On the Resolution of Orbifold Singularities in Heterotic String Theory

Denis Klevers

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Referent:	Prof. Dr. Hans-Peter Nilles
Korreferent :	Prof. Dr. Albrecht Klemm

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Chapter 1

Introduction

Two major challenges of modern theoretical physics are the explanation for the existence of matter and its properties and the unification of quantum mechanics and gravity. The accepted theoretical framework to describe the dynamics of the fundamental constituents of nature, the so-called elementary particles, is the Standard Model of elementary particle physics. It is a quantum field theory, that describes all relevant interactions of matter at the accessible energy scales to high precision. However, an inclusion of gravity, i.e. quantum gravity, seems not to be possible. Furthermore, the standard model has many free parameters, that are put in by hand to fit the experimental data. There are also a few unsolved fundamental problems like the strong CP-problem and further more aesthetic issues that guide the way to physics beyond the established and verified scope of the standard model.

A first step towards the solution of these problems is to formulate new types of theories taking into account all possible symmetries of nature, namely Supersymmetry. Then one is able to formulate genuine extensions of the standard model, that also provide the opportunity to unify the three forces contained in the standard model as well as the observed elementary particles according to some new principles. These theories are the so-called Supersymmetric Grand Unified Theories, that provide e.g. the possibility to solve the so-called Hierarchy Problem. Then one can even proceed further and try to include also gravity, what leads to the appealing theories of Supergravity.

String Theory goes far beyond the concepts arising from quantum field theory and seems to provide the solutions for the major problems of theoretical physics, at least in principle. The main assumption of the so-called Bosonic String Theory is that the fundamental constituents of nature are one-dimensional vibrating strings. However, the bosonic string alone seems not to be consistent and one has again to take into account supersymmetry to construct the so-called Superstring Theories. The general assumption of the existence of strings is very restrictive. There are only five different consistent superstring theories, namely the Type I, Type IIA, Type IIB and the SO(32) and $E_8 \times E_8$ heterotic strings. All these theories demand spacetime to be 10-dimensional. However, we are living in a four-dimensional world, i.e. four-dimensional spacetime. Thus one may ask how to make contact with the real world from the point of view of superstring theories.

The solution of this problem is called Compactification. The general ansatz is the claim, that the extra dimensions are compactified and shrunk to unobservably small size, such that they seem to be not present at all. The idea of compactification is known to physicists since the work of Kaluza and Klein [34, 36] in the nineteen-twenties, where they unified electromagnetism and Einstein's theory of gravity by adding one compact extra dimension to the four usual dimensions of spacetime. It is still the attempt of modern physics to explain the various properties and parameters of the four-dimensional theories in terms of compactified higher-dimensional theories like the superstrings. The aim is to make very few assumptions for the construction of the higher-dimensional theories, that is the case for the superstring theories, and to explain the whole complicated physical reality in four dimensions by the right choice of the compactification space.

For the heterotic string theories the phenomenologically interesting compactifications are those on the so-called Calabi-Yau manifolds [6]. Many interesting concepts and connections to further aspects of string theory as well as mathematics occur in the context of Calabi-Yau compactification. Furthermore, whole research areas for four-dimensional model building emerge from the consideration of the heterotic strings on Calabi-Yau manifolds and related geometries.

We will discuss a few exciting aspects of these issues here, focusing on special considerations and explicit calculations in the context of the heterotic strings, that are of special phenomenological interest. We will encounter many physically as well as mathematically interesting concepts and ideas, that often even exceed the context of string theory and prove to be relevant for different physical theories and situations.

Overview

This work is divided into two parts, one for the presentation of the preliminaries, that are necessary to understand the subject of our considerations, which we present in the second part. In chapter 2 we begin with the discussion of superstrings, where we emphasise the construction of the heterotic string and introduce our notation. Then we turn to the issue of compactification in chapter 3. We first discuss the general concepts as known from Kaluza-Klein theory, then we will address the compactification of the heterotic string on Calabi-Yau spaces and orbifolds. Chapter 4 is dedicated to the various aspects of anomalies, starting with their basic definitions and proceeding to the their application in string theory. Then we start with the main part of this work in chapter 5 and 6, where we describe in detail first the local orbifold and resolution models and then the general matching description. There we present already some of our calculations, that are necessary for the explicit matching of the six-dimensional heterotic models in chapter 7. We will present all details and calculations of this matching there. Finally, we apply the local analysis to match also global orbifold models with appropriately constructed resolution models in chapter 8. In the appendix we show the tables of the four-dimensional model matching and some trace relations necessary for our calculations.

