $\alpha' \to 0$) do not commute (see fig. 1 of [26]). If we take the field theory limit first and compactify afterwards, we get only the untwisted matter states of the effective four dimensional theory. Therefore we have to include the twisted states predicted by string theory compactification (see 4.3.2) as additional four dimensional chiral matter at the fixed points. One important feature of the orbifolding procedure is of course the breaking of supersymmetry. This works in analogy to the five dimensional example considered in 3.2.2 and we implement $\mathbb{Z}_3$ in such a way, that in the effective four dimensional theory only 1/4 of the ten dimensional supersymmetry is left.

[1] The notation is the same as in chapter 2, for the definition of group lattices see app. B.3.

3.3.1 Gauge multiplet transformation

We are interested in gauge anomalies, therefore we want to know which gauge fields and gauginos are present at the fixed points. Hence we consider only the transformation of the ten dimensional gauge multiplet under the orbifold twist. The complete Lagrangian contains of course also the supergravity multiplet (see [15, 12, 16]), but in the context of chiral anomalies the transformations of this multiplet are not relevant for us. The gauge multiplet contains the gauge fields A_M and left-handed Majorana-Weyl gauginos χ, both transforming in the adjoint of $\mathrm{E}_8 \times \mathrm{E}_8{}'$. Under a local gauge transformation with gauge element $g(x,z)$ they transform as

$$\begin{aligned} i\,{}^g\!A_M(x,z) &= g(x,z)\Big(iA_M(x,z) + \partial_M\Big)g^{-1}(x,z), &\quad (3.23a)\\ {}^g\chi(x,z) &= g(x,z)\chi(x,z)g^{-1}(x,z). &\quad (3.23b) \end{aligned}$$

Consider first the generalization of (3.18) to four non-compact and six compact dimensions, described by Minkowski coordinates x^μ and complex coordinates $z^i, z^{\underline{i}}$, with $i, \underline{i} = 1,2,3$. For the transformation of a generic vector field we obtain

$$V_M(x,\Theta z) = (\Theta^{-1})_M{}^N\, V_N(x,z) = \begin{cases} V_\mu(x,z) \\ \theta^2\, V_i(x,z) \\ \theta\, V_{\underline{i}}(x,z) \end{cases} . \qquad (3.24)$$

The generalization of (3.19) to ten dimensions and twist group $\mathbb{Z}_3$ reads

$$-\Gamma_0 R^\dagger \Gamma_0 \Gamma_M R = (\Theta^{-1})_M{}^N \Gamma_N \quad , \quad R^3 = \mathbb{1}, \qquad (3.25)$$

with solution[2]

$$R = \exp(-\pi\phi^j\Sigma_j) \quad , \quad \Sigma_j = \tfrac{1}{2}[\Gamma_{2j+2}, \Gamma_{2j+3}] = \begin{cases} j=1: i\mathbb{1}\times\mathbb{1}\times\sigma_3 \\ j=2: i\mathbb{1}\times\sigma_3\times\mathbb{1} \\ j=3: i\sigma_3\times\mathbb{1}\times\mathbb{1} \end{cases} . \qquad (3.26)$$

A six dimensional spinor η^α (with chirality $\alpha = \alpha_1\alpha_2\alpha_3$ as defined in A.2) transforms as

$$R\,\eta^\alpha = \exp(-2\pi i\,\tfrac{1}{2}\phi^i\alpha_i)\eta^\alpha. \qquad (3.27)$$

With the choice of ϕ as in (2.31) we see that $\eta^0 := \eta^{+++}$ is invariant, while

$$\eta^a := \left(\eta^{+--}, \eta^{-+-}, \eta^{--+}\right) \qquad (3.28)$$

transforms as a triplet of the holonomy group $\mathrm{SU}(3)_H$. The components η^{-++}, η^{+-+} and η^{++-} transform as an anti-triplet, but they are not independent. This follows from the Majorana condition in ten dimensions. We have chosen ϕ exactly such that

[2] For the definition of gamma matrices and spinors used here see app. A.2.

this result is obtained, because if we apply this to the supersymmetry generators, we get one constant spinor in four dimensions, responsible for $N = 1$ supersymmetry.

The space group acts on the internal coordinates according to (2.26) and (2.29). For the embedding into the gauge degrees of freedom we restrict ourselves to *shift embeddings* of Θ. This is the case in which the twist embedding commutes with the Wilson lines a_i^I $(i = 1, 2, 3)$, as explained below (3.22). Hence we can use a shift representation v^I instead of $\tilde{\Theta}$. We can implement a maximum of three Wilson lines, although we have six internal dimensions. To explain this, let us consider one of the three two-tori and switch on a Wilson line a_i along the lattice vector $\hat{\imath}$. A Wilson line in direction of the second lattice vector $\theta_i \hat{\imath}$ has to be identical to a_i, in order to be consistent with the $\mathbb{Z}_3$ twist which acts as $\hat{\imath} \to \theta_i \hat{\imath}$.

Representing the generators of $\mathrm{E}_8 \times \mathrm{E}_8{}'$ in the Cartan-Weyl basis (see app. B.1) with normalization

$$\exp(2\pi i\, H^\alpha) = \exp(2\pi i\, E^\alpha) = 1, \quad \forall \alpha \in \Gamma_{\mathrm{E}_8^2}, \tag{3.29}$$

we write the gauge and gaugino fields as

$$\begin{aligned} A_M &= A_M^I H_I + A_M^\alpha E_\alpha, && (3.30\text{a}) \\ \chi &= \chi^I H_I + \chi^\alpha E_\alpha. && (3.30\text{b}) \end{aligned}$$

The transformation of the gauge field under the orbifold twist is given by

$$A_M(x, \Theta z) = (\Theta^{-1})_M{}^N\, U\, A_N(x, z)\, U^{-1} \quad , \quad U = \exp(2\pi i\, v^I H_I). \tag{3.31a}$$

In general (if we allow for Wilson lines) the gauge fields have to be invariant going round one of the compact dimensions (the non-contractible loops of the torus) only up to gauge transformations

$$A_M(x, z + \hat{\imath}) = T_i\, A_M(x, z)\, T_i^{-1} \quad , \quad T_i = \exp(2\pi i\, a_i^I H_I). \tag{3.31b}$$

For both transformations the constraints (denoting the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8$ with $\Gamma_{\mathrm{E}_8^2}$, see app. B.3)

$$3v \in \Gamma_{\mathrm{E}_8^2} \quad , \quad 3a \in \Gamma_{\mathrm{E}_8^2}, \tag{3.32}$$

guarantee that U and T_i are compatible with the gauge group lattice.

Analogous relations hold for the gauginos

$$\begin{aligned} \chi(x, \Theta z) &= U\, R \chi(x, z)\, U^{-1} && (3.33\text{a}) \\ \chi(x, z + \ell_i) &= T_i\, \chi(x, z)\, T_i^{-1}, && (3.33\text{b}) \end{aligned}$$

where R is defined according to (3.26).

3.3.2 Untwisted states

Having defined the transformation properties of the gauge multiplet, we want to know which states are present at the four dimensional fixed points $\mathfrak{Z}_s$, defined by (2.32). We know from (2.24) that

$$A_M(x, \Theta\mathfrak{z}_s) = A_M(x, \mathfrak{z}_s - \ell_s), \qquad \ell_s^i = \begin{cases} 0 & s_i = 0 \\ R_i & s_i = 1 \\ (1+\theta)R_i & s_i = 2 \end{cases} . \tag{3.34}$$

A similar relation holds for the gauginos. Using (3.31) and (3.33) this can be rewritten as

$$A_M(x, \Theta\mathfrak{z}_s) = (\Theta^{-1})_M{}^N \, U_s \, A_N(x, z) \, U_s^{-1} \tag{3.35a}$$
$$\chi(x, \Theta z) = U_s \, R\chi(x, z) \, U_s^{-1}, \tag{3.35b}$$

with the fixed point dependent transformations

$$U_s = \prod_{i=1}^{3} (T_i)^{s_i} \, U = \exp\left(2\pi i \, v_s^I H_I\right), \qquad v_s^I = v^I + \sum_i s_i a_i^I . \tag{3.36}$$

Now we can read off the states at each fixed point $\mathfrak{z}_s$, using (3.24), (3.26) and the gauge algebra (B.4). We get the following relations for states corresponding to the roots α of $\Gamma_{\mathrm{E}_8{}^2}$, labeling the adjoint representation of the fixed point gauge group G_s with $\mathbf{Ad}_s$ and the chiral matter states with $\mathbf{R}_s$

$$\textbf{Gauge:} \quad \mathbf{Ad}_s : \quad H^I, (v_s, \alpha) \equiv 0, \tag{3.37a}$$
$$\textbf{Matter:} \quad (\mathbf{3}_H, \mathbf{R}_s) : \quad (v_s, \alpha) \equiv \tfrac{2}{3}, \tag{3.37b}$$

where $\mathbf{3}_H$ is the representation under the holonomy group SU(3). In principle there are also chiral multiplets including gauginos with negative four dimensional chirality and the $A_{\underline{i}}^\alpha$ with $(v_s, \alpha) \equiv \frac{1}{3}$, transforming under $\overline{\mathbf{R}}_s$, but they are not independent.

An important point is that in particular the 16 Cartan elements of $\mathrm{E}_8 \times \mathrm{E}_8$ are present, which means that the rank of the gauge group can not be reduced. The gauge group $G_s \subset \mathrm{E}_8 \times \mathrm{E}_8$ at the fixed point is determined by these Cartan generators and the set of generators E^α that commute with U_s. All gauge multiplets are singlets under $\mathrm{SU}(3)_H$. The possible gauge groups at the fixed points are classified in app. D.2. It is found that there are only five different possibilities, one of which is the trivial case (no gauge group breaking).

The untwisted matter transforms always in a **3** of the holonomy group. It consists of chiral multiplets containing complex scalars $A_i^\alpha, A_{\underline{i}}^\alpha$ and left-handed fermions χ_a^α.

Fixed point classes

Depending on the number of Wilson lines, we obtain a different number of fixed points with the same spectrum. In the examples of chapter 6 we will denote classes of fixed points with $T(s_1, \{s_2\}, \{s_3\})$, depending on the number of Wilson lines used, as shown in table 3.2. For pure shift models all 27 fixed points are of course equivalent.

Wilson lines $\neq 0$	# of classes	# of fixed points per class	notation
a_1	3	9	$T(s_1)$
a_1, a_2	9	3	$T(s_1, s_2)$
a_1, a_2, a_3	27	1	$T(s_1, s_2, s_3)$

Table 3.2: *Notation for fixed points with equivalent spectrum, as used in the examples.*

Four dimensional zero modes

Considering the effective theory by projecting out all massive states, we have to combine the constraints from all 27 fixed points. From this we conclude, that the four dimensional gauge group is given by

$$G = \bigcap_s G_s. \tag{3.38}$$

For the fields we get the conditions $\forall\, i = 1, 2, 3$

$$\textbf{Gauge:} \quad \textbf{Ad}: \quad H^I, (v, \alpha) \equiv (a_i, \alpha) \equiv 0, \tag{3.39a}$$
$$\textbf{Matter:} \quad (\mathbf{3}_H, \mathbf{R}): \quad (v, \alpha) \equiv \tfrac{2}{3},\ (a_i, \alpha) \equiv 0, \tag{3.39b}$$

which are the same conditions that can be found in string theory (4.24). Furthermore we notice, that for a pure shift model (i.e. $a_i = 0\ \forall\, i$) the fixed point spectrum (3.37) at each of the 27 fixed points is identical to the zero mode spectrum.

Chapter 4

Orbifolds in heterotic string theory

In this chapter, which is mainly based on [52, 63, 64, 40], we will give a brief introduction to compactifications of heterotic string theory, focusing on the orbifold case.

Orbifold compactifications have been first used in the context of string theory [18, 19]. The main advantage in contrast to field theory is that the spectra at the orbifold fixed points are determined unambiguously. Therefore we get, in addition to the usual string states and winding modes, so called *twisted states*, which lead to localized fields at the fixed points. These states can not be obtained from the field theory compactification outlined in the previous chapter.

4.1 Heterotic strings

Heterotic string theory was introduced in [32]. It is a hybrid construction of a left-moving 26-dimensional bosonic string and a right-moving 10-dimensional superstring. In the bosonic construction it is described by coordinates[1] (in light-cone gauge)

$$\left(X_L^i(\tau+\sigma), X_L^I(\tau+\sigma)\right), \qquad i=1,\dots,8, \quad I=1,\dots,16 \tag{4.1}$$

for the left-movers and

$$\left(X_R^i(\tau-\sigma), S^a(\tau-\sigma)\right), \qquad i,a=1,\dots,8 \tag{4.2}$$

for the right-movers. The X_L^I represent the coordinates which are compactified on a 16-torus with radii $R^2=\alpha'$. They have to fulfill boundary conditions

$$X^I(\sigma=2\pi) = X^I(\sigma=0), \tag{4.3}$$

and can be expanded in terms of the quantized momenta p^I as

$$X_L^I(\tau+\sigma) = X_0^I + p^I(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0} \tfrac{1}{n}\tilde{\alpha}_n^I e^{-in(\tau+\sigma)}. \tag{4.4}$$

[1] We choose units in which the slope parameter $\alpha' = \frac{1}{2}$

The p^I form a 16-dimensional lattice, which is restricted by modular invariance to be even and self-dual. There exist only two possibilities, namely the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$, leading to two different heterotic string theories.

We will consider the $\mathrm{E}_8 \times \mathrm{E}_8$ case in the following, a description of the E_8 lattice can be found in appendix B.3.1. The mass formula for the left-movers is given by

$$\frac{1}{4}m_L^2 = \frac{1}{2}(p^I)^2 + \tilde{N} - 1, \tag{4.5}$$

where $\tilde{N}$ counts all left-moving oscillators. For the right-movers we have simply

$$\frac{1}{4}m_R^2 = N, \tag{4.6}$$

with the right-moving oscillator number N. The right-moving vacuum is massless and given by a ten-dimensional vector supermultiplet $|i\rangle_R, |a\rangle_R$ with $i, a = 1, \ldots, 8$. The complete mass operator is given by $m^2 = m_L^2 + m_R^2$ and the spectrum is constrained by left-right level matching, required by modular invariance. This means $m_L^2 = m_R^2$, therefore we do not get a tachyonic vacuum state, because (4.6) restricts m^2 to be positive for our choice of the right-moving vacua.

The massless spectrum, containing all states obtained by requiring (4.5) and (4.6) to be zero, is given by tensor products of the left- and right-moving states. It corresponds to ten-dimensional supergravity coupled to $\mathrm{E}_8 \times \mathrm{E}_8$ super Yang-Mills theory and contains the supergravity multiplet, including the graviton, antisymmetric tensor field, dilaton and the corresponding superpartners

$$\left.\begin{array}{c} |i\rangle_R \\ |a\rangle_R \end{array}\right\} \otimes \tilde{\alpha}_L^j |0\rangle_L. \tag{4.7}$$

In addition we have the gauge bosons of $\mathrm{E}_8 \times \mathrm{E}_8$ and the corresponding gauginos. The states corresponding to the Cartan subalgebra (see app. B) and to the roots of $\mathrm{E}_8 \times \mathrm{E}_8$ with $p^2 = 2$, respectively, are given by

$$\left.\begin{array}{c} |i\rangle_R \\ |a\rangle_R \end{array}\right\} \otimes \tilde{\alpha}_L^I |0\rangle_L, \qquad \left.\begin{array}{c} |i\rangle_R \\ |a\rangle_R \end{array}\right\} \otimes |p^I\rangle_L. \tag{4.8}$$

4.2 Compactification to four dimensions

The heterotic string and it's field theory limit, given by the massless modes described above, are ten-dimensional theories. In the last chapter we described how to compactify higher dimensional field theories, to obtain a four dimensional theory, as required by our experience of a four dimensional world. In this section we want to outline how to do the same with heterotic string theory.