Part I

Preliminaries: Strings, Compactification and Anomalies

Chapter 2

Superstrings and their Effective Theories

There are many excellent text books on string theory. Therefore, we review the very basics of the construction of superstrings to provide the necessary background for later sections and to introduce our notation. The methods of conformal field theory are omitted for simplicity. Furthermore, we address only the closed string theories in detail.

The standard text books are [26, 27], [50, 51] and [5], which we will follow. The labels of the SO(8) representations applied for the various states of the massless spectra are according to [44].

2.1 The Construction of Superstrings

Our description of the superstring will be those of the so-called Ramond-Neveu-Schwarz string, a formulation using world-sheet supersymmetry, i.e. supersymmetry of the two-dimensional field theory on the string worldsheet, not of spacetime itself.

2.1.1 The Action and Boundary Conditions

We start with the action given in an already convenient form

$$S = \frac{1}{\pi} \int\limits_{\Sigma} d^2\sigma \left(2\partial_+ X^\mu \partial_- X_\mu + i\Psi_-^\mu \partial_+ \Psi_{-\mu} + i\Psi_+^\mu \partial_- \Psi_{+\mu} \right) \tag{2.1}$$

using units, such that the string tension $T \equiv 1/\pi$ [1]. This is a two-dimensional superconformal field theory. The D functions $X : \Sigma \to M_D$ describe the embedding of the two-dimensional worldsheet Σ with coordinates (τ, σ) into the D-dimensional spacetime M_D with coordinates x^μ, $\mu = 0, \ldots, D-1$. For simplicity M_D is chosen to be flat Minkowski

[1] We assume a topologically trivial worldsheet for simplicity, i. e. a vanishing Euler number $\chi(\Sigma) = 0$. Hence, there exist global coordinates and the considerations remain simple.

space with metric $\eta_{\mu\nu}$ of signature $(D-1,1)$. More general situations would lead to the so-called non-linear σ-models, which we do not consider here. It is convenient to use $\partial_\pm = 1/2\,(\partial_\tau \pm \partial_\sigma)$ for light-cone coordinates $\sigma^\pm := \tau \pm \sigma$ on the worldsheet. Ψ^μ denote the superpartners of the X^μ, i.e. D Majorana fermions of the two-dimensional Lorentz group SO$(1,1)$, which are split into the two chirality components $\Psi_\pm{}^\mu$, the Majorana-Weyl spinors.

This action leads to the equations of motion

$$\partial_+\partial_- X^\mu = 0\,, \qquad \partial_+\Psi_- = 0\,, \qquad \partial_-\Psi_+ = 0\,. \tag{2.2}$$

These are solved by left- and right-moving wavesolutions $\Psi^\mu_\pm(\tau,\sigma) \equiv \Psi^\mu_\pm(\tau\pm\sigma)$ and $X^\mu(\tau,\sigma) \equiv X^\mu_{\mathrm{L}}(\tau+\sigma) + X^\mu_{\mathrm{R}}(\tau-\sigma)$.

It is possible to describe closed as well as open strings by imposing appropriate boundary conditions. First of all we introduce a convenient coordinate range $0 \le \sigma \le \pi$.

- Closed strings are subject to the following boundary conditions

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)\,, \tag{2.3}$$
$$\Psi^\mu_+(\tau,\sigma+\pi) = \pm\Psi^\mu_+(\tau,\sigma)\,,\ \Psi^\mu_-(\tau,\sigma+\pi) = \pm\Psi^\mu_-(\tau,\sigma) \tag{2.4}$$

The requirement of (anti-)periodicity of $\Psi^\mu_\pm$ is referred to as Ramond R (Neveu-Schwarz NS) boundary conditions. The mode expansions of the fields then read as

$$X^\mu_{\mathrm{R}} = \tfrac{1}{2}x^\mu_0 + \tfrac{1}{2}p^\mu(\tau-\sigma) + \tfrac{i}{2}\textstyle\sum_{n\neq 0}\tfrac{1}{n}\alpha^\mu_n e^{-2in\sigma^-}\,, \tag{2.5}$$
$$\Psi^\mu_- = \textstyle\sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in\sigma^-} \quad \text{or} \quad \Psi^\mu_- = \textstyle\sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-2ir\sigma^-}\,, \tag{2.6}$$

and for the left-movers X^μ_{L}, Ψ^μ_+ analogously by replacing $\sigma^- = \tau-\sigma \to \sigma^+ = \tau+\sigma$ and taking a new set of expansion coefficients denoted by $\tilde\alpha^\mu_n$, $\tilde d^\mu_n$ and $\tilde b^\mu_r$. Reality of the solutions constrains $(\alpha^\mu_{-n})^\dagger = \alpha^\mu_n$, $(d^\mu_{-n})^\dagger = d^\mu_n$, etc. . Set $\alpha^\mu_0 := \frac{1}{2}p^\mu$ for convenience. The two distinct boundary conditions for the fermions can be imposed for the left- and right-movers $\Psi^\mu_\pm$ independently. Therefore, it is possible to choose R-R, R-NS, NS-R or NS-NS boundary conditions, which are called sectors.