4.2.1 Toroidal compactification

The left-moving part of heterotic string theory is already compactified on a 16 dimensional torus. An obvious possibility to construct a four dimensional theory is to go one step further and compactify 6 of the remaining uncompactified coordinates on a torus. The difference is, that we have to compactify left- and right-movers simultaneously [54]. In analogy to (4.4) we get

$$X_L^A(\tau+\sigma) = X_0^A + 2p_L^A(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\tfrac{1}{n}\tilde{\alpha}_n^A e^{-in(\tau+\sigma)}, \tag{4.9a}$$

$$X_R^i(\tau-\sigma) = X_0^i - 2p_R^i(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\tfrac{1}{n}\alpha_n^i e^{-in(\tau-\sigma)}, \tag{4.9b}$$

where $A=(i,I)$, $i=1,\dots,6$ and $I=1,\dots,16$. Modular invariance restricts the quantized momenta p_L^A and p_R^i, just as in the case of the uncompactified heterotic string. They have to span an even, selfdual lattice Γ of Lorentzian signature $(22,6)$. All lattices of this form are isomorphic to $\Gamma_{\mathrm{E}_8{}^2}\oplus 6P_2$, where P_2 is the lattice with signature $(1,1)$ and metric $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Every lattice with the desired properties can be obtained from this combination through a $\mathrm{SO}(22,6)$ transformation.

The mass formula reads for the left- and right-movers, respectively,

$$\frac{1}{4}m_L^2 = \frac{1}{2}(p_L^A)^2 + \tilde{N} - 1, \qquad \frac{1}{4}m_R^2 = \frac{1}{2}(p_R^i)^2 + N. \tag{4.10}$$

This is invariant under $\mathrm{SO}(6)\times\mathrm{SO}(22)$ transformations, therefore different models can be obtained by a variation of $\dim(\mathrm{SO}(22,6)/(\mathrm{SO}(6)\times\mathrm{SO}(22))) = 6\times 22$ parameters, which can be interpreted as background fields [56]. They describe the size and shape of the torus g_{ij}, the presence of an antisymmetric background field B_{ij} and six Wilson lines a_i^I, corresponding to the non-contractible loops of the torus. Using these fields the internal momenta in (4.10) are given by

$$p_L^A = (p_L^i, p_L^I) = (L^i + n^i R_i e_i,\, p^I + a_i^I n^i), \tag{4.11a}$$

$$p_R^i = L^i - n^i R_i e_i, \tag{4.11b}$$

with

$$L^i = \frac{m_i}{2R_i}e^{*i} - B^{ij}n_j + \frac{a_i^I}{2R_i}e^{*i}(p^I - \tfrac{1}{2}a_j^I n^j), \tag{4.12}$$

where the six-torus is defined by the lattice vectors e_i and radii R_i. The e^{*i} form the dual lattice and n, m correspond to winding numbers and Kaluza-Klein momenta.

All of these compactifications lead to four dimensional string models which have massless particles corresponding to $N=4$ supergravity. This follows immediately, because all the massless graviton and gravitino states from the uncompactified theory remain massless after compactification. In the gauge sector we get additional states, because we have at least 22 Cartan generators from $|a,i\rangle_R \otimes \tilde{\alpha}_L^A|0\rangle_L$. Therefore the gauge group has rank 22 and one would naively expect it to be $\mathrm{E}_8 \times \mathrm{E}_8 \times \mathrm{U}(1)^6$. But it turns out, that for special values of the background field B_{ij} the gauge symmetry can be enlarged, because we get additional massless states filling up the adjoint representation of a rank 22 gauge group. We will not go into detail here and assume in the following that this is not the case.

4.2.2 Compactification on orbifolds

It turned out in the last section, that we get $N = 4$ supersymmetry from toroidal compactification. This is too much to be compatible with the standard model in four dimensions. The maximum of supersymmetry is $N = 1$, because the gauge couplings are chiral. If we chose an orbifold instead of the torus, as introduced in [18, 19], we can break $N = 4$ to $N = 1$ supersymmetry. As we have seen studying field theory in the last chapter, unwanted states can be projected out by appropriate discrete symmetries. In the case of string theory we have the possibility to consider two different point groups for the left- and right-movers separately, leading to *asymmetric orbifolds* [55]. We will discuss only *symmetric orbifolds* here, where the point groups for left- and right-movers are the same.

We have to choose an implementation of the space group action, containing lattice shifts and point group transformations on torus coordinates, and of the corresponding actions in the gauge group lattice (see 3.22). We will restrict ourselves, as in the field theory case (see 3.3.1), to shift-embeddings of the twist Θ, but will allow for Wilson lines [42]. Therefore we have a general element $(\Theta, \ell; v^I, a^I)$ acting on the string coordinates. General boundary conditions for the closed string coordinates (split into six "torus"- and sixteen "gauge"-coordinates and using the torus lattice vectors e^i as above) read now

$$\begin{aligned} X^i(\sigma = 2\pi) &= (\Theta X(\sigma = 0))^i + n^j e^i_j, && (4.13a) \\ X^I(\sigma = 2\pi) &= X(\sigma = 0) + v^I + n^i a^I_i. && (4.13b) \end{aligned}$$

We can differentiate between *untwisted* and *twisted* strings, where the former are already closed in $\mathbb{R}^6$ or on the torus T^6 (before twisting), therefore the boundary conditions reduce to

$$X^i(\sigma = 2\pi) = X^i(\sigma = 0) + \ell^i. \tag{4.14}$$

The untwisted modes have expansions according to (4.9), including nontrivial winding vectors ℓ^i for strings which close only on T^6. The mass formula is given by (4.10).

The twisted modes have boundary conditions (4.13a) that do not allow for winding modes or torus momenta, their coordinate expansions are given by

$$X^i(\sigma, \tau) = \mathfrak{z}^i + \frac{i}{2} \sum_m \left(\frac{\alpha^i_{m+1/n}}{m + 1/n} e^{-i(m+1/n)(\tau-\sigma)} + \frac{\tilde{\alpha}^i_{m-1/n}}{m - 1/n} e^{-i(m-1/n)(\tau+\sigma)} \right), \tag{4.15}$$

where the center-of-mass position $\mathfrak{z}$ corresponds to the fixed points of the orbifold and n is the order of the point group in question (see 2.1.2). Therefore we get a number of twisted sectors which corresponds to the number of fixed points of the orbifold. The oscillators α^i and $\tilde{\alpha}^i$ are modded, which leads to fractional oscillator numbers $N, \tilde{N} = 1/n, 1+1/n, 2+1/n, \ldots$.

The gauge coordinates, with boundary conditions (4.13b), have an expansion according to (4.4), where the quantized momenta are given by

$$p^I = w^I + v^I + s^i a_i^I, \qquad w^I \in \Gamma_{\mathrm{E_8}^2}, \tag{4.16}$$

and are therefore shifted due to the orbifolding. This shifting and the fact that we have modded oscillators in (4.15) leads to a shift δc in the zero point energy of the left-movers. If we consider the restrictions of modular invariance, we find that a level matching condition between left- and right-moving states has to be satisfied.

Instead of clinging to the general case, we will discuss the implications of the orbifolding procedure by means of the $\mathbb{Z}_3$ orbifold, which we already dealt with in the last chapter and which will be used for considerations of anomaly cancellation later on.

4.3 The $\mathbb{Z}_3$ orbifold

In this section we will specify to the case of the $\mathbb{Z}_3$ orbifold. This particular case has been studied by various authors, mainly to obtain gauge groups close to the standard model (see e.g. [41, 21], for a classification of models see [46, 45]).

By constructing $T^6/\mathbb{Z}_3$ as outlined in section 2.2, we fix the lattice vectors of the torus and therefore the background g_{ij}, up to a choice of the radii R_i. These can be chosen arbitrarily, as long as we do not select the self-dual radius, which would lead to additional massless states, as explained at the end of 2.2. Furthermore We allow for nontrivial Wilson lines. As mentioned below (4.16) the zero-point energy of the left-movers changes in the case of the twisted sectors. The change can be given in terms of the vector ϕ, defined in (2.31), as

$$\delta c = \frac{1}{2} \sum_i |\phi_i| \left(1 - |\phi_i|\right) = \frac{1}{3}. \tag{4.17}$$

The level-matching condition reads for this twist, which is of order 3,

$$\frac{3}{2}\left[\phi^2 - \left(v + s^i a_i\right)^2\right] \equiv 0. \tag{4.18}$$

It is worth noting here, that this is an additional condition for the choice of shifts and Wilson lines, which we did not find from the field theoretical considerations in the last chapter.

Now we have all necessary information to calculate the massless untwisted and twisted states. In the following we will use the same notation as in the field theory chapter, describing the six torus coordinates by three complex variables and their conjugates. For Minkowski and ten-dimensional indices we will use uppercase Latin and Greek letters from the middle of the alphabet, such that an $10d$ oscillator state is described by $\alpha^M|0\rangle = (\alpha^\mu, \alpha^i, \alpha^{\underline{i}})|0\rangle$.

4.3.1 Untwisted zero modes

The mass formula for the untwisted modes is given by

$$\frac{1}{4}m^2 = \frac{1}{2}(p^I)^2 + \tilde{N} - 1. \tag{4.19}$$

The four-dimensional massless states can be obtained from the torus states, but although the untwisted strings are already closed on T^6, they propagate on $\mathbb{R}^6_{L+R}/S \otimes T^{16}_L/G$. Therefore we get a reduction of the torus spectrum, because we have to project out all states which are not singlets under $S \times G$, which is a consequence of modular invariance. Since all states are products of the form $|\,\rangle_L \otimes |\,\rangle_R$, we have to consider the action of $S \times G$ on these states, where G acts on the left-movers only. The oscillator states α^M transform as a generic vector field according to (3.24), while the left-handed winding states transform as

$$G|p^I\rangle_L = e^{2\pi i\, p^J v_s^J}|p^I\rangle_L, \tag{4.20}$$

at the fixed point $\mathfrak{z}_s$.

Using this we can construct the four-dimensional zero modes as follows[2]

1. The $N = 1$ *supergravity multiplet*, containing left- and right-moving singlets
$$|\mu\rangle_R \otimes \tilde{\alpha}^\nu_L|0\rangle_L, \tag{4.21}$$
2. The *moduli*, invariant states of the form
$$|i\rangle_R \otimes \tilde{\alpha}^{\underline{j}}_L|0\rangle_L \quad \text{and} \quad |\underline{i}\rangle \otimes \tilde{\alpha}^j_L|0\rangle_L, \tag{4.22}$$
3. *Gauge fields*, given by all 16 Cartan generators (which keeps the rank of the Gauge group fixed) and the roots of $E_8 \times E_8$
$$|\mu\rangle_R \otimes \tilde{\alpha}^I_L|0\rangle_L \quad \text{and} \quad |\mu\rangle_R \otimes |p^I\rangle_L. \tag{4.23}$$
Not all roots p^I are possible, because they have to be invariant under the transformation (4.20), which gives the conditions
$$(p, v) \equiv 0 \quad \text{and} \quad (p, a_i) \equiv 0. \tag{4.24}$$
4. *Chiral matter fields*, given by
$$|\underline{i}\rangle_R \otimes |\tilde{p}^I\rangle_L \quad \text{with} \quad (\tilde{p}, v) \equiv \tfrac{2}{3}, \quad (\tilde{p}, a_i) \equiv 0. \tag{4.25}$$
There are also the states $|i\rangle \otimes |\tilde{p}^I\rangle$ with $(\tilde{p}^I, v_s) \equiv \frac{1}{3}$, but these are not independent, as argued in the last chapter.

The conditions for the gauge fields and chiral matter states in (4.24) and (4.25) are identical to the ones we found in the field theory approach (3.39).

[2]We specify the bosonic states only, the fermionic ones are just the superpartners of these.

4.3.2 Twisted zero modes

Due to the shifted zero mode energy (4.17) the mass formula for the twisted modes reads

$$\frac{1}{4}m^2 = \frac{1}{2}(p^I)^2 + \tilde{N} - \frac{2}{3}, \tag{4.26}$$

with fractional oscillator numbers as explained in 4.2.2. The momenta p^I are given according to (4.16), therefore we get two possibilities for zero modes

$$(p^I)^2 = \begin{cases} \frac{4}{3} & \text{for } \tilde{N} = 0 \\ \frac{2}{3} & \text{for } \tilde{N} = \frac{1}{3}. \end{cases} \tag{4.27}$$

The corresponding twisted chiral matter states are localized at the fixed points $\mathfrak{z}_s$, defined in (2.32). The states with $\tilde{N} = \frac{1}{3}$ transform as triplets under the holonomy group, because of the oscillators in the compactified dimensions. For the calculation of anomalies it is convenient to distinguish both cases and we will label singlet and triplet states at a specific fixed point $\mathfrak{z}_s$ by $\mathbf{S}_s$ and $\mathbf{T}_s$, respectively.

Chapter 5

Anomalies

In this chapter the main topic of this thesis, anomaly cancellation, is addressed. In the first two sections we review the origin of chiral anomalies in quantum field theory and describe how to calculate them. After that the situation on orbifolds is analyzed. Similar to the last chapter we will first deal with the (simpler) five dimensional case and proceed afterwards to the $\mathbb{Z}_3$ orbifold.

5.1 Calculation of chiral anomalies

Anomalies occur in quantum field theory, if a symmetry of the classical equations of motion is destroyed by higher order quantum corrections, i.e. loop corrections in perturbation theory[1]. We are interested in *chiral anomalies* which occur in even space-time dimensions, because only in this case it is possible to define chiral (left- and right-handed) fermions. An anomaly of this type has first been found in massless four dimensional QED, in which it corresponds to a triangle diagram describing the coupling between an axial and two vector currents. This is known as the Adler-Bell-Jackiw anomaly [2, 11].

If a theory contains local gauge symmetries the conservation of the currents corresponds to gauge invariance. This is expressed in terms of the Ward identities in the quantized theory. The presence of an anomaly in this case would make the theory inconsistent, because it induces a violation of gauge invariance. Hence a consistent gauge theory has to be free of chiral anomalies.

In the following we will describe how to calculate the four dimensional anomaly using path integrals. Consider a theory of massless left-handed fermions coupled to (non-abelian) gauge fields A_μ and transforming in some representation $\mathbf{R}$. We write the gauge fields contracted with the generators acting on the fermions as $A_\mu = A_\mu^a T_a$. We can write the action in terms of Dirac fermions Ψ, if we add non-interacting right-

[1] For a textbook introduction to the subject see e.g. chapter 19 of [61], chapter 22 of [70] or [13].

handed fermions to the theory. The fermionic functional integral is given by

$$Z[\psi] = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{iS[\Psi]}, \quad \text{with} \quad S = \int \mathrm{d}^4x\, \overline{\psi}(i\not{D})\psi, \tag{5.1}$$

with a Dirac operator defined by

$$\not{D} = \not{\partial} + i\not{A}\, P_+. \tag{5.2}$$

An infinitesimal gauge transformation with generator $\Lambda(x) = \Lambda^a(x)T_a$, where the gauge group generators T_a have to be taken in the representation $\mathbf{R}$, acts on the Dirac fermions as

$$\psi'(x) = (1 - i\Lambda(x))P_+\psi(x) \quad \text{and} \quad \overline{\psi}'(x) = \overline{\psi}(x)P_-(1 + i\Lambda(x)). \tag{5.3}$$

The projection operators $P_\pm = \frac{1}{2}(1 \pm \tilde{\gamma})$ in (5.2) and (5.3) take into account that the right handed fermions do not interact.