- Open strings are described by demanding boundary conditions

$$\begin{array}{ccc} \text{Neumann} & & \text{Dirichlet} \\ \partial_\sigma X^\mu(\tau,\sigma=0,\pi) = 0 & \text{or} & X^\mu(\tau,\sigma=0,\pi) = X^\mu_{0,\pi} \end{array} \tag{2.7}$$

$$\begin{array}{ccc} \text{Ramond} & & \text{Neveu-Schwarz} \\ \Psi^\mu_+(\tau,\pi) = \Psi^\mu_-(\tau,\pi) & \text{or} & \Psi^\mu_+(\tau,\pi) = -\Psi^\mu_-(\tau,\pi) \\ & \Psi^\mu_+(\tau,0) = \Psi^\mu_-(\tau,0)\,, & \end{array} \tag{2.8}$$

with constant values $X^\mu_{0,\pi}$, which are assigned to the end points $\sigma = 0,\pi$. The boundary conditions fix the various expansion coefficients in (2.5) and (2.6) for left-and right-movers to be equal. One can imagine this as a condition for a standing wave. Hence, there are only the modes α_n^μ, b_r^μ and d_n^μ. The form of the expansions remains the same up to an overall factor of $\frac{1}{\sqrt{2}}$ and the absence of the factor 2 in the exponential. Set also $\alpha_0^\mu := p^\mu$ for convenience. We will only draw the massless spectrum of the open string and just note, that the Dirichlet condition leads to the notion of D-branes. The construction of the Type I string assuring tadopole cancellation is omitted.

The theory is quantised by imposing the canonical equal time (anti-)commutation relations for the fields by setting

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.9}$$
$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, \quad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}. \tag{2.10}$$

for the various oscillator modes. Analogous relations have to be demanded for the right-movers in case of the closed strings. Mixed commutators vanish. This algebra is completely decoupled. Thus we can study the left- and right-moving sectors separately. For later purposes and the analogy to the harmonic oscillator we will refer to oscillators $\alpha_{-n}^\mu, b_{-r}^\mu, d_{-n}^\mu$, $n > 0$ with negative mode label as creation operators.

2.1.2 Constraints and Light-Cone Gauge

There are the so-called generalised Virasoro constraints[2] that one has to require in the classical theory

$$\begin{aligned} J_+ &= \Psi_+^\mu \partial_+ X_\mu = 0\,, \quad J_- = \Psi_-^\mu \partial_- X_\mu = 0\,, \qquad (2.11)\\ T_{++} &= \partial_+ X_\mu \partial_+ X^\mu + \tfrac{i}{2}\Psi_+^\mu \partial_+ \Psi_{+\mu} = 0\,, \quad T_{--} = \partial_- X_\mu \partial_- X^\mu + \tfrac{i}{2}\Psi_-^\mu \partial_- \Psi_{-\mu} = 0\,. \end{aligned}$$

Here J, T denote the supercurrent of world-sheet supersymmetry and the energy momentum tensor, respectively. The labels $+,-$ refer to the components with respect to $\sigma^\pm$.

Instead of imposing these constraints for the corresponding operators on all physical states of the Hilbert space we solve them explicitely. We use the reparametrisation invariance $(\tau, \sigma) \to (\tau, \sigma) + \left(\xi^0(\tau, \sigma), \xi^1(\tau, \sigma)\right)$ of the classical action with

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda(\tau, \sigma)\,\eta^{\alpha\beta}\,, \tag{2.12}$$

where $\alpha, \beta = 0, 1$ denote the worldsheet coordinates and $\eta = \mathrm{diag}(-1, 1)$ the world-sheet metric. This yields $\xi^\pm \equiv \xi^+(\sigma^\pm) = \frac{1}{2}e^{2in\sigma^\pm}$, which precisely corresponds to the conformal symmetry of the two-dimensional field theory. Insertion of a special transformation $\tilde{\tau} = \frac{1}{2}\left[\xi^+(\sigma^+) + \xi^-(\sigma^-)\right]$ into (2.5) cancels all α_n^+, $n \neq 0$, yielding

$$X^+(\sigma, \tau) = x^+ + p^+\tau\,. \tag{2.13}$$

[2]These are just the classical equations of motion of the worldsheet metric and gravitino, which emerge in a more general treatment of the action (2.1), cf. chapter 4.3.5 of [26].

Here we have introduced the light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right)$ and X^i, $i = 1, \ldots, D-2$, for the transverse coordinates. A further fermionic symmetry is used to obtain

$$\Psi^+ \equiv 0 \tag{2.14}$$

for the NS-sector and the non-zero modes in the R-sector. This is the so-called light-cone gauge. Note that this gauge choice is not covariant as we have broken the spacetime Lorentz group SO$(D-1,1)$ explicitely to SO$(D-2)$ (stabiliser or little group). Hence Lorentz symmetry is now a severe constraint for the quantum theory. Consistency requires $D = 10$, otherwise the Lorentz algebra of SO$(D-2)$ does not close: $[J^{i-}, J^{j-}] \stackrel{!}{=} 0$. However, we can be sure about the absence of negative norm states in this gauge, because all symmetries have been fixed. The unphysical polarisation states are no more present.

Finally, inserting (2.13) into the constraints (2.11) we can express the fields X^- and Ψ^- in terms of the transverse fields X^i, Ψ^i. We obtain an algebraic expression for p^- in terms of the transverse oscillators and momenta. This enables us to deduce the mass formula $M^2 = -p^2 = 2p^+p^- - p^ip^i$ for the closed string

$$\frac{M_{\mathrm{R}}^2}{8} = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - a_{\mathrm{NS}}\,, \qquad \frac{M_{\mathrm{R}}^2}{8} = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} n d_{-n}^i d_n^i - a_{\mathrm{R}}\,. \tag{2.15}$$

Replacing 1/8 by 1/2 yields the mass formula for the open string. The mass equations (2.15) apply for the NS- and R-sector, respectively, and normal ordering constants a_{NS}, a_{R} have been introduced. An analogous formula can be derived for M_{L}^2 in terms of the left-moving oscillators. Note that the mass is proportional to the total number $N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i$ of creation operators acting on a given state. Furthermore, we obtain two equations for p^-, one in terms of the modes of the left-moving, the other in terms of the modes of the right-movers. This yields the so-called the level matching condition

$$M_{\mathrm{L}}^2 = M_{\mathrm{R}}^2\,. \tag{2.16}$$

It is valid for all possible combinations of NS- and the R-boundary conditions for the left- and right-moving waves of the closed string. As only the transverse modes $i = 1, \ldots, 8$ are independent degrees of freedom, the corresponding modes create the physical excitations.

2.1.3 The GSO-Projection and the Massless Spectrum

The groundstates $|0\rangle_{\mathrm{R}}$, $|0\rangle_{\mathrm{NS}}$ of the Ramond and Neveu-Schwarz sectors are defined by

$$\alpha_m^i\,|0,k\rangle_{\mathrm{R}} = d_m^i\,|0,k\rangle_{\mathrm{R}} = 0\,, \quad \alpha_m^i\,|0,k\rangle_{\mathrm{NS}} = b_r^i\,|0,k\rangle_{\mathrm{NS}} = 0\,, \quad m, r > 0\,, \tag{2.17}$$

$$\alpha_0^\mu\,|0,k\rangle_{\mathrm{R,NS}} = k^\mu\,|0,k\rangle_{\mathrm{R,NS}}\,. \tag{2.18}$$

The vacuum[3] $|0\rangle_{\mathrm{NS}}$ is degenerated, because d_0^i does not change the mass (2.15) of the state. As the d_0^i satisfy (after some rescaling) the eight-dimensional Clifford algebra, $d_0^i\,|0\rangle_{\mathrm{R}}$ furnishes a spinor representation of SO(8) and can be labelled by a spinor index $|a\rangle$. Therefore,

[3]We will drop the momentum label k in the following.

sector	G-parity	states	SO(8)-representation	field-content
NS	-1	$\|0\rangle_{\mathrm{NS}}$	$\mathbf{1}$	ϕ
	1	$b^i_{-1/2}\,\|0\rangle_{\mathrm{NS}}$	$\mathbf{8}_{\mathrm{V}}$	A_i
R	1	$\|0\rangle_{\mathrm{R}}$	$\mathbf{8}_{\mathrm{S}}$	$\lambda_i^{(+)}$

Table 2.1: The spectrum of the open string. The ground state of the NS-sector is tachyonic and removed by GSO. Positive chirality is chosen for the R-vacuum. The last two columns show the representations under the Lorentz group and the corresponding fields, respectively.

states in the R-sector correspond to spacetime fermions, whereas spacetime bosons occur in the NS-sector. Excited states are obtained by acting on the two groundstates with the creation operators.