The classical action S is invariant under this transformation, but the measure in general is not. It changes according to

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} \to \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(i\int \mathrm{d}^4x\, \mathcal{A}[\Lambda, A^\mu]\right). \tag{5.4}$$

The anomaly $\mathcal{A}[\Lambda, A^\mu]$ is formally defined by

$$\mathcal{A} = -(\mathrm{Tr}_{\{\phi\}}\Lambda - \mathrm{Tr}_{\{\phi^\dagger\}}\Lambda), \tag{5.5}$$

where $\{\phi\}$ and $\{\phi^\dagger\}$ are eigenfunctions of $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$, respectively.

To calculate the anomaly we have to perform a Wick rotation into Euclidean space-time to make the path integral well defined. We can expand ψ and $\overline{\psi}$ in terms of eigenfunctions of $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$ as

$$\psi(x) = \sum_n a_n\phi_n(x) \quad \text{and} \quad \overline{\psi} = \sum_m \phi_m^\dagger(x)\bar{b}_m. \tag{5.6}$$

The Grassmann coefficients a_n and $\bar{b}_m$ can be used to express the path integral measure

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} = \prod_n \mathrm{d}a_n\, \mathrm{d}\bar{b}_n. \tag{5.7}$$

Using the orthonormality of the eigenfunctions and (5.3) we obtain for the transformation of the Grassmann variables under an infinitesimal gauge transformation

$$a_n' = \sum_m C^+_{nm}a_m \quad \text{and} \quad \bar{b}_n' = \sum_m C^-_{nm}\bar{b}_m, \tag{5.8}$$

with the transformation matrices

$$C^\pm_{nm} = \delta_{nm} \pm i\int \mathrm{d}^4x\, \phi^\dagger(x)\Lambda(x)P_\pm\phi(x). \tag{5.9}$$

Because we have a fermionic path integral the measure transforms with the inverse determinant $(\det C^+ \det C^-)^{-1}$. We rewrite this using $\det C = \exp(\mathrm{Tr}\,\ln C)$ and the expansion of the logarithm $\ln(1+x) = x + \mathcal{O}(x^2)$. We obtain

$$(\det C^+ \det C^-)^{-1} = \exp\left\{ i \int \mathrm{d}^4 x\, \mathrm{Tr} \sum_n \left(\phi_n^\dagger(x) \Lambda(x) \tilde{\gamma} \phi_n(x) \right) \right\}, \tag{5.10}$$

where the trace has to be taken over spinor and gauge representations. The anomaly defined by (5.5) is therefore given by the trace, but the exponent in (5.10) is not well defined. The completeness of the eigenfunctions introduces a factor $\delta(0)$, which is infinite. To regularize the sum, we follow the approach of Fujikawa [23, 24]. The idea is to introduce a Gaussian cut-off $\exp(-\not{D}^2/M^2)$ and take the limit $M \to \infty$ after the evaluation of the expression. Together with a Fourier transformation we obtain for the anomaly

$$\mathcal{A}[\Lambda, A^\mu] = \lim_{M\to\infty} \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \mathrm{Tr}\, \left(e^{-ikx} \Lambda(x) \tilde{\gamma} e^{-\not{D}^2/M^2} e^{ikx} \right). \tag{5.11}$$

We need at least four gamma matrices from the expansion of the exponential to get a non-vanishing spinor trace, which means also a factor of M^{-4}. To compensate for this we rescale the momenta $k \to Mk$, which exactly gives a factor M^4. So this is the only term that survives the limit $M \to \infty$. Using this we finally find for the anomaly (a detailed evaluation can be found for example in [70])

$$\mathcal{A}[\Lambda, A^\mu] = -\frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \Lambda^a F^b_{\mu\nu} F^c_{\rho\sigma}\, \mathrm{Tr}\, (T_a \{T_b, T_c\}). \tag{5.12}$$

5.2 Anomalies of nonabelian gauge groups

Considering four dimensional theories containing nonabelian gauge groups we have to make sure that the anomaly given in (5.12) vanishes. It is proportional to the totally symmetric quantity

$$d_{abc} := \mathrm{Tr}\, \left(T_a \{T_b, T_c\} \right), \tag{5.13}$$

where the trace implies a sum over all left-handed fermion fields. The generators T_i have to be taken in the representation of the corresponding fields. To have a consistent (anomaly free) gauge theory, we must have

$$d_{abc} = 0, \tag{5.14}$$

which is fulfilled identically for the so called *safe groups*, which include all simple Lie groups, except SU(N) with $N \geq 3$ and U(1) factors [25]. This result can be obtained by examination of the corresponding Lie algebras. It is easy to see for algebras which have real or pseudoreal representations only, because in that case we have (the generators T_a are taken to be hermitian)

$$T_a^T = T_a^* = -S T_a S^{-1}, \tag{5.15}$$

which gives $d_{abc} = -d_{abc}$ when inserted into (5.13), using the cyclic property of the trace.

If the gauge group is the direct product of simple algebras, one has to consider the combinations of generators for which the product of T_a, T_b and T_c is neutral under the complete group, otherwise d_{abc} vanishes. For example in the standard model, with gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, one has to consider the combinations $\mathrm{SU}(2)^3$, $\mathrm{SU}(3)^3$, $\mathrm{SU}(2)^2\mathrm{U}(1)$, $\mathrm{SU}(3)^2\mathrm{U}(1)$ and $\mathrm{U}(1)^3$, because the **3** of SU(2) and the **8** of SU(3) do not transform as singlets.

To calculate the contributions of various representations, it is convenient that the traces over a product of generators in some specific representation of a simple Lie algebra can be related to the trace over some reference representation (usually one chooses the fundamental representation) by means of *indices*. Relevant for the discussion of anomalies in four dimensions are the second and third order indices of representations of SU(N) groups, which are discussed in appendix C.

5.3 Anomalies on $S^1/\mathbb{Z}_2$

In this section we want to describe how to deal with anomalies on orbifolds. The five dimensional case considered here has been covered in recent publications [4, 67, 62, 8]. Therefore we want to indicate briefly how the anomaly calculation of section 5.1 can be performed on this orbifold.

Considering five dimensional theories no chiral anomalies should occur, because chirality is not defined. Applying (5.5) to a theory on the circle, where we have $D = D^\dagger$, we get zero. For orbifold theories this argument does not hold, because chiral projections are possible at the fixed points. In the $S^1/\mathbb{Z}_2$ case we have two four dimensional fixed points and, as we have seen in chapter 3, the fermions transform chirally under the reflection $y \to -y$,

$$\psi(-y) = \gamma_4 \psi(y) \quad \text{and} \quad \overline{\psi}(-y) = -\overline{\psi}(y)\gamma_4. \tag{5.16}$$

To calculate the anomaly we have to restrict the trace to states left invariant under this reflection. Therefore we have to insert a projection operator which is here given by $\frac{1}{2}(1 \pm \gamma_4)$ for ψ and $\overline{\psi}$, respectively. Because the eigenfunctions are equivalent we get from (5.5)

$$\mathcal{A}^{(5)}[\Lambda] = -\mathrm{Tr}_{\{\phi\}} \Lambda \gamma_4. \tag{5.17}$$

Using the expansion (3.8) for the five dimensional fields and the orthogonality and completeness relations we can rewrite (5.17) in terms of the four dimensional modes present at the fixed points 0 and πR. We can use the regularization for the trace as in the previous section. Taking into account that $\gamma_4 = \tilde{\gamma}$, which is the chirality operator in four dimensions, we get[2]

$$\mathcal{A}^{(5)}[\Lambda] = \frac{1}{2}\left(\delta(y) + \delta(y - \pi R)\right)\mathcal{A}^{(4)}[\Lambda]. \tag{5.18}$$

This result seems plausible, because the two fixed points are indistinguishable, thus the anomaly should be distributed equally between them.

5.4 The $\mathbb{Z}_3$ orbifold

Now we want to analyze the anomaly structure of the Z orbifold. This is a generalization of the results for five dimensional orbifolds to higher dimensions[3], applied to $T^6/\mathbb{Z}_3$. The models under consideration here are derived from a consistent string theory, therefore the effective quantum field theory limit is globally anomaly free[4]. We know from string theory, that the possible anomalies are canceled by the Green-Schwarz mechanism [29]. As in the five-dimensional case, the question of local anomaly cancellation arises. Are there local anomalies at the fixed points, although the theory is globally (i.e. at the zero-mode level) anomaly free, and if yes, how are they canceled?

We will outline in the following how local anomalies on $T^6/\mathbb{Z}_3$ can be described in terms of fixed point equivalent models, providing an alternative description of the anomaly contribution of states present at each class of fixed points. The global anomaly contribution from the untwisted sector will be given by the sum of all the local anomalies. In addition we will get anomalies from the twisted states, which are localized at the fixed points.

5.4.1 Untwisted and twisted anomalies

To calculate the gauge anomalies in the untwisted sector, i.e. from the gaugino field, we can in principle use the same method as in the five dimensional case, inserting a projection operator into (5.5), that restricts the fermions to be consistent with the orbifold conditions (3.33). Special care has to be taken, because there might be the possibility that we get not only a sum of four dimensional anomalies at the fixed points, but also higher dimensional contributions. We will not go into the details of the calculation, it can be found in [26]. The result contains an intrinsic ten-dimensional contribution which is canceled by the Green-Schwarz mechanism and a contribution from the states at the various fixed points. We will not discuss the first part here, since we are interested in the localized anomalies only. The result is given by

$$\mathcal{A}^{(10)}_{\mathrm{UT}} = \frac{1}{27}\sum_s 3\mathcal{A}^{(4)}_s(\mathbf{R}_s)\delta(z - \xi_s - \Gamma), \tag{5.19}$$

[2] For a complete derivation of this result see [8]. In [30] the same result was obtained using topological arguments.

[3] Some work in this direction can be found in [36, 7].

[4] Speaking of "global" and "local" anomalies in the context of orbifolds, we want to distinguish between the anomalies at the zero-mode level (global) and those at the four dimensional fixed points (local). In both cases we are considering anomalies of the local gauge symmetry.

where $\mathcal{A}_s^{(4)}(\mathbf{R}_s)$ is the usual four dimensional gauge anomaly. It can be obtained from (5.12), including all states of $\mathbf{R}_s$ transforming under the local gauge group G_s.

From (5.19) we conclude, that the relevant part of the anomaly is given entirely in terms of the four dimensional anomalies at the 27 fixed points. These are generated by the local gaugino states present there, which are determined by (3.37b). The factor 3 takes into account, that the states transform as a triplet under the holonomy group and have therefore to be included with a multiplicity of 3. We have to be careful about the field content of the fixed point models, because the sum runs over all states at the fixed points, not only the zero modes.

The anomaly in the twisted sector is much simpler to calculate, because these are already four dimensional states, localized at the fixed points (see section 4.2.2). In the field theoretical approach we just put these states at the fixed points "by hand", using the results from string theory. The "twisted" anomaly is given by

$$\mathcal{A}_{\mathrm{T}}^{(10)} = \sum_s \mathcal{A}(\mathbf{S}_s, 3\mathbf{T}_s)\delta(z - \xi_s - \Gamma), \tag{5.20}$$

labeling with $\mathbf{S}_s, \mathbf{T}_s$ the holonomy group singlets and triplets at the fixed point $\mathfrak{z}_s$, as given in (4.27).

The result for the four dimensional zero modes is obtained by taking only states present at all fixed points simultaneously. For the untwisted anomaly we get therefore

$$\mathcal{A}_{\mathrm{UT0}}^{(4)} = 3\mathcal{A}(\mathbf{R}), \tag{5.21}$$

where $\mathbf{R}$ is the representation of the untwisted zero modes as given in (3.39b). The twisted anomaly is already located at the fixed points, so it is not changed in the zero-mode limit.

5.4.2 Local anomaly cancelation

With the results from the last section we can consider the problem of local anomaly cancellation. For every given model, using the spectrum at a given fixed point, we are able to calculate the anomaly, but in order to make a general statement about all possible models including Wilson lines, the concept of *fixed point equivalent models*[5] is very useful. Consider the local anomaly contribution from a model with Wilson lines at one fixed point $\mathfrak{z}_s$. Using (5.19) and (5.20), it is given by

$$\mathcal{A}_s = \frac{1}{27}\mathcal{A}(\mathbf{R}_s) + \mathcal{A}(\mathbf{T}_s, \mathbf{S}_s). \tag{5.22}$$

This can be compared with the zero mode spectrum of a pure shift model with shift $v' = v_s = v + s^i a_i$,

$$\mathcal{A}_0 = \mathcal{A}(\mathbf{R}) + \sum_s \mathcal{A}(\mathbf{T}_s, \mathbf{S}_s). \tag{5.23}$$

[5]This concept has already been introduced in [51], but for the twisted states only. In [26] it has been extended for general anomaly calculations.

The zero-mode matter $\mathbf{R}$ of (5.23) is determined by $(v', \alpha) \equiv \frac{2}{3}$ (see 3.39b) and is thus identical to the representation $\mathbf{R}_s$ of (5.22), that is given by $(v_s, \alpha) \equiv \frac{2}{3}$. The twisted sectors of the shift-only model are all identical with matter content equal to $\mathbf{T}_s$ and $\mathbf{S}_s$ at the fixed point $\mathfrak{z}_s$. Thus we conclude that the localized anomaly at fixed point $\mathfrak{z}_s$ is the same as 1/27 of the zero-mode anomaly of a "fixed point equivalent" model with $v = v_s$.

The question of local anomaly cancellation reduces therefore to the question of zero mode anomaly cancellation for the equivalent shift-only models. There are only five of them (or eight, if one includes the permutation $\mathrm{E}_8 \leftrightarrow \mathrm{E}_8{}'$) and their spectrum is given in appendix D.2.

Nonabelian anomalies

It can be checked explicitly that all nonabelian zero-mode anomalies cancel in the pure shift models. Using the concept of fixed point equivalent models, we can conclude from this observation that the localized nonabelian anomalies cancel in every possible model of $T^6/\mathbb{Z}_3$ including Wilson lines.

Abelian anomalies

Applying (5.20) and (5.21) to the mixed gravitational U(1) anomalies given by Tr U(1), we get nonzero results only for two models, E_7 and $\mathrm{SU}(9)$[6]. We know from string theory, that the arising global anomalies are canceled by the Green-Schwarz mechanism. So far it is an open question, whether the antisymmetric tensor field (responsible for the anomaly cancellation) can also be used to cancel the localized anomalies. Further investigations of this point would be very interesting.

Mixed anomalies

Before showing some explicit examples in the next chapter, we want to justify why we are caring about the mixed gravitational anomalies only in the abelian case. In four dimensions there are two more gauge anomalies involving U(1)s, pure U(1) and mixed nonabelian anomalies, given by $\sim \mathrm{Tr}\, Q^3$ and $\sim \sum_i \mathrm{Tr}\, QG_i^2$, respectively (where G_i is one of the nonabelian gauge groups at one fixed point). It has been shown [66, 51], that these three types of anomalies are not independent, but there is the following relation between them, fulfilled for every U(1) generator Q and local gauge group G_i.

$$\frac{1}{24}\mathrm{Tr}\, Q = \frac{1}{k_i} \sum_{\Phi(G_i)} D_2(\Phi) \mathrm{Tr}\,{}_\Phi Q = \frac{1}{3k_q}\mathrm{Tr}\, Q^3, \tag{5.24}$$

[6] The calculation is done explicitly in appendix D.2.

where

$$\begin{array}{rcl} \Phi & : & \text{Representations of } G_i \text{ at the fixed point,} \\ Q, q & : & \mathrm{U}(1) \text{ generator and corresponding dual vector,} \\ k_q & : & \text{Level of } Q, \text{ given by } k_q = 2q^2, \\ k_i & : & \text{Kaz-Moody level of } \Phi, \\ D_2(\Phi) & : & \text{Second order index of } \Phi(G_i). \end{array}$$

This can be explicitly checked for the two shift models (E_7 and SU(9)) containing U(1) generators, as shown in table 5.1. Therefore, by means of the equivalent models, this is valid for all models including Wilson lines.

	$G = \mathrm{SU}(9)$	$G = \mathrm{E}_7$
U(1) vectors	$q = (0^8)(1, 0^8)$	$q_1 = (1, 1, 0^6)(0^8)$ $q_2 = (0^8)(1, 0^7)$
levels k_q	2	$(4, 2)$
$\mathrm{Tr}\, Q$	$3(18) + 27(6) = 216$	$3(54, 18) + 27(10, -2) = (432, 0)$
$\mathrm{Tr}\, Q^3$	$3(-6) + 27(\frac{8}{3}) = 54$	$3(48, -6) + 27(\frac{8}{3}, \frac{2}{3}) = (216, 0)$
$\mathrm{Tr}_{\,G} D_2 Q$	$27\, D_2(\overline{\mathbf{9}})(\frac{2}{3}) = 9$	$3D_2(\mathbf{56})(1, 0) = (18, 0)$
$\mathrm{Tr}_{\,\mathrm{SO}(14)} D_2 Q$	$3\left(D_2(\mathbf{64})\frac{1}{2} + D_2(\mathbf{14})(-1)\right) = 9$	$3\left(D_2(\mathbf{64})(0, \frac{1}{2}) + D_2(\mathbf{14})(0, -1)\right)$ $+27\, D_2(\mathbf{14})(\frac{2}{3}, -\frac{1}{3}) = (18, 0)$

Table 5.1: *Confirmation of (5.24) for the two shift models containing* U(1)*s. Both models have the gauge group* $\mathrm{SO}(14) \times \mathrm{U}(1)$ *in the hidden sector. In the case of* E_7 *the values for both* U(1) *generators are given simultaneously as* $(1, 2)$. *For the computation of the quadratic indices see appendix C.*

Chapter 6

Explicit examples

In this chapter four $\mathbb{Z}_3$ orbifold models are described in detail[1]. In the first section three models with one Wilson line are investigated to show the different issues of anomaly cancellation for both nonabelian and abelian anomalies. In the second part a more involved model with two Wilson lines is discussed, which was first constructed in [48] and used to point out the problem of nonabelian anomalies in [27].

The following notation is used: Representations belonging to the two different E_8 sectors (sometimes called "visible" and "hidden" sector) are distinguished by a prime for the second (hidden) sector. Of course it is always possible to transform every model to a new one, in which the visible and hidden sectors are exchanged. U(1) factors are commonly denoted as subscripts to the representations of the nonabelian groups.

6.1 Examples with one Wilson line

In the following three orbifold models with one Wilson line are described, which illustrate the different aspects of local cancellation of abelian and nonabelian anomalies. In a first example we want to analyze the nonabelian structure only, because it is somewhat simpler and (due to the fact that everything can be reduced to the shift-only models) both globally and locally anomaly free.

In principle there exist four types of models relevant to the discussion of nonabelian anomalies:

1. Models with safe groups only, so trivially no nonabelian anomaly occurs. Safe groups are all simple Lie groups except SU(N) with $N \geq 3$ and U(1) (see 5.2).
2. Models where all anomalous contributions cancel already on the zero mode level, locally at each fixed point and in the bulk.
3. Models where the bulk is anomaly free and the zero mode fixed point anomalies

[1] All models have been calculated using a program written by the author, see appendix D.1.

cancel each other. In this case the power of the concept of fixed point equivalent models can be demonstrated and such a model will be presented in section 6.1.1.

4. Models where the bulk anomaly is canceled by contributions from the fixed points. These models usually show a complicated embedding structure, an example with this behaviour is considered in the appendix D.3.1.

The case of abelian anomalies, which will be considered in the subsections 6.1.2 and 6.1.3, is more difficult. As it was argued in the last chapter local anomaly cancellation is not guaranteed. In the two examples presented here we will consider the implications that arise[2].

6.1.1 Nonabelian anomalies in the twisted sectors

This model has the feature that the gauge group contains no $U(1)$ factors. Therefore only nonabelian anomalies can occur and in this case they are present in the twisted sectors only. In the untwisted sector zero modes do not appear, which is achieved by using the same vector for the shift and one Wilson line.

Shift and Wilson lines

$$\begin{aligned} v &= \tfrac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0 \,|\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,)\,, \\ a_1 &= \tfrac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0 \,|\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \end{aligned}$$

Zero mode gauge group

$$\mathrm{SU}(3) \times \mathrm{E}_6 \times \mathrm{E}_8{}'$$

The gauge group is the common result for the standard embedding, because the Wilson line, being equal to the shift, gives no additional restriction on the gauge group generators.

Zero mode matter content[3]

$$\begin{array}{lcl} \textbf{UT}\ \textit{(3)} & : & \emptyset \\ \textbf{T(0)}\ \textit{(9)} & : & (\mathbf{1},\mathbf{27})(1)' + 3\cdot(\mathbf{3},\mathbf{1})(\mathbf{1})' \\ \textbf{T(1)}\ \textit{(9)} & : & (\mathbf{1},\overline{\mathbf{27}})(1)' + 3\cdot(\overline{\mathbf{3}},\mathbf{1})(\mathbf{1})' \\ \textbf{T(2)}\ \textit{(9)} & : & 3\cdot(\mathbf{1},\mathbf{1})(\mathbf{1})' \end{array} \qquad (6.2)$$

[2] In the appendix (see D.3.2) we present an additional model with an abelian anomaly.

[3] In this (and the following models) we list the global zero mode matter content at each fixed point labeled by UT for "UnTwisted sector" and $T(i)$ for the i-th twisted sector, as explained in tabular 3.2. The multiplicities of the different sectors are given in parenthesis.

Anomaly cancellation and equivalent models

Nonabelian anomalies could only be generated by nontrivial representations of SU(3), because E_6 and E_8 are anomaly free. If one looks at the matter content of this model at the zero mode level (see 6.2), where the representations at the fixed points are classified in terms of the global gauge group, the $SU(3)$ anomalies seem to cancel only globally after integrating over the extra dimensions. Then the anomalies of T(0) and T(1), which have opposite signs, cancel each other.

If the massive modes are taken into account, which can be done by looking at the induced models at each fixed point (tab. 6.1 and fig. 6.1.1), local anomaly cancellation is achieved.

Fixed point	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
T(0)	$E_6 \times SU(3)$ $\times E_8{}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(\mathbf{1})'$ $+3\cdot(\mathbf{1},\mathbf{3})(\mathbf{1})'$
T(1)	$E_6 \times SU(3)$ $\times E_8{}'$	$(\overline{\mathbf{27}},\mathbf{3})(1)'$	$(\overline{\mathbf{27}},\mathbf{1})(\mathbf{1})'$ $+3\cdot(\mathbf{1},\overline{\mathbf{3}})(\mathbf{1})'$
T(2)	$E_8 \times E_8{}'$	nothing	$3\cdot(\mathbf{1})(\mathbf{1})'$

Table 6.1: *Induced models at the fixed points.*

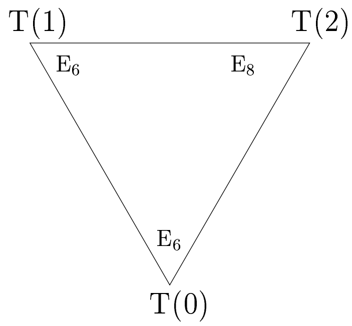


Figure 6.1: *Equivalent models of 6.1.1. Each edge of the triangle represents one of the three different classes of fixed points. The corresponding equivalent models are described in the appendix (see D.2).*

At T(0) and T(1) we get the standard embedding where the anomaly cancellation is well known, while at T(2) the original gauge group $E_8 \times E_8{}'$ remains unbroken.

This model shows, having a simple setup of shift and Wilson lines, how the massive modes contribute to the nonabelian anomaly. In this case the cancellation works only with the massive modes of the untwisted sector, because no zero modes are present.

6.1.2 Symmetric distribution of local U(1) anomalies

After the first simple example of nonabelian anomaly cancellation we want to include the abelian case now. The situation is somewhat more difficult, because global anomaly cancellation is not guaranteed for the abelian anomaly. This follows directly from the fact that two of the shift-only models contain a global U(1) anomaly (see appendix D.2).

Adding Wilson lines opens up three possibilities. If the model without Wilson lines is already anomalous, the addition of Wilson lines could alter the anomaly, but never remove it. If the shift-only model is anomaly free, the Wilson line can nevertheless induce an anomalous U(1). This possibility will be discussed in the next model (6.1.3). In the following we will analyze a model that has local U(1) anomalies in two twisted sectors which exactly cancel each other.

Shift and Wilson lines

$$\begin{aligned} v &= \tfrac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0 \mid\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,)\,, \\ a_1 &= \tfrac{1}{3}(\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0 \mid\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \end{aligned}$$

Zero mode gauge group

$$\mathrm{SU}(3) \times \mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$$

The gauge group of the visible sector follows from a sequence of reductions induced by the shift and Wilson line. One can think of the process split up in two gauge breakings, treating the shift and Wilson line action in an independent way. In a first step (due to the shift) E_8 is broken to $\mathrm{SU}(3) \times \mathrm{E}_6$ and in a second step E_6 is reduced to $\mathrm{SU}(6) \times \mathrm{U}(1)$. The hidden sector gauge group is just the standard SO(14) as in the shift only models E_7 or SU(9).

Zero mode matter content

$$\begin{array}{lcl} \mathbf{UT}\ \mathit{(3)} & : & (\overline{\mathbf{3}},\overline{\mathbf{15}})_0(\mathbf{1})'_0 \\ \mathbf{T(0)}\ \mathit{(9)} & : & (\mathbf{1},\overline{\mathbf{15}})_0(1)'_0 + (\mathbf{1},\mathbf{6})_{-1}(\mathbf{1})'_0 + (\mathbf{1},\mathbf{6})_1(\mathbf{1})'_0 + 3\cdot(\mathbf{3},\mathbf{1})_0(\mathbf{1})'_0 \\ \mathbf{T(1)}\ \mathit{(9)} & : & (\mathbf{1},\mathbf{6})_{-1/3}(\mathbf{1})'_{2/3} + (\mathbf{3},\mathbf{1})_{2/3}(\mathbf{1})'_{2/3} \\ \mathbf{T(2)}\ \mathit{(9)} & : & (\mathbf{1},\mathbf{6})_{1/3}(\mathbf{1})'_{-2/3} + (\mathbf{3},\mathbf{1})_{-2/3}(\mathbf{1})'_{-2/3} \end{array} \tag{6.4}$$

Anomaly cancellation and equivalent models

The nonabelian anomalies cancel locally and globally in the usual way, for remarks on the anomaly coefficients see the corresponding subsection of the model given in appendix D.3.2.

We will focus on the abelian anomaly. We have two U(1) generators in the global gauge group, but looking at the zero mode matter content we see that the U(1) charges in the visible sector cancel locally between all representations and only T(1) and T(2) give a nonvanishing contribution to the trace of the hidden sector generator,

$$\mathrm{Tr}_{\,T(1)}\mathrm{U}(1) = -\mathrm{Tr}_{\,T(2)}\mathrm{U}(1) = 54. \tag{6.5}$$

Fixed point	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
T(0)	$\mathrm{E}_6 \times \mathrm{SU}(3)$ $\times \mathrm{E}_8{}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(\mathbf{1})'$ $+3\cdot(\mathbf{1},\mathbf{3})(\mathbf{1})'$
T(1)	$\mathrm{SU}(9)$ $\times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(\overline{\mathbf{84}})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_{1/2}$ $+(\mathbf{1})(\mathbf{14})'_{-1}$	$(\mathbf{9})(\mathbf{1})'_{2/3}$
T(2)	$\mathrm{SU}(9)$ $\times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(\overline{\mathbf{84}})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_{-1/2}$ $+(\mathbf{1})(\mathbf{14})'_1$	$(\mathbf{9})(\mathbf{1})'_{-2/3}$

Table 6.2: *Induced models at the fixed points.*

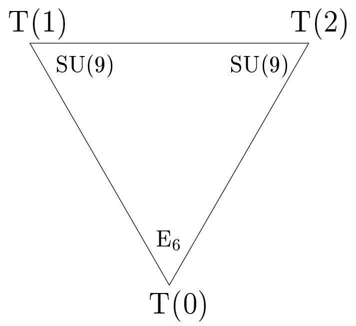


Figure 6.2: *Equivalent models of 6.1.2.*

To understand this locally, we take the equivalent models at each fixed point (tab. 6.2, fig. 6.1.2) into account. Both fixed points have the structure of the SU(9) shift model and contain therefore an anomalous U(1). The U(1) generators are equal (tab. 6.3), but their contributions to the global U(1) differ by a sign, so the whole model is anomaly free, although we have anomalous U(1)s at two fixed point classes.

6.1.3 Generation of a global U(1) anomaly with a Wilson line

The following model has the interesting property that the shift, which corresponds to the $\mathrm{E}_6{}^2$ standard shift of table D.1, leads to a U(1) anomaly free model, but a global anomaly is generated by the introduction of a Wilson line.

T(1)	**T(2)**
Simple Roots (E8' sector)	Simple Roots (E8' sector)
$\alpha'_1 = \left(0^8 \,\|\|\, 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$	$\beta'_1 = \left(0^8 \,\|\|\, 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$
$\alpha'_2 = \left(0^8 \,\|\|\, 0,\ 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0\right)$	$\beta'_2 = \left(0^8 \,\|\|\, 0,\ 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0\right)$
$\alpha'_3 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0,\ 0\right)$	$\beta'_3 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0,\ 0\right)$
$\alpha'_4 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0\right)$	$\beta'_4 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0\right)$
$\alpha'_5 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0\right)$	$\beta'_5 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0\right)$
$\alpha'_6 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1\right)$	$\beta'_6 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1\right)$
$\alpha'_7 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ \text{-}1\right)$	$\beta'_7 = \left(0^8 \,\|\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ \text{-}1\right)$
U(1) Charges (E8' sector)	U(1) Charges (E8' sector)
$q'_1 = \left(0^8 \,\|\|\, 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right)$	$p'_1 = \left(0^8 \,\|\|\, 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right)$

Table 6.3: *Embedding structure of 6.1.2 for the fixed points T(1) and T(2). The simple roots of the hidden sector (marked with primes) and the* U(1) *generators (orthogonal to the roots) are given. It can be seen that the embedding into the* E_8 *lattice is completely equal at the two fixed points and therefore the* U(1) *contributions cancel each other exactly.*

Shift and Wilson lines

$$\begin{aligned} v &= \tfrac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0 \mid 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0\ )\,, \\ a_1 &= \tfrac{1}{3}(\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0 \mid 0,\ 0,\ 0,\text{-}2,\ 0,\ 0,\ 0,\ 0\ ) \end{aligned}$$

Zero mode gauge group

$$\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SO}(8)' \times \mathrm{SU}(3)' \times \mathrm{U}(1)'^2$$

As in the previous example we can think of the gauge group breaking as a two-step operation. The SU(3) subgroups are generated by the shift breaking, the SU(6), SO(8) and U(1) groups are produced by the Wilson line from the E_6 subgroups:

$$\begin{aligned} \mathrm{E}_6 &\rightarrow \mathrm{SU}(6) \times \mathrm{SU}(2) \rightarrow \mathrm{SU}(6) \times \mathrm{U}(1) \\ \mathrm{E}_6{}' &\rightarrow \mathrm{SO}(9)' \times \mathrm{U}(1)'^2 \rightarrow \mathrm{SO}(8)' \times \mathrm{U}(1)'^2 \end{aligned}$$

Zero mode matter content

$$\begin{aligned} \mathbf{UT}\ \mathit{(3)} &: (\overline{\mathbf{15}},\overline{\mathbf{3}})_0(\mathbf{1},\mathbf{1})'_{0,0} + (\mathbf{1},\mathbf{1})_0(\mathbf{8},\overline{\mathbf{3}})'_{1,0} + (\mathbf{1},\mathbf{1})_0(\mathbf{1},\overline{\mathbf{3}})'_{-2,0} \\ \mathbf{T(0)}\ \mathit{(9)} &: (\mathbf{1},\mathbf{3})_0(\mathbf{1},\mathbf{3})'_{0,0} \\ \mathbf{T(1)}\ \mathit{(9)} &: (\mathbf{6},\mathbf{1})_{-1/3}(\mathbf{1},\mathbf{1})'_{1,1/3} + (\mathbf{1},\mathbf{3})_{2/3}(\mathbf{1},\mathbf{1})'_{1,1/3} \\ \mathbf{T(2)}\ \mathit{(9)} &: (\mathbf{6},\mathbf{1})_{1/3}(\mathbf{1},\mathbf{1})'_{1,-1/3} + (\mathbf{1},\mathbf{3})_{-2/3}(\mathbf{1},\mathbf{1})'_{1,-1/3} \end{aligned} \tag{6.7}$$

Anomaly cancellation and equivalent models

The contributions to nonabelian anomalies cancel in the usual manner (see D.3.2).