Now it is in principal possible to analyse the spectrum for every possible mass level. But this naively constructed spectrum is not tachyon-free in the NS-sector and also not supersymmetric. Hence, we introduce the so-called G-parity by

$$G = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1} \quad (\mathrm{NS})\,, \quad G = \Gamma_{10}\,(-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r} \quad (\mathrm{R}) \tag{2.19}$$

to project out states with $G = -1$ in the NS-sector and $G = 1$ or $G = -1$ in the R-sector. This is the GSO-projection. As $\Gamma_{10} := \Gamma_0 \cdots \Gamma_9$ is the chirality operator the different choices of G in the R-sector correspond to different chiralities of the ground state spinor. Fortunately, this is consistent with string perturbation theory, i. e. G-parity is preserved in interactions.

Now we are ready to analyse the massless spectrum of the superstring theories. First we discuss the open string. There are the two distinct vacua corresponding to the NS- and R-sector. The spectrum is shown up to the massless level in Tab. 2.1. Consider the state $b^i_{-1/2}\,|0\rangle_{\mathrm{NS}}$ in the NS-sector. This is the $\mathbf{8}_{\mathrm{V}}$ of SO(8) corresponding to the on-shell degrees of freedom of a massless vector field in 10 dimensions. For Lorentz covariance of the theory we have to demand a vanishing mass of this state. This fixes the normal ordering constant a_{NS} in the NS-sector to $a_{\mathrm{NS}} = \frac{1}{2}$. Spacetime supersymmetry fixes also $a_{\mathrm{R}} = 0$ providing the vacuum $|0\rangle_{\mathrm{R}}$, i. e. the Majorana-Weyl spinor denoted by $|+\rangle_{\mathrm{R}}$, in $\mathbf{8}_{\mathrm{S}}$ as its massless superpartner. Note that the tachyonic vacuum $|0\rangle_{\mathrm{NS}}$ is removed by the GSO-projection.

As open strings produce also closed strings in interactions, the above discussion is incomplete. However, the complete construction of a consistent open string theory is more involved and omitted here. The resulting theory is referred to as Type I superstring theory. Its massless spectrum contains also the states of the 10-dimensional $\mathcal{N} = 1$ supergravity multiplet. Furthermore, the massless states in Tab. 2.1 have internal degrees of freedom of a SO(32) gauge group. They constitute a $\mathcal{N} = 1$ vector multiplet in 10 dimensions. Hence, the low energy effective theory for the massless fields of type I is described in terms of $\mathcal{N} = 1$ supergravity coupled to a SO(32)-Super-Yang-Mills theory. Note that the occurrence of only one supersymmetry corresponds to the fact there are only the fermionic fields Ψ^i_+ yielding just one supercharge transforming as $\mathbf{8}_{\mathrm{S}}$ in spacetime.

Type IIA

sector	G-parity	states	SO(8)-representation	field-content
NS-NS	$(1,1)$	$\tilde{b}^i_{-1/2}\,\lvert 0\rangle_{\rm NS}\otimes b^i_{-1/2}\,\lvert 0\rangle_{\rm NS}$	$\mathbf{35}_{\rm V}+\mathbf{28}+\mathbf{1}$	$G_{ij}{+}B_{ij}{+}\Phi$
NS-R	$(1,1)$	$\tilde{b}^i_{-1/2}\,\lvert 0\rangle_{\rm NS}\otimes\lvert +\rangle_{\rm R}$	$\mathbf{56}_{\rm S}+\mathbf{8}_{\rm C}$	$\Psi^{(+)}_i{+}\Psi^{(+)}$
R-NS	$(-1,1)$	$\lvert -\rangle_{\rm R}\otimes b^i_{-1/2}\,\lvert 0\rangle_{\rm NS}$	$\mathbf{56}_{\rm C}+\mathbf{8}_{\rm S}$	$\Psi^{(-)}_i{+}\Psi^{(-)}$
R-R	$(-1,1)$	$\lvert -\rangle_{\rm R}\otimes\lvert +\rangle_{\rm R}$	$\mathbf{56}_{\rm V}+\mathbf{8}_{\rm V}$	$C_{ijk}{+}C_i$

Type IIB

sector	G-parity	states	SO(8)-representation	field-content
NS-NS	$(1,1)$	$\tilde{b}^i_{-1/2}\,\lvert 0\rangle_{\rm NS}\otimes b^i_{-1/2}\,\lvert 0\rangle_{\rm NS}$	$\mathbf{35}_{\rm V}+\mathbf{28}+\mathbf{1}$	$G_{ij}{+}B_{ij}{+}\Phi$
NS-R	$(1,1)$	$\tilde{b}^i_{-1/2}\,\lvert 0\rangle_{\rm NS}\otimes\lvert +\rangle_{\rm R}$	$\mathbf{56}_{\rm S}+\mathbf{8}_{\rm C}$	$\Psi^{(+)}_i{+}\Psi^{(+)}$
R-NS	$(1,1)$	$\lvert +\rangle_{\rm R}\otimes b^i_{-1/2}\,\lvert 0\rangle_{\rm NS}$	$\mathbf{56}_{\rm S}+\mathbf{8}_{\rm C}$	$\tilde{\Psi}^{(+)}_i{+}\tilde{\Psi}^{(+)}$
R-R	$(1,1)$	$\lvert +\rangle_{\rm R}\otimes\lvert +\rangle_{\rm R}$	$\mathbf{35}_{\rm s}+\mathbf{28}+\mathbf{1}$	$C_{ijkl}{+}C_{ij}{+}C$

Table 2.2: The spectrum of the two Type II superstrings. The second column shows the G-parity of the left- and right-moving sector, respectively. The various $C_{\dots}$-fields denote p-form fields. C_{ijkl}, i.e. $\mathbf{35}_{\rm s}$, denotes the self-dual part only.

Considering the closed string we have to combine the left- and right-moving states, apply the GSO-projection according to (2.19), the level matching condition (2.16) and the mass formula $M^2 = M^2_{\rm L} + M^2_{\rm R}$. One can combine the two different choices for the chirality (G-parity) of the vacuum $\lvert 0\rangle_{\rm R}$ denoted by $\lvert\pm\rangle_{\rm R}$ for the left- and right-moving waves in two inequivalent ways. These two theories are called the Type IIA and Type IIB superstring, respectively. The corresponding massless spectra are given in Tab. 2.2. The SO(8) representations are determined by simply tensoring the relevant representations[4] $\mathbf{8}_{\rm V}$, $\mathbf{8}_{\rm S}$ and $\mathbf{8}_{\rm C}$ of SO(8).

In both cases there are two supercharges corresponding precisely to the independent left- and right-moving fermionic waves $\Psi^i_\pm$. Hence, there are two gravitini corresponding to the two $\mathcal{N}=2$ supergravity theories in $D=10$, also called Type IIA and Type IIB supergravity. Hence, the states in the first (second) box of Tab. 2.2 are precisely organised in the corresponding supergravity multiplets in $D=10$ with the splitting into fermionic and bosonic matter:

$$\textbf{Type IIA}:\ G_{ij}, B_{ij}, \Phi, C_i, C_{ijk}\quad\text{(bosonic)};\ \Psi^{(+)}_i, \Psi^{(-)}_i, \Psi^{(+)}, \Psi^{(-)}\ \text{(fermionic)} \tag{2.20}$$

$$\textbf{Type IIB}:\ G_{ij}, B_{ij}, \Phi, C, C_{ij}, C_{ijkl}\ \text{(bosonic)};\ \Psi^{(+)}_i, \tilde{\Psi}^{(+)}_i, \Psi^{(-)}, \tilde{\Psi}^{(-)}\ \text{(fermionic)}. \tag{2.21}$$

The effective theories of the massless states of these superstrings are described by the two $\mathcal{N}=2$ supergravity theories in 10 dimensions. The bosonic fields are the 10D metric G_{ij}, the B-field, the dilaton Φ and additional form fields. The Type IIA theory is vector-like and therefore trivially anomaly-free as it contains spinors of both chiralities and consequently

[4]The decomposition into irreducible representations is according to section 10.5 of [51].

is parity symmetric. However, Type IIB is chiral containing only spinors of one chirality. Anomaly freedom has to be analysed as done in section 4.2.

2.2 The Heterotic Strings

The construction of the superstring shows explicitely, (2.1) and (2.9), (2.10), that left- and right-moving sectors are completely decoupled. Treating the two-dimensional world-sheet action more abstractly as an action for a conformal field theory (CFT) one might try to choose arbitrary and different CFTs for each of the two sectors. In many cases this generalised string action has no geometrical interpretation in terms of a string propagating in spacetime. Then the consistency of the two-dimensional CFT restricts the possible choices for a generalised action. One obvious consistency condition is the requirement of conformal invariance of the two-dimensional quantum theory. A break-down of conformal symmetry during quantization is expressed by the so-called Weyl anomaly. This anomaly has to vanish, which requires a vanishing central charge of the underlying CFT. The requirement of anomaly freedom is also an alternative way to determine the dimensionality of spacetime D for the bosonic string or the superstring. Using covariant quantization rather than the light-cone gauge (2.13) one has to use Fadeev-Popov ghosts to cancel unphysical degrees of freedom. They contribute a central charge of $c_{\text{ghost}} = -26$ or $c_{\text{ghost}} = -15$ for the bosonic string or superstring, respectively. As each bosonic field X^μ contributes $c = 1$ and each fermionic Ψ^μ a charge $c = 1/2$ the number of fields D, i. e. the dimensionality of spacetime, is fixed by $D = 26$ or $D = 10$ yielding $c_{\text{tot}} = 0$.