More interesting is the case of the U(1) anomalies. The visible sector U(1) cancels trivially at each fixed point and is not present in the bulk. The second of the two hidden sector U(1)s is also only present at the fixed points and cancels between them (this can be understood in terms of the equivalent models, see below). But the first hidden sector U(1) creates a global anomaly with

$$\mathrm{Tr}\,\mathrm{U}(1) = 216, \tag{6.8}$$

although the shift does not produce an anomalous model and the Wilson line is orthogonal to the shift vector. This seems to contradict a statement made in [51], where it is claimed that no global abelian anomalies should exist under these conditions[4].

Fixed point	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
T(0)	$\mathrm{E}_6 \times \mathrm{SU}(3)$ $\times \mathrm{E}_6{}' \times \mathrm{SU}(3)'$	$(\mathbf{27},\overline{\mathbf{3}})(\mathbf{1},\mathbf{1})'$ $+(\mathbf{1},\mathbf{1})(\mathbf{27},\overline{\mathbf{3}})'$	$(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{3})'$
T(1)	$\mathrm{SU}(9)$ $\times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(\overline{\mathbf{84}})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_1$ $+(\mathbf{1})(\mathbf{14})'_{-2}$	$(\mathbf{9})(\mathbf{1})'_{4/3}$
T(2)	$\mathrm{SU}(9)$ $\times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(\overline{\mathbf{84}})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_{-1}$ $+(\mathbf{1})(\mathbf{14})'_2$	$(\mathbf{9})(\mathbf{1})'_{-4/3}$

Table 6.4: *Equivalent models at the fixed points of model 6.1.3. The absence of* U(1) *anomalies at T(0) gets trivial because of the absence of* U(1) *generators at this class of fixed points. The* U(1) *in the* E_8 *sector cancels, because the contributions of T(1) and T(2) have an opposite sign. This is not true for the* U(1) *in the* $\mathrm{E}_8{}'$ *sector (see text).*

If we look at the equivalent models at the fixed points T(1) and T(2) (tab. 6.4, fig. 6.1.3), we see that they correspond to the shift-only model SU(9) and contain an anomalous U(1). At first sight it seems likely that the two anomalies exactly cancel, because they have opposite signs. The fact that the anomaly does not cancel between T(1) and T(2) can be traced back to the different embeddings of SO(14) into E_8. As a consequence different U(1) generators are present at the two classes of fixed points. To see this we have to construct the model explicitly, find the roots of the local gauge groups and the U(1) generators. The relevant results are collected in table 6.5 and it turns out, that the generator of the global U(1) anomaly is a linear combination

[4]This point has been clarified with the author [50], the statement made in [51] is only valid if the Wilson lines do not induce additional abelian anomalies in the direction of the shift vector.

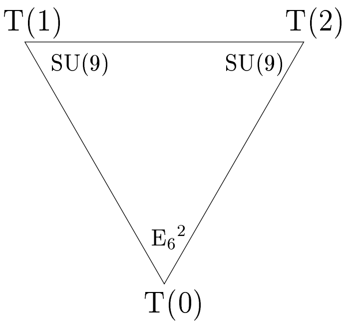


Figure 6.3: *Equivalent models of 6.1.3.*

of the two local generators. The second U(1) in the $E_8{}'$ sector is just the orthogonal combination and therefore anomaly free.

T(1)	**T(2)**
Simple Roots (E8' sector)	Simple Roots (E8' sector)
$\alpha'_1 = \left(0^8 \,\|\, 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$	$\beta'_1 = \left(0^8 \,\|\, 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$
$\alpha'_2 = \left(0^8 \,\|\, 0,\ 0,\ 1,\text{-}1,\ 0,\ 0,\ 0,\ 0\right)$	$\beta'_2 = \left(0^8 \,\|\, 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0\right)$
$\alpha'_3 = \left(0^8 \,\|\, \frac{1}{2},\frac{-1}{2},\frac{-1}{2},\ \frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right)$	$\beta'_3 = \left(0^8 \,\|\, \frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\ \frac{1}{2}\right)$
$\alpha'_4 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1\right)$	$\beta'_4 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ \text{-}1\right)$
$\alpha'_5 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0\right)$	$\beta'_5 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0\right)$
$\alpha'_6 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0\right)$	$\beta'_6 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1\right)$
$\alpha'_7 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ \text{-}1\right)$	$\beta'_7 = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 0,\ 1,\text{-}1,\ 0,\ 0\right)$
U(1) Charges (E8' sector)	U(1) Charges (E8' sector)
$q'_1 = \left(0^8 \,\|\, 1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0\right)$	$p'_1 = \left(0^8 \,\|\text{-}1,\text{-}1,\text{-}1,\ 1,\ 0,\ 0,\ 0,\ 0\right)$
Combined generators of anomalous (Q'_1) and non anomalous (Q'_2) U(1)s	
$Q'_1 = \frac{1}{2}\left(q'_1 - p'_1\right) = \left(0^8 \,\|\, 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$	
$Q'_2 = \frac{1}{2}\left(q'_1 + p'_1\right) = \left(0^8 \,\|\, 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0\right)$	

Table 6.5: *Embedding structure of 6.1.3. The simple roots of the hidden sector (marked with primes) and the* U(1) *generators (orthogonal to the roots) are given. It can be seen that the embedding into the* E_8 *lattice is different at the two fixed points and a linear combination of the* U(1) *generators gives rise to a global anomaly.*

6.2 The Kim^2 model

This model, introduced in [48], has been part of the motivation for our work, because the zero mode level spectrum seems to show local nonabelian anomalies. The model is also interesting from another point of view. The gauge group of the visible sector resembles the standard model gauge group, if one is able to break the nonabelian part of the gauge group, which is SU(3) × SU(3), to SU(3) × SU(2) × U(1). In [48] it is argued, that the anomalous U(1) gets massive by absorbing the model independent axion and can therefore be used to achieve the gauge group breaking.

The structure of the model is more complicated then the cases investigated so far, because we have two Wilson lines. This leads to nine different classes of twisted sectors and six U(1) factors in total. The role of the additional U(1)s in the context of the Standard model is not clear.

Shift and Wilson lines

$$
\begin{aligned}
v &= \tfrac{1}{3}(\ 1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 0,\ 0 \,|\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ )\,, \\
a_1 &= \tfrac{1}{3}(\ 0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 1,\ 1 \,|\, 1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0\ )\,, \\
a_2 &= \tfrac{1}{3}(\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2 \,|\, 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\ )
\end{aligned}
$$

Zero mode gauge group

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)^4 \times \mathrm{SO}(12)' \times \mathrm{U}(1)^{2\prime}$$

Zero mode matter content

$$
\begin{array}{lll}
\mathbf{UT}\ \mathit{(3)} & : & (\mathbf{1},\mathbf{3})_{0,-1,-1,0}(\mathbf{1})'_{0,0} \quad +(\mathbf{3},\mathbf{1})_{2,0,0,0}(\mathbf{1})'_{0,0} \\
\mathbf{T(0,0)}\ \mathit{(3)} & : & (\bar{\mathbf{3}},\bar{\mathbf{3}})_{0,0,0,0}(\mathbf{1})'_{0,0} \quad +(\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2}}(\mathbf{1})'_{0,0} \quad +(\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}}(\mathbf{1})'_{0,0} \\
 & & (\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2}}(\mathbf{1})'_{0,0} \quad +(\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}}(\mathbf{1})'_{0,0} \quad +(\mathbf{3},\mathbf{1})_{1,-1,0,0}(\mathbf{1})'_{0,0} \\
 & & (\mathbf{1},\mathbf{3})_{-1,1,0,0}(\mathbf{1})'_{0,0} \quad +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}}(\mathbf{1})'_{0,0} \quad +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(\mathbf{1})'_{0,0} \\
 & & +3\cdot(\mathbf{1},\mathbf{1})_{-1,-1,0,0}(\mathbf{1})'_{0,0} \\
\mathbf{T(1,0)}\ \mathit{(3)} & : & (\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{-1}{2},\frac{-1}{6},\frac{-1}{6}}(\mathbf{1})'_{\frac{4}{3},0} \quad +(\mathbf{1},\bar{\mathbf{3}})_{\frac{1}{2},\frac{1}{2},\frac{-1}{6},\frac{-1}{6}}(\mathbf{1})'_{\frac{4}{3},0} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{5}{6}}(\mathbf{1})'_{\frac{4}{3},0} \\
 & & (\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{5}{6},\frac{-1}{6}}(\mathbf{1})'_{\frac{4}{3},0} \quad +(\mathbf{1},\mathbf{1})_{-1,1,\frac{1}{3},\frac{1}{3}}(\mathbf{1})'_{\frac{4}{3},0} \\
\mathbf{T(2,0)}\ \mathit{(3)} & : & (\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{1}{6}}(\mathbf{1})'_{\frac{-4}{3},0} \quad +(\bar{\mathbf{3}},\mathbf{1})_{0,0,\frac{-1}{3},\frac{-1}{3}}(\mathbf{1})'_{\frac{-4}{3},0} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-3}{2},\frac{1}{6},\frac{1}{6}}(\mathbf{1})'_{\frac{-4}{3},0} \\
 & & (\mathbf{1},\mathbf{1})_{-1,0,\frac{-1}{3},\frac{2}{3}}(\mathbf{1})'_{\frac{-4}{3},0} \quad +(\mathbf{1},\mathbf{1})_{-1,0,\frac{2}{3},\frac{-1}{3}}(\mathbf{1})'_{\frac{-4}{3},0} \\
\mathbf{T(0,1)}\ \mathit{(3)} & : & (\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{6}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \quad +(\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{6}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \quad +(\mathbf{1},\bar{\mathbf{3}})_{-1,0,0,\frac{-1}{3}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \\
 & & +(\bar{\mathbf{3}},\mathbf{1})_{0,-1,0,\frac{-1}{3}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{6}}(\mathbf{1})'_{\frac{4}{3},\frac{4}{3}} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-5}{6}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,-1,0,\frac{2}{3}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \quad +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{6}}(\mathbf{1})'_{\frac{-2}{3},\frac{-2}{3}} \\
\mathbf{T(0,2)}\ \mathit{(3)} & : & (\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{6}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \quad +(\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{6}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \quad +(\mathbf{1},\bar{\mathbf{3}})_{-1,0,0,\frac{1}{3}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \\
 & & +(\bar{\mathbf{3}},\mathbf{1})_{0,-1,0,\frac{1}{3}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{5}{6}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \quad +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{6}}(\mathbf{1})'_{\frac{-4}{3},\frac{-4}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,-1,0,\frac{-2}{3}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}} \quad +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{6}}(\mathbf{1})'_{\frac{2}{3},\frac{2}{3}}
\end{array}
$$

$$
\begin{array}{llll}
\mathbf{T(1,1)}\ \textit{(3)} & : & (\mathbf{1},\mathbf{3})_{-1,0,\frac{1}{3},0}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} & +(\mathbf{3},\mathbf{1})_{\frac{-1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{2}}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} & +(\bar{\mathbf{3}},\mathbf{1})_{0,1,\frac{1}{3},0}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} \\
 & & +(\mathbf{1},\bar{\mathbf{3}})_{\frac{1}{2},\frac{1}{2},\frac{-1}{6},\frac{1}{2}}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{-1}{2}}(\mathbf{1})'_{\frac{-4}{3},\frac{4}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{5}{6},\frac{1}{2}}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,1,\frac{-2}{3},0}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} & +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{-1}{2}}(\mathbf{1})'_{\frac{2}{3},\frac{-2}{3}} & \\
\mathbf{T(1,2)}\ \textit{(3)} & : & (\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}}(\mathbf{12})'_{0,\frac{2}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}}(\mathbf{1})'_{-2,\frac{2}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}}(\mathbf{1})'_{2,\frac{2}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,1,\frac{1}{3},\frac{-1}{3}}(\mathbf{1})'_{0,\frac{-4}{3}} & +3\cdot(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}}(\mathbf{1})'_{0,\frac{-4}{3}} & \\
\mathbf{T(2,1)}\ \textit{(3)} & : & (\mathbf{1},\bar{\mathbf{3}})_{\frac{1}{2},\frac{-1}{2},\frac{1}{6},\frac{-1}{6}}(\mathbf{1})'_{0,\frac{4}{3}} & +(\bar{\mathbf{3}},\mathbf{1})_{0,0,\frac{-1}{3},\frac{1}{3}}(\mathbf{1})'_{0,\frac{4}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{3}{2},\frac{1}{6},\frac{-1}{6}}(\mathbf{1})'_{0,\frac{4}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,0,\frac{-1}{3},\frac{-2}{3}}(\mathbf{1})'_{0,\frac{4}{3}} & +(\mathbf{1},\mathbf{1})_{-1,0,\frac{2}{3},\frac{1}{3}}(\mathbf{1})'_{0,\frac{4}{3}} & \\
\mathbf{T(2,2)}\ \textit{(3)} & : & (\mathbf{1},\mathbf{3})_{\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{-1}{2}}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} & +(\mathbf{1},\bar{\mathbf{3}})_{\frac{1}{2},\frac{-1}{2},\frac{1}{6},\frac{1}{2}}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} & +(\mathbf{3},\mathbf{1})_{1,0,\frac{-1}{3},0}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} \\
 & & +(\bar{\mathbf{3}},\mathbf{1})_{0,0,\frac{2}{3},0}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{3}{2},\frac{1}{6},\frac{1}{2}}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} & +(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{-3}{2},\frac{1}{6},\frac{-1}{2}}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} \\
 & & +(\mathbf{1},\mathbf{1})_{-1,0,\frac{-1}{3},0}(\mathbf{1})'_{\frac{4}{3},\frac{-4}{3}} & +3\cdot(\mathbf{1},\mathbf{1})_{-1,0,\frac{-1}{3},0}(\mathbf{1})'_{\frac{-2}{3},\frac{2}{3}} &
\end{array}
$$

Anomalies and equivalent models

The fixed point equivalent models are shown in figure 6.2. The abelian factors that appear in the various local gauge groups combine to the set of zero mode U(1) factors given above. Taking appropriate linear combinations, only one of these U(1)s is anomalous (for details see [48]).

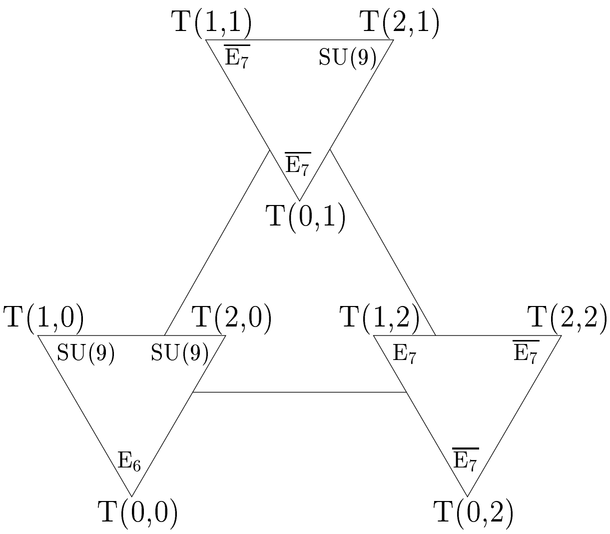


Figure 6.4: *Equivalent models of 6.2. The different fixed point classes with respect to the first and second Wilson line are indicated by the triangle structure. An overline above an equivalent model indicates that the* E_8 *and* $\mathrm{E}_8{}'$ *sector are exchanged with respect to tab. D.1.*

Chapter 7

Conclusions and outlook

The results presented in this thesis can be summarized as follows. We considered the anomaly structure of heterotic string models compactified on $T^6/\mathbb{Z}_3$. It turned out that the relevant part of the anomaly is given in terms of localized anomalies at the fixed points. We showed how models with arbitrary Wilson lines can be locally described in terms of fixed point equivalent models. The local anomaly contribution is therefore given by the zero mode anomaly of one of the five different shift-only models. We concluded from this observation, that all models are locally free of nonabelian anomalies. This is not true in the abelian case, because two of the shift-only models have an anomalous U(1). Explicit examples have been given to illustrate the implications of this. In particular it has been shown how the abelian zero-mode anomaly can be related to the anomalies of the equivalent models. Unexpected consequences arise from the fact that the local U(1) generators can be embedded differently into E_8, such that the anomalous U(1) at zero mode level is a linear combination of the local ones.

For the consideration of local anomalies in field theoretical models we find two consequences. The contribution of heavy Kaluza-Klein modes, which played an important role for the cancellation of nonabelian anomalies in our setup, arise also in pure field theory models and should be taken into account. Secondly we have seen that consistent field theory models can be obtained from string theory orbifolds and they might be therefore a good starting point for orbifold models in ten dimensions.

It would be very interesting to apply the results obtained here to orbifold models which resemble the standard model gauge group. It has been shown in one of the examples (section 6.2), that it is possible to obtain such models, but the role of the additional Cartan generators is not clear. In [43] it has been pointed out, that the rank of the gauge group can be broken by more general twist embeddings, therefore a more detailed study of the embedding structure would be desirable.

On the other hand the issue of local anomaly cancellation in string theory orbifolds is interesting in its own right. An important, but up to now unsolved, problem is how the Green-Schwarz mechanism, responsible for the cancellation of the zero mode anomaly, can be related to the localized abelian anomalies. This is an interesting topic for future work. Another line of research would be the generalization of the results to more general orbifolds, in the first place to general point groups $\mathbb{Z}_n$. In the case of n being prime, the results should be applicable with only trivial modifications concerning additional twisted sectors. For non-prime $\mathbb{Z}_n$ orbifolds the argumentation has to be altered. In the case of $\mathbb{Z}_4$ for example, we get an additional $\mathbb{Z}_2$ subgroup, which modifies the configuration of the second twisted sectors.

The examples of this work have been generated with a computer program, which is able to calculate the properties of $\mathbb{Z}_3$ orbifold models. The possibility to test ideas quickly has stimulated the research and it would be attractive to extend the code to deal with the more general models described above.

Appendix A

Notations and Conventions

A.1 Metric and indices

We will use uppercase Latin indices running over all coordinates of the d dimensional space, Greek indices with values $0, \ldots, 3$ to indicate the usual four Minkowski spacetime dimensions and lowercase Latin indices with values $4, \ldots, d-1$ to label the extra (compact) dimensions.

The metric is defined to be spacelike,

$$g_{MN} = g^{MN} = \text{diag}(\text{-}1, 1, \ldots, 1), \quad M, N = 0, 1, \ldots, d. \tag{A.1}$$

A.2 Spinors in various dimensions

A general approach to Clifford algebras and spinors can be found in [64, 69], we will only cite the relevant results for our applications and follow the conventions of [26].

In general the Γ matrices represent the Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}. \tag{A.2}$$

In two (Euclidean) dimensions they are given by the Pauli matrices $\boldsymbol{\sigma}$, explicitly

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \text{-}i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & \text{-}1 \end{pmatrix}, \tag{A.3}$$

where σ_3 acts as chirality generator. The charge conjugation matrix is given by $s = \sigma_1$ and the Majorana condition reads $s\psi^* = \psi$.

In four Minkowski dimensions we will use the γ matrices

$$\gamma_0 = i\mathbb{1} \otimes \sigma_1, \quad \gamma_1 = \mathbb{1} \otimes \sigma_2, \quad \gamma_2 = \sigma_1 \times \sigma_3, \quad \gamma_3 = \sigma_2 \otimes \sigma_3, \tag{A.4}$$

with chirality generator $\tilde{\gamma} = \sigma_3 \otimes \sigma_3$ and charge conjugation matrix $C = \sigma_1 \otimes \sigma_2$.

Going to five dimensions we can use the same set of γ matrices, using $\tilde{\gamma}$ as the additional generator γ_4.

When considering supersymmetry in five dimensions it is useful to rewrite 4-component Dirac spinors as symplectic spinors that fulfill a Majorana condition, introducing an additional SU(2) spinor index $i = 1, 2$ and the constraint

$$\psi^i = s^{ij} C \bar{\psi}^{jT}. \tag{A.5}$$

We can decompose this spinor in two-component spinors $\psi^i_{L,R}$ as

$$\psi^i = \begin{pmatrix} \psi^i_L \\ \psi^i_R \end{pmatrix}, \tag{A.6}$$

where (A.5) reads now

$$\psi^i_L = s^{ij} s \psi^{j*}_R \quad \text{and} \quad \psi^i_R = s^{ij} s \psi^{j*}_L. \tag{A.7}$$

For the discussion of spinors on T^6, we need spinors in six Euclidean dimensions, explicitly given by

$$\begin{aligned} \tilde{\sigma}_1 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_1, \quad & \tilde{\sigma}_2 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_2, \quad & \tilde{\sigma}_3 &= \mathbb{1} \otimes \sigma_1 \otimes \sigma_3, \\ \tilde{\sigma}_4 &= \mathbb{1} \otimes \sigma_2 \otimes \sigma_3, \quad & \tilde{\sigma}_5 &= \sigma_1 \otimes \sigma_3 \otimes \sigma_3, \quad & \tilde{\sigma}_6 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3. \end{aligned} \tag{A.8}$$

The chirality operator reads now $\tilde{\sigma} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3$, and charge conjugation is generated by $c = \sigma_2 \otimes \sigma_1 \otimes \sigma_2$.

In ten dimensions with Minkowski signature we can use gamma matrices

$$\Gamma_\mu = \mathbb{1}_6 \otimes \gamma_\mu, \quad \Gamma_a = \tilde{\sigma}_a \otimes \tilde{\gamma}. \tag{A.9}$$

Chirality and charge conjugation are generated by $\tilde{\Gamma} = \tilde{\sigma} \otimes \tilde{\gamma}$ and $S = c \otimes C$, respectively. We have furthermore the possibility to impose Majorana and Weyl conditions on a spinor simultaneously.

If we go from $(1, 9)$ to $(1, 3)$ dimensions, with compactification on T^6, we can decompose a ten dimensional Majorana-Weyl spinor Ψ with chirality $\beta = \pm$ in terms of six dimensional spinors $\eta^{\alpha_1\alpha_2\alpha_3}$, built from a tensor product of three two dimensional spinors with chirality $\alpha_i = \pm$, as

$$\Psi^\beta = \frac{1}{\sqrt{2}} \sum_\alpha \eta^{\alpha_1\alpha_2\alpha_3} \otimes \chi^{\alpha_1\alpha_2\alpha_3}. \tag{A.10}$$

The four dimensional spinors are given by $\chi^{\alpha_1\alpha_2\alpha_3}$.

Appendix B

Lie algebra

In this chapter some basic facts about Lie algebras and their representations are presented[1] and applications of some of the methods to SU(3) and the exceptional group E_8 are given. The treatment is far from complete and focuses on the points, that are relevant for this work.

B.1 Cartan Matrices and Dynkin Diagrams

The abelian Cartan subalgebra h of a simple Lie algebra G of dimension d is given by the maximal commuting set of hermitian generators $\{H^I\}$, where $I = 1, \dots, r$ and r is the *rank* of G. As they all commute, they can be diagonalized simultaneously.

The remaining $d - r$ generators can be cast into the form of raising and lowering generators $E^\alpha, E^{-\alpha}$, satisfying the eigenvalue equation

$$[H^I, E^\alpha] = \alpha^I E^\alpha, \tag{B.1}$$

α being an r-dimensional vector and an element of the dual of the Cartan subalgebra h^* given by the isomorphism

$$\alpha(H^I) = K(H^I, H^\alpha) = \alpha^I \quad , \quad H^\alpha := \alpha \cdot H = \sum_I \alpha^I H^I, \tag{B.2}$$

induced by the *Killing form*

$$K(A, B) := \frac{1}{2g} \mathrm{Tr}\,(\mathrm{ad}(A)\mathrm{ad}(B)). \tag{B.3}$$

Here $\mathrm{ad}(A)$ means to take A in the *adjoint representation* of G, in which the states are given by the generators and the action of a generator on a state is given by the commutator. The normalization constant g is called the *dual Coxeter number*.

[1] The notation of ch. 13,14 of [22], which gives a good review of the mathematical tools, is used. For a more physical introduction with many useful tables see [68]. Part of the original work was done by Dynkin, see [20] and references therein.

We will denote the set of all roots by Δ. The complete commutation relations of G can be written in this *Cartan-Weyl basis* as

$$[H^I, H^J] = 0, \tag{B.4a}$$
$$[H^I, E^\alpha] = \alpha^I \cdot E^\alpha, \tag{B.4b}$$
$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta \\ \frac{2}{|\alpha|^2} H^\alpha, & \text{if } \alpha = -\beta \\ 0, & \text{otherwise,} \end{cases} \tag{B.4c}$$

where $N_{\alpha,\beta}$ is a constant. The scalar product of roots is dual to the action of the Killing form, in the sense that

$$(\alpha, \beta) = K(H^\alpha, H^\beta). \tag{B.5}$$

The hermitian conjugate of (B.4b) shows that $E^{\alpha\dagger} = E^{-\alpha}$, so if α is a root, $-\alpha$ is also one. After fixing a (r-dimensional) basis in the root space, one can define *positive roots* by the requirement that the first nonvanishing coefficient of the expansion in this basis is positive. This splits the root space into positive roots $\in \Delta_+$ and negative roots $\in \Delta_-$ with $\Delta_- = -\Delta_+$. *Simple roots* are defined by requiring that they are positive roots which can not be written as a sum of two other positive roots. They provide a basis $\{\alpha_i\}$ in the root space and are used to define the *Cartan matrix*, which encodes all information of the structure of a simple Lie algebra. Its elements are given by

$$A_{ij} = (\alpha_i, \alpha_j^\vee), \tag{B.6}$$

with the definition of the *simple coroots* as

$$\alpha_j^\vee = \frac{2}{|\alpha_j|^2}. \tag{B.7}$$

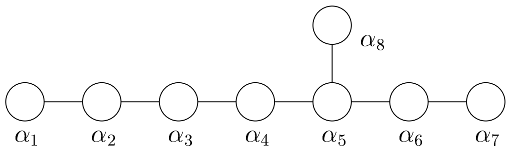
The normalization of roots can be chosen arbitrarily. We will fix it by assigning a length of 2 to the *longest root* θ, which is defined to be the unique element of Δ, given by an expansion in the simple roots as $\sum_i m_i \alpha_i$, for which the sum of the expansion coefficients $\sum_i m_i$ is maximized. If all roots have the same length, the algebra is called *simply laced*, otherwise there are at maximum two different length possible, so one differentiates between "long" and "short" roots.

To classify a simple Lie algebra the graphical description with Dynkin diagrams is very useful. They can be produced from the Cartan matrix by the following algorithm:

- Draw a dot for every root α_i, choose different colors for different length (the usual convention is to choose white dots for long roots and black dots for short ones).
- Draw $A_{ij}A_{ji}$ lines between the dots corresponding to α_i and α_j.

For the maximal exceptional group E_8 the Cartan matrix and Dynkin diagram are given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{B.8a}$$



(B.8b)

B.2 Representations

To classify states of representations different from the adjoint (discussed above), it is useful to expand their *weights*, the analogue of the roots of the adjoint representation, in the *simple weight basis* $\{\omega_i\}$. This is given by the dual to the simple coroot basis as

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}. \tag{B.9}$$

The expansion coefficients of a weight in this basis are called *Dynkin labels*. By comparing this to (B.6) we see that the elements of the Cartan matrix are the Dynkin labels of the simple roots.

An arbitrary representation can be classified by its *highest weight* λ, defined similar to the highest root in the previous section. As a result every raising operator applied to λ gives 0, which means that all states of a representation can be reached by successive application of lowering operators. The algorithm goes like this:

- Write the highest weight in the simple weight basis with coefficients λ_i.
- For every nonnegative coefficient λ_i the corresponding simple root α_i can be subtracted at most λ_i times.
- Every subtraction of a simple root gives a state of the representation.

For SU(N) there exists a diagrammatic representation of highest weights in terms of *Young tableaux*[2], which is used in appendix C for the computation of anomaly coefficients. The Young tableau representing a representation is related to the highest weight, written in Dynkin labels as $\lambda = (\lambda_1, \ldots, \lambda_r)$, in the following way. For each label λ_i a column with i boxes is drawn, beginning with the longest column to the left.

[2]For a detailed discussion and relations to Gelfand-Tsetlin patterns see e.g. [22].

As an example consider the algebra of SU(3) with Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{B.10}$$

and highest root $\theta = (1, 1)$ (in Dynkin labels), which labels the adjoint representation **8**. The application of the algorithm to compute all states of this representation is shown in figure B.1, where the $(0, 0)$ state counts twice, representing the two dimensional Cartan subalgebra (SU(3) has rank 2). The Young tableau is given by one column with two boxes and one single box as $\lambda \simeq \ydiagram{2,1}$.

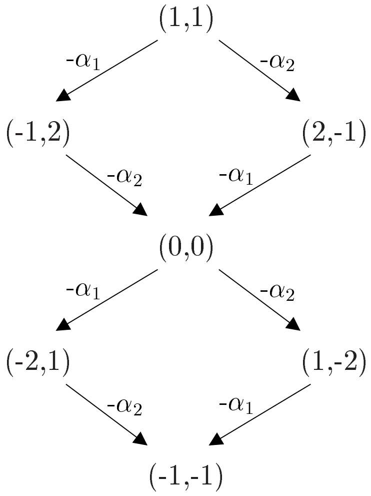


Figure B.1: *The adjoint representation of* SU(3) *with all states written in the simple weight basis.*

B.3 Lattices

Every weight of a finite dimensional representation can be expressed in the simple weight basis with integer coefficients (Dynkin labels). The simple weights define therefore an r-dimensional lattice, called the *weight lattice*

$$\Gamma_G = \{\textstyle\sum_i n_i \omega_i;\ n_i \in \mathbb{Z}\}\,. \tag{B.11}$$

With help of the simple roots and coroots we can define the *root lattice* $\tilde{\Gamma}_G$, which is a subset of the weight lattice

$$\tilde{\Gamma}_G = \{\textstyle\sum_i n_i \alpha_i;\ n_i \in \mathbb{Z}\}\,, \qquad \tilde{\Gamma}_G \subseteq \Gamma_G, \tag{B.12}$$

and the *coroot lattice* $\Gamma_G^\vee$, which is dual to the weight lattice.