This few statements about conformal field theory are the necessary ingredients to understand the two different constructions of the heterotic strings[5].

2.2.1 Two Constructions of the Heterotic String

The heterotic string theories are more abstract formulations of consistent superstring theories. There are two inequivalent heterotic string theories, the SO(32) and $E_8 \times E_8$ heterotic string, both being closed string theories. They are constructed using the degrees of freedom of the bosonic string for the left-movers and those of the superstring for the right-movers. Therefore, the fields of the left-movers have to contribute $c = 26$ for a vanishing central charge $c_{\text{tot}} = c_{\text{ghost}} + c$ and Weyl anomaly. This can be obtained in two ways, either using the so-called fermionic formulation with 32 left-moving fermions λ^A ($c = 32 \times 1/2 = 16$) or the so-called bosonic formulation with 16 left-moving bosons X_{L}^I ($c = 16 \times 1 = 16$), cf. Tab. 2.3. The fermionic description is clearly a 10-dimensional theory as there are both 10 left- and right-movers $X_{\text{L},}^\mu$, X_{R}^μ, that can be interpreted as the coordinates of the string in spacetime. The additional fields λ^A simply correspond to internal degrees of freedom. However, this geometric picture breaks down for the bosonic formulation, at least at first glance, because there are 26 left-movers $X_{\text{R}}^{\mu,I}$ and 10 right-movers X_{L}^μ. To overcome this

[5]For more details about conformal field theory, see e.g. [22].

	left-movers	right-movers	#(bosons X)
Bosonic formulation	$X_{\mathrm{L}}^{\mu}(\sigma^{+})$, $X_{\mathrm{L}}^{I}(\sigma^{+})$	$X_{\mathrm{R}}^{\mu}(\sigma^{-})$, $\Psi_{-}(\sigma^{-})$	$(26, 10)$
Fermionic formulation	$X_{\mathrm{L}}^{\mu}(\sigma^{+})$, $\lambda_{+}^{A}(\sigma^{+})$	$X_{\mathrm{R}}^{\mu}(\sigma^{-})$, $\Psi_{-}(\sigma^{-})$	$(10, 10)$

Table 2.3: The massless spectrum of the heterotic string. $\mu = 0, \ldots, 9$, $I = 1, \ldots, 16$ and $A = 1, \ldots, 32$ label the fields. The naive dimension of spacetime according to the left- and right-movers $X_{\mathrm{L,R}}$ is stated in the last column.

lack of interpretability one compactifies the 16 fields X_{L}^{I} on a 16-dimensional torus T^{16}. Then only the fields X_{L}^{μ} are considered to correspond to coordinates in a 10-dimensional spacetime. We choose this bosonic construction of the heterotic strings to provide the appropriate language for the description of the orbifold in section 3.2.2.

2.2.2 The Bosonic Formulation

A torus T^{16} is constructed by $\mathbb{R}^{16}/(2\pi\Lambda)$ with a 16-dimensional lattice Λ, spanned by vectors e_i, $i = 1, \ldots, 16$. The boundary conditions of a closed string now include also a winding around the torus, i. e.

$$X_{\mathrm{L}}^{I}(\tau, \sigma + \pi) = X_{\mathrm{L}}^{I}(\tau, \sigma) \mod 2\pi L^{I} \tag{2.22}$$

for $L = \sum_{i=1}^{16} n_i e_i$, $n_i \in \mathbb{Z}$. This is the winding number, describing how many times the string winds around the respective torus cycle. Then we use a mode expansion of the X_{L}^{I} to obtain

$$p_{\mathrm{L}} = L \in \Lambda \,. \tag{2.23}$$

Thus p_{L} is characterised by the lattice Λ. Consistency of the string theory, namely modular invariance, requires the following conditions for the lattice:

$$\begin{aligned} \text{self-dual} \quad &: \quad (\Lambda)^{*} = \Lambda \,, && (2.24) \\ \text{even} \quad &: \quad \lambda \in \Lambda \Rightarrow \lambda^{2} = 0 \mod 2 \,. && (2.25) \end{aligned}$$

There are only two 16-dimensional lattices fulfilling this conditions:

- the Spin(32)/$\mathbb{Z}_2$ lattice containing only the conjugacy class of the root lattice ($\lambda^2 = 2$) and of one of the two spinors ($\lambda^2 = 4$).
- the $\mathrm{E}_8 \times \mathrm{E}_8$-root lattice.