For the algebras $\mathrm{G}_2, \mathrm{F}_4$ and E_8 we have that $\tilde{\Gamma}_G = \Gamma_G$ and only in the case of E_8 the weight lattice is selfdual, $\Gamma_{\mathrm{E}_8}^\vee = \Gamma_{\mathrm{E}_8}$.

B.3.1 The E_8 Lattice

The weight lattice of E_8 can be given in an orthogonal basis as (see for example [64], chapter 11.4)

$$\Gamma_{\mathrm{E}_8} = \Big\{ \big(n^I\big)\,, \big(\tfrac{1}{2} + n^I\big) \mid n^I \in \mathbb{Z},\, \textstyle\sum_{I=1}^{8} n^I = 0 \mod 2 \Big\}\,, \tag{B.13}$$

which leads to the following vectors of $\mathbb{R}^8$ with length 2:

$$(\ \underline{\pm 1, \pm 1, 0^6}\ ), \qquad (\ \underline{(\pm\tfrac{1}{2})^8}\ ), \tag{B.14}$$

where the underlining means all possible permutations and in the second case the restriction to an even number of $(-)$ signs is made. Notice that E_8 is simply laced and weight and root lattice are identical. Therefore Γ_{E_8} is selfdual.

The first set of (B.14) corresponds to the rootset of SO(16) and the vectors of the second set are the weights of the **128** spinor representation of SO(16), corresponding to the fact that SO(16) is a maximal subgroup of E_8 and the **248** of E_8 transforms as $\mathbf{120} + \mathbf{128}$ under this subgroup.

B.4 Weyl reflections

A *Weyl reflection* of a weight β at a root α is defined by

$$\mathsf{w}_\alpha \beta = \beta - (\alpha, \beta)\alpha, \qquad \alpha \in \Delta. \tag{B.15}$$

Obviously reflections at α and $-\alpha$ give the same result, so one can restrict the considerations to positive roots.

The set of all Weyl reflections forms a group, the so called *Weyl group* of G. It is generated by the simple Weyl reflections, defined as reflections at the simple roots

$$s_i \beta = \mathsf{w}_{\alpha_i} \beta. \tag{B.16}$$

We are especially interested in the case of E_8 to characterize shift embeddings. Therefore we consider the action of the different elements of (B.14) on E_8 weights. Here we show only the action of root elements of SO(16), i.e. from the first set in (B.14). For the action of the spinor weights, which can be used to change the number of zero entries in a weight vector, see [26]. We have to distinguish two cases,

$$(\ldots, 1, \ldots, 1, \ldots)\,, \tag{B.17a}$$

$$\text{and} \qquad (\ldots, 1, \ldots, -1, \ldots)\,, \tag{B.17b}$$

where the two non-zero entries are at positions i and j, respectively.

Reflections at roots of type (B.17a) exchange two entries of a weight vector and change their sign simultaneously, i.e.

$$(\ldots, a_i, \ldots, a_j, \ldots) \to (\ldots, -a_j, \ldots, -a_i, \ldots)\,, \tag{B.18}$$

while the second type (B.17b) does only the exchange

$$(\ldots, a_i, \ldots, a_j, \ldots) \to (\ldots, a_j, \ldots, a_i, \ldots)\,. \tag{B.19}$$

This offers the possibility, combining operations of both types, to permute the entries of a given weight vector and change their signs, but one has to be careful that an arbitrary change of signs is only possible if the weight contains zeros.

Appendix C

Casimir invariants and indices

In this appendix it will be shown how to calculate indices of SU(N) representations, which are relevant for the calculation of anomalies, as mentioned in section 5.2. The discussion is based on [58], see also [59, 60].

Let G be a Lie algebra and ϕ some representation. The definition of the p-th order index $D_p(\phi)$ is given by

$$D_p(\phi) = \mathrm{Tr}_{\,\phi} J_p, \tag{C.1}$$

where J_p is the p-th order Casimir invariant, which exists for SU(N) with $N \geq p$.

For the discussion of four dimensional anomalies only the second and third order indices are important, and we are interested in relations between traces over different representations, therefore it is useful to introduce a normalized index as

$$\tilde{D}_p(\phi) = \frac{D_p(\phi)}{D_p(\mathbf{N})}, \tag{C.2}$$

where $\mathbf{N}$ is the fundamental representation of SU(N). Then we get for the pure nonabelian and mixed nonabelian/abelian anomalies

$$\mathcal{A}_3 \sim \sum_{\phi} \tilde{D}_3(\phi), \qquad \mathcal{A}_2 \sim \sum_{\phi} \tilde{D}_2(\phi) Q(\phi), \tag{C.3}$$

respectively, where $Q(\phi)$ is the U(1) charge of the representation ϕ. Of course, if we want to compare the mixed anomaly with pure U(1) anomalies, as it is done in section 5.4.2, we have to multiply our result with the quadratic index of the fundamental representation of SU(N), which is given by $\frac{1}{2}$ in the normalization of roots chosen in appendix B.

The rules to calculate $\tilde{D}_2(\phi)$ and $\tilde{D}_3(\phi)$ can be given in terms of the Young tableau of ϕ (see app. B.2), as described in detail in [58]. First notice that the indices obey a sum rule for tensor products of representations

$$\phi_A \otimes \phi_B = \bigoplus_C \phi_C. \tag{C.4}$$

In this case the indices are related by

$$d(\phi_A)\tilde{D}_p(\phi_B) + d(\phi_B)\tilde{D}_p(\phi_A) = \sum_C \tilde{D}_p(\phi_C), \tag{C.5}$$

with the dimension of a representation denoted by $d(\phi)$. In general one can give $\tilde{D}_p$ for representations with complete symmetric, $\mathbf{S}_k = (\square\cdots\square)$, or antisymmetric, $\mathbf{A}_k = \begin{pmatrix}\square\\ \vdots\\ \square\end{pmatrix}$, Young tableau, both with k boxes, as

$$\begin{aligned}\tilde{D}_p(\mathbf{S}_k) &= \sum_{j=0}^{k-1}(k-j)^{p-1}\frac{(N+j-1)!}{j!\,(N-1)!} && \text{(C.6a)}\\ \tilde{D}_p(\mathbf{A}_k) &= -\sum_{j=0}^{k-1}(-1)^{k-j}(k-j)^{p-1}\frac{N!}{j!\,(N-i)!} && \text{(C.6b)}\end{aligned}$$

From the definition it is obvious that the antifundamental representation $\overline{\mathbf{N}}$ has $\tilde{D}_2 = 1$ and $\tilde{D}_3 = -1$. With (C.5) and the tensor product $\mathbf{N}\otimes\overline{\mathbf{N}} = \mathbf{1}\oplus\mathbf{Ad}$, we find for the adjoint representation

$$\tilde{D}_2(\mathbf{Ad}) = 2N, \qquad \tilde{D}_3(\mathbf{Ad}) = 0. \tag{C.7}$$

For some more representations, characterized by their Young tableaux, the indices are given in table C.1.

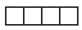
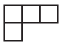
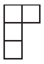
Repr.	$\tilde{\mathbf{D}}_\mathbf{2}$	$\tilde{\mathbf{D}}_\mathbf{3}$
	$N+2$	$N+4$
	$N-2$	$N-4$
	$\frac{1}{2}(N+2)(N+3)$	$\frac{1}{2}(N+3)(N+6)$
	$\frac{1}{2}(N-2)(N-3)$	$\frac{1}{2}(N-3)(N-6)$
	N^2-3	N^2-9
	$\frac{1}{6}(N+2)(N+3)(N+4)$	$\frac{1}{6}(N+3)(N+4)(N+8)$
	$\frac{1}{6}(N-2)(N-3)(N-4)$	$\frac{1}{6}(N-3)(N-4)(N-8)$
	$\frac{1}{3}N(N^2-4)$	$\frac{1}{3}N(N^2-16)$
	$\frac{1}{2}(N+2)(N^2+N-4)$	$\frac{1}{2}(N+4)(N^2+N-8)$
	$\frac{1}{2}(N-2)(N^2-N-4)$	$\frac{1}{2}(N-4)(N^2-N-8)$

Table C.1: *Expressions for the quadratic and cubic indices* $\tilde{D}_2$ *and* $\tilde{D}_3$ *for different representations of* SU(N)*, characterized by their Young tableaux.*

Appendix D

Orbifold models

In the first part of this appendix we will describe shortly how the explicit models presented in this work have been calculated. Afterwards we present the five possible shift embeddings of the $\mathbb{Z}_3$ orbifold (without Wilson lines). In a third part we give two more examples for models with one Wilson line, which illustrate some aspects of the examples presented in chapter 6.

D.1 Calculation of models

All models of this thesis have been calculated with a computer program, written in JAVA, which has an interface accessible through the World Wide Web[1]. It is possible to calculate models of $\mathbb{Z}_3$ orbifolds, where the compactification is done to $D = 4$ or $D = 6$. Up to two Wilson lines can be included and all information concerning the gauge groups (global and induced), untwisted and twisted zero mode matter representations and U(1) factors, is calculated. A detailed analysis of the embedding structure with explicit calculation of the simple roots and abelian generators is performed and the highest weights of the different representations are identified. The output is given as LaTeX document and can be converted online to postscript.

D.2 Shift embeddings

In table D.1 the five possible shift embeddings [19], of the $\mathbb{Z}_3$ orbifold are listed and their spectrum on the zero mode level is given.

These embeddings are only unique up to Weyl reflections (see app. B.4) and shifts in the $E_8 \times E_8$ lattice [14]. With these operations every shift can be transformed into one of the five standard shifts. To judge easily to which model a specific shift belongs, it is enough to count the number of zeros in the two eight-dimensional parts of the shift

[1] Please contact the author, if you want to access the corresponding web page.

Model	**Shift**	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
E_8	$\frac{1}{3}\left(0^8 \,\vert\, 0^8\right)$	E_8 $\times \mathrm{E}_8{}'$	none	$3 \cdot (\mathbf{1})(\mathbf{1})'$
E_6	$\frac{1}{3}\left(1,1,-2,0^5 \,\vert\, 0^8\right)$	$(\mathrm{E}_6 \times \mathrm{SU}(3))$ $\times \mathrm{E}_8{}'$	$(\mathbf{27},\overline{\mathbf{3}})(\mathbf{1})'$	$(\mathbf{27},\mathbf{1})(\mathbf{1})'$ $+3 \cdot (\mathbf{1},\mathbf{3})(1)'$
$\mathrm{E}_6{}^2$	$\frac{1}{3}\left(1,1,-2,0^5 \,\vert\, 1,1,-2,0^8\right)$	$(\mathrm{E}_6 \times \mathrm{SU}(3))$ $\times (\mathrm{E}_6 \times \mathrm{SU}(3))'$	$(\mathbf{27},\overline{\mathbf{3}})(\mathbf{1},\mathbf{1})'$ $+(\mathbf{1},\mathbf{1})(\mathbf{27},\overline{\mathbf{3}})'$	$(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{3})'$
E_7	$\frac{1}{3}\left(1,1,0^6 \,\vert\, 2,0^7\right)$	$(\mathrm{E}_7 \times \mathrm{U}(1))$ $\times (\mathrm{SO}(14) \times \mathrm{U}(1))'$	$(\mathbf{1})_0(\mathbf{64})'_{\frac{1}{2}}$ $+(\mathbf{56})_1(\mathbf{1})'_0$ $+(\mathbf{1})_0(\mathbf{14})'_{-1}$ $+(\mathbf{1})_{-2}(\mathbf{1})'_0$	$(\mathbf{1})_{\frac{2}{3}}(\mathbf{14})'_{\frac{-1}{3}}$ $+(\mathbf{1})_{\frac{-4}{3}}(\mathbf{1})'_{\frac{2}{3}}$ $+3 \cdot (\mathbf{1})_{\frac{2}{3}}(\mathbf{1})'_{\frac{2}{3}}$
SU(9)	$\frac{1}{3}\left(1^4,2,0^3 \,\vert\, 2,0^7\right)$	SU(9) $\times (\mathrm{SO}(14) \times \mathrm{U}(1))'$	$(\mathbf{84})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_{\frac{1}{2}}$ $+(\mathbf{1})(\mathbf{14})'_{-1}$	$(\overline{\mathbf{9}})(\mathbf{1})'_{\frac{2}{3}}$

Table D.1: *Shift embeddings of the $\mathbb{Z}_3$ orbifold and zero mode matter content.*

vector [26]. If no zeros are present, the sign of the product of all entries is relevant. The corresponding models are listed in table D.2.

# of zeros in v	8	7, 4, 1	$6, 0_+$	5, 2	$3, 0_-$
model	E_8	SO(14)	E_7	E_6	SU(9)

Table D.2: *Correspondence between the number of zeros in a generic shift vector and the models given in table D.1. If the shift vector contains no zeros, the sign of the product of all entries is relevant, denoted here by $(0_+, 0_-)$.*

In table D.3 the gauge group breaking has been illustrated in terms of the (extended) Dynkin diagrams (see appendix B).

Three of these embeddings (E_6, E_7 and SU(9)) can be realized in two different ways, because their action on E_8 and $\mathrm{E}_8{}'$ is not symmetric. Therefore the first and second eight entries of the shift vector given in table D.1 can be flipped to obtain models where the gauge and matter content of the visible and hidden sector are exchanged.

All models are globally free of nonabelian anomalies. Two of the embeddings (E_7 and SU(9)) have an abelian anomaly, which can be obtained summing up the U(1) charges of all matter fields,

$$\mathrm{Tr}\,\mathrm{U}(1) = \mathrm{Tr}_{\,\mathrm{UT}}\mathrm{U}(1) + \mathrm{Tr}_{\,\mathrm{T}}\mathrm{U}(1) = 3\,\mathrm{Tr}_{\,\mathbf{R}}\mathrm{U}(1) + \sum_s \left(\mathrm{Tr}_{\,\mathbf{S}_s}\mathrm{U}(1) + 3\,\mathrm{Tr}_{\,\mathbf{T}_s}\mathrm{U}(1)\right), \quad \text{(D.1)}$$

where the trace has to be taken over all representations, defined by (4.27). The results for the two models are

$$
\begin{aligned}
\mathrm{E}_7 \quad &: \quad \mathrm{Tr}\,\mathrm{U}(1) = 3 \cdot 54 + 27 \cdot 10 = 432 \quad \textit{(anomalous)} \\
& \mathrm{Tr}\,\mathrm{U}(1)' = 3 \cdot 18 + 27 \cdot (-2) = 0 \\
\mathrm{SU}(9) \quad &: \quad \mathrm{Tr}\,\mathrm{U}(1)' = 3 \cdot 18 + 27 \cdot 6 = 216 \quad \textit{(anomalous)}
\end{aligned}
$$

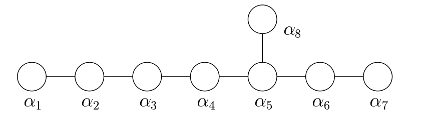
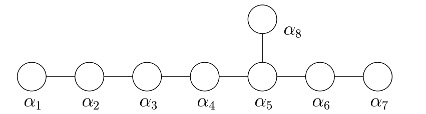
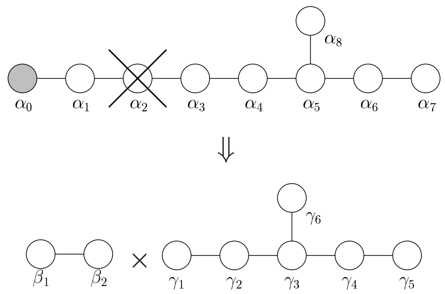
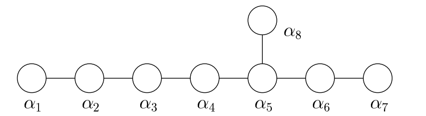
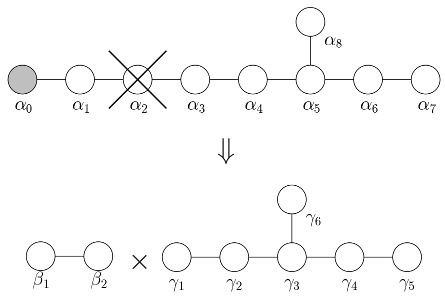
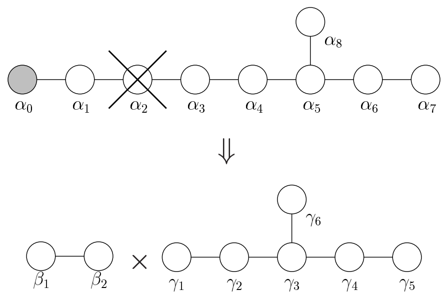
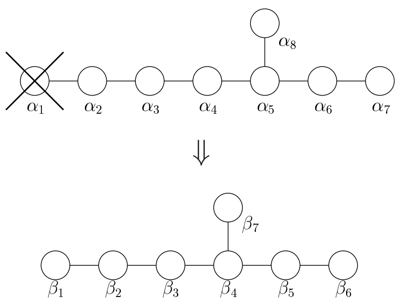
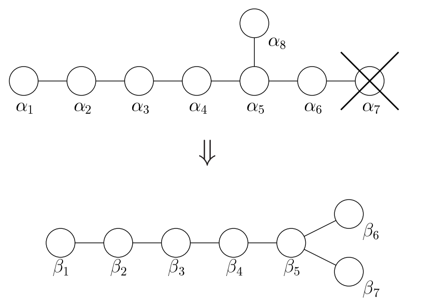
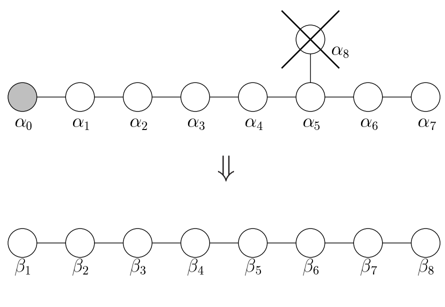
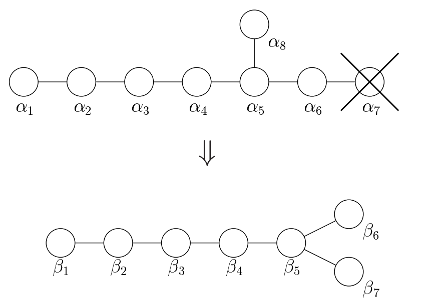
Model	Visible sector	Hidden sector
E_8		
E_6		
${E_6}^2$		
E_7		
SU(9)		

Table D.3: *Dynkin diagrams illustrating the gauge groups in the five shift embeddings of the $\mathbb{Z}_3$ orbifold. Shaded circles mark the lowest root in an extended diagram. Crossed circles are projected out by the orbifolding.*

D.3 More examples with one Wilson line

D.3.1 Different embeddings of SU(3)

We are going to analyze a model with a more complicated nonabelian structure. We do not get U(1) factors and we want to focus on the visible sector only, thus the shift vector and the Wilson line have been chosen such that the $E_8{}'$ is left unbroken. This model is interesting, because the embedding structure of the local gauge groups into the E_8 group is nontrivial.

Shift and Wilson lines

$$\begin{array}{rcl} v &=& \frac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ 0\,|\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \\ a_1 &=& \frac{1}{3}(\ 0,\ 0,\ 0,\ 1,\ 1,\text{-}2,\ 0,\ 0\,|\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \end{array}$$

Zero mode gauge group

$$\mathrm{SU}(3)^4 \times \mathrm{E}_8{}'$$

The gauge group of the visible sector contains four SU(3) groups, representing the maximal embedding of SU(3) into E_8. This is realized with a standard shift of E_6 type and a further reduction of the E_6 with an appropriately chosen Wilson line.

Zero mode matter content

$$\begin{array}{lcl} \textbf{UT}\ \textit{(3)} &:& (\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{1},\overline{\mathbf{3}})(\mathbf{1})' \\ \textbf{T(0)}\ \textit{(9)} &:& (\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})(\mathbf{1})' + (\mathbf{3},\mathbf{1},\overline{\mathbf{3}},\mathbf{1})(\mathbf{1})' + (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{3})(\mathbf{1})' + 3\cdot(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})(\mathbf{1})' \\ \textbf{T(1)}\ \textit{(9)} &:& (\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1})(\mathbf{1})' + (\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{3})(\mathbf{1})' + (\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\overline{\mathbf{3}})(\mathbf{1})' + 3\cdot(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})(\mathbf{1})' \\ \textbf{T(2)}\ \textit{(9)} &:& (\mathbf{1},\mathbf{3},\overline{\mathbf{3}},\mathbf{1})(\mathbf{1})' + (\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{1})(\mathbf{1})' + (\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})(\mathbf{1})' + 3\cdot(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})(\mathbf{1})' \end{array} \tag{D.3}$$

Anomaly cancellation and equivalent models

Considering the zero mode spectrum (D.3) and summing up all SU(3) representations we have the (expected) global anomaly cancellation. Local cancellation does not seem to be realized. At each fixed point we find that the last term gives a nonvanishing contribution to one of the SU(3)s. This picture changes after inclusion of the massive modes (tab. D.4). We see that we get a standard E_6 model at each individual fixed point, which is anomaly free.

Although the local gauge groups are identical, the global gauge group turns out to be $\mathrm{SU}(3)^4$, because all 3 fixed point classes have a different embedding into E_8 (tab. D.5).

Fixed point	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
T(0)	$E_6 \times SU(3)$ $\times {E_8}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(1)'$ $+3 \cdot (\mathbf{1},\mathbf{3})(\mathbf{1})'$
T(1)	$E_6 \times SU(3)$ $\times {E_8}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(1)'$ $+3 \cdot (\mathbf{1},\mathbf{3})(\mathbf{1})'$
T(1)	$E_6 \times SU(3)$ $\times {E_8}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(1)'$ $+3 \cdot (\mathbf{1},\mathbf{3})(\mathbf{1})'$

Table D.4: *Induced models at the fixed points.*

Therefore the E_6 at each fixed point reduces globally to $SU(3)^3$. This example shows, that it is important to consider the embedding structure of the local gauge group, to understand the zero mode limit. It has already been pointed out in 6.1.3 that this is also crucial to obtain the U(1) factor which is anomalous in the effective theory.

T(0)	**T(1)**	**T(2)**
Simple Roots (E_6)	Simple Roots (E_6)	Simple Roots (E_6)
$\alpha_1 = (0, 0, 0, 1, -1, 0, 0, 0 \,\|\, 0^8)$	$\beta_1 = (0, 0, 0, 0, 0, 0, 1, -1 \,\|\, 0^8)$	$\gamma_1 = (0, 0, 0, 0, 0, 0, 1, 1 \,\|\, 0^8)$
$\alpha_2 = (0, 0, 0, 0, 1, -1, 0, 0 \,\|\, 0^8)$	$\beta_2 = (\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2} \,\|\, 0^8)$	$\gamma_2 = (\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \,\|\, 0^8)$
$\alpha_3 = (0, 0, 0, 0, 0, 1, -1, 0 \,\|\, 0^8)$	$\beta_3 = (0, 0, 0, 1, -1, 0, 0, 0 \,\|\, 0^8)$	$\gamma_3 = (0, 0, 0, 0, 1, -1, 0, 0 \,\|\, 0^8)$
$\alpha_4 = (0, 0, 0, 0, 0, 0, 1, -1 \,\|\, 0^8)$	$\beta_4 = (0, 0, 1, -1, 0, 0, 0, 0 \,\|\, 0^8)$	$\gamma_4 = (0, 0, 1, 0, 0, 1, 0, 0 \,\|\, 0^8)$
$\alpha_5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2} \,\|\, 0^8)$	$\beta_5 = (0, 1, -1, 0, 0, 0, 0, 0 \,\|\, 0^8)$	$\gamma_5 = (0, 1, -1, 0, 0, 0, 0, 0 \,\|\, 0^8)$
$\alpha_6 = (0, 0, 0, 0, 0, 0, 1, 1 \,\|\, 0^8)$	$\beta_6 = (0, 0, 0, 0, 1, -1, 0, 0 \,\|\, 0^8)$	$\gamma_6 = (0, 0, 0, 1, -1, 0, 0, 0 \,\|\, 0^8)$
Simple Roots (SU(3))	Simple Roots (SU(3))	Simple Roots (SU(3))
$\alpha_7 = (0, 1, -1, 0, 0, 0, 0, 0 \,\|\, 0^8)$	$\beta_7 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \,\|\, 0^8)$	$\gamma_7 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2} \,\|\, 0^8)$
$\alpha_8 = (1, -1, 0, 0, 0, 0, 0, 0 \,\|\, 0^8)$	$\beta_8 = (0, 0, 0, 0, 0, 0, 1, 1 \,\|\, 0^8)$	$\gamma_8 = (0, 0, 0, 0, 0, 0, 1, -1 \,\|\, 0^8)$

Table D.5: *Embedding structure of D.3.1 for all fixed points (visible sector only). It shows that each class of fixed points (with induced gauge group* $E_6 \times SU(3)$*) is embedded in a different way into* E_8*.*

D.3.2 Different induced models at each fixed point

In this model the nonabelian and the abelian anomaly are interesting. Variations of this embedding have already been used in chapter 6. The choice of shift and Wilson lines used here turns out to be special, because the zero mode representations at the fixed points $T(1)$ and $T(2)$ are quite different, which can only be understood in terms of the equivalent models.

Shift and Wilson lines

$$\begin{array}{rcl} v &=& \frac{1}{3}(\ 1,\ 1,\text{-}2,\ 0,\ 0,\ 0,\ 0,\ \ 0 \,|\ \ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \ 0\,) \\ a_1 &=& \frac{1}{3}(\ 0,\ 0,\ 0,\ 1,\ 1,\text{-}2,\ 1,\ \ 1 \,|\ \ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \ 0\,) \end{array}$$

Zero mode gauge group

$$\mathrm{SU}(3) \times \mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$$

The gauge sector corresponds exactly to the model considered in section 6.1.2, but the embedding structure of the visible sector is very different. This can be checked by looking at the roots and U(1) generators of both models. The embedding is responsible for the changes in the zero mode matter content compared to 6.1.2.

Zero mode matter content

$$\begin{array}{lcl} \mathbf{UT}\ \mathit{(3)} &:& (\overline{\mathbf{3}},\overline{\mathbf{6}})_{-2}(\mathbf{1})'_0 \\ \mathbf{T(0)}\ \mathit{(9)} &:& (\mathbf{1},\mathbf{15})_0(1)'_0 + (\mathbf{1},\overline{\mathbf{6}})_{-2}(\mathbf{1})'_0 + (\mathbf{1},\overline{\mathbf{6}})_2(\mathbf{1})'_0 + 3\cdot(\mathbf{3},\mathbf{1})_0(\mathbf{1})'_0 \\ \mathbf{T(1)}\ \mathit{(9)} &:& (\mathbf{1},\mathbf{1})_{-4/3}(\mathbf{14})'_{-1/3} + (\mathbf{1},\mathbf{1})_{8/3}(\mathbf{1})'_{2/3} + 3\cdot(\mathbf{1},\mathbf{1})_{-4/3}(\mathbf{1})'_{2/3} \\ \mathbf{T(2)}\ \mathit{(9)} &:& (\mathbf{1},\mathbf{6})_{-2/3}(\mathbf{1})'_{-2/3} + (\overline{\mathbf{3}},\mathbf{1})_{-4/3}(\mathbf{1})'_{-2/3} \end{array} \tag{D.5}$$

Anomaly cancellation and equivalent models

The cancellation of the nonabelian anomalies at the zero mode level is not obvious in this model. There are two possible anomalous gauge groups, SU(3) and SU(6). Summing up all contributions from (D.5) we find that there are equal numbers of **3** and $\overline{\mathbf{3}}$ representations of SU(3), thus the anomaly vanishes. Calculating the anomaly for SU(6), we have to take into account that the anomaly coefficent of the **15** representation is 2 in units of the fundamental representation (see app. C). Hence the contributions to a potential SU(6) anomaly are

$$\begin{array}{rcl} \sum\limits_{\mathrm{SU}(6)} &=& 3\cdot\Big(3\tilde{D}_3(\overline{\mathbf{6}})\Big) + 9\cdot\Big(\tilde{D}_3(\mathbf{15}) + 2\tilde{D}_3(\overline{\mathbf{6}}) + \tilde{D}_3(\mathbf{6})\Big) \\ &=& 3\cdot(-3) + 9\cdot(2 + (-2) + 1) \\ &=& 0 \end{array}$$

The fixed point equivalent models are shown in tab. D.6 and fig. D.3.2. We find three different equivalent models for the three classes of fixed points. This explains why $T(1)$ and $T(2)$ have a different matter content, although they seem to be conjugated to each other with respect to the Wilson line contribution.

This model contains an abelian anomaly at the zero mode level. A naive sum of the charges in (D.5) gives -288 and -72 for U(1) and U(1)$'$, respectively. Now we can

Fixed point	**Gauge group**	**Untwisted matter** *(Multiplicity 3)*	**Twisted matter** *(Multiplicity 27)*
T(0)	$E_6 \times SU(3)$ $\times {E_8}'$	$(\mathbf{27},\overline{\mathbf{3}})(1)'$	$(\mathbf{27},\mathbf{1})(\mathbf{1})'$ $+3\cdot(\mathbf{1},\mathbf{3})(\mathbf{1})'$
T(1)	$E_7 \times U(1)$ $\times SO(14)' \times U(1)'$	$(\mathbf{1})_0(\mathbf{64})'_{1/2}$ $+(\mathbf{56})_{-2}(\mathbf{1})'_0$ $+(\mathbf{1})_0(\mathbf{14})'_{-1}$ $+(\mathbf{1})_4(\mathbf{1})'_0$	$(\mathbf{1})_{-4/3}(\mathbf{14})'_{-1/3}$ $+(\mathbf{1})_{8/3}(\mathbf{1})'_{2/3}$ $3\cdot(\mathbf{1})_{-4/3}(\mathbf{1})'_{-1/3}$
T(2)	$SU(9)$ $\times SO(14)' \times U(1)'$	$(\mathbf{84})(\mathbf{1})'_0$ $+(\mathbf{1})(\mathbf{64})'_{-1/2}$ $+(\mathbf{1})(\mathbf{14})'_1$	$(\overline{\mathbf{9}})(\mathbf{1})'_{-2/3}$

Table D.6: *Induced models at the fixed points.*

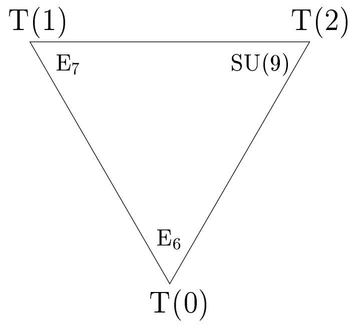


Figure D.1: *Local gauge groups of model D.3.2. Each edge of the triangle represents one of the three fixed points. The gauge groups correspond to the equivalent models.*

take linear combinations to make only one of the two U(1)s anomalous. We could have expected this result from the equivalent model picture, because the equivalent models at $T(1)$ and $T(2)$ are both anomlous at the zero mode level. A cancellation between both contributions, as it occured in the example of section 6.1.2, is not possible here, because the equivalent models are not identical.

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