This will explain the presence of gauge bosons in the spectrum of the heterotic string. Note first that the mass formula is now given by

$$\frac{M_{\mathrm{L}}}{8} = \sum_{n=1}^{\infty} \left(\tilde{\alpha}_{-n}^{i}\tilde{\alpha}_{n}^{i} + \tilde{\alpha}_{-}^{I}\tilde{\alpha}_{n}^{I}\right) + \frac{1}{2}p_{\mathrm{L}}^{I}p_{\mathrm{L}}^{I} - 1 \,, \tag{2.26}$$

Heterotic String

sector	states	SO(8)-representation	R	field-content
external-NS	$\tilde{\alpha}^i_{-1}\|0\rangle_{\rm L} \otimes b^i_{-1/2}\|0\rangle_{\rm NS}$	$\mathbf{35}_{\rm V}+\mathbf{28}+\mathbf{1}$	$\mathbf{1}$	$G_{ij}+B_{ij}+\Phi$
external-R	$\tilde{\alpha}^i_{-1}\|0\rangle_{\rm L} \otimes \|+\rangle_{\rm R}$	$\mathbf{56}_{\rm S}+\mathbf{8}_{\rm C}$	$\mathbf{1}$	$\Psi^{(+)}_i+\Psi^{(+)}$
internal-NS	$\left.\begin{matrix}\tilde{\alpha}^I_{-1}\|0\rangle_{\rm L}\\ \|p_{\rm L}^2=2\rangle\end{matrix}\right\} \otimes b^i_{-1/2}\|0\rangle_{\rm NS}$	$\mathbf{8}_{\rm V}$	$\mathbf{Adj}$	A^I_i, A^p_i
internal-R	$\left.\begin{matrix}\tilde{\alpha}^I_{-1}\|0\rangle_{\rm L}\\ \|p_{\rm L}^2=2\rangle\end{matrix}\right\} \otimes \|+\rangle_{\rm R}$	$\mathbf{8}_{\rm S}$	$\mathbf{Adj}$	$\chi^{(+)I}$, $\chi^{(+)p_{\rm L}}$

Table 2.4: The spectrum of the two heterotic superstrings. G denotes the respective gauge group, namely SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$ and R the representation of the states under G. "external" and "internal" denotes, whether the oscillator $\tilde{\alpha}_{-1}$ is tangential to T^{16} or to spacetime.

because we consider only the bosonic string and the $X^I_{\rm L}$ enter also into the solution of (2.11). The level matching condition (2.16) still applies even for mass formulæ of the bosonic string for $M^2_{\rm L}$ and of the superstring for $M^2_{\rm R}$. This guarantees for example a tachyon free spectrum.

Massless States and the Effective Theory

The massless states of the left-movers are obtained either by acting with one oscillator $\tilde{\alpha}_{-1}$ or by demanding $p^2_{\rm L}=2$. The states are given in Tab. 2.4.

It is known from the theory of Lie algebras that the roots can be represented by weight vectors λ with $\lambda^2=2$. Hence, the massless states with $p^2_{\rm L}=2$ correspond to gauge bosons $A^{p_{\rm L}}_i$ of SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$, because of the occurring lattices Λ given above. The neutral gauge bosons A^I_i are given by the states with one oscillator $\tilde{\alpha}^I_{-1}$, $I=1,\ldots 16$, and correspond to the Cartan subalgebra of dimension 16. All these fields, generically denoted by A^a_i, compose the gauge bosons of a SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$ gauge theory. The spectrum is supersymmetric due to the right-movers yielding a complete $\mathcal{N}=1$ vector multiplet. The fields $\chi^{(+),a}=\chi^{(+)I}$, $\chi^{(+)p_{\rm L}}$ are the corresponding gauginos. Futhermore, there is the $\mathcal{N}=1$ supergravity multiplet. Hence, the low energy effective theory of the heterotic strings is described by $\mathcal{N}=1$ supergravity coupled to a Super-Yang-Mills theory. The field content of this theory is given by

$$\textbf{Heterotic String}: \; G_{ij}, B_{ij}, \Phi, A^a_i \;\; \text{(bosonic)}; \;\; \Psi^{(+)}_i, \Psi^{(+)}, \chi^{(+),a} \;\; \text{(fermionic)}\,. \tag{2.27}$$

It will turn out that anomaly freedom of this effective theory is highly non-trivial and puts severe constraints on the choices of a gauge group G of its Super-Yang-Mills sector. At a first naive glance G could be arbitrary. Although $\mathcal{N}=1$ supergravity alone is not anomaly-free, the coupled supergravity/Super-Yang-Mills theory[6] is, but only for very special choices of G. It is a promising coincidence at this point that the gauge groups of the heterotic string, SO(32) and $E_8 \times E_8$, are among those allowed gauge groups G, compare section 4.2. However, in the limited context of supergravity theories the occurrence of $496 = \dim(G)$ is a very surprising numerical coincidence. But it is the existence of the underlying consistent heterotic strings that clarifies the mystery of the anomaly freedom of these coupled supergravity/Super-Yang-Mills theories.

[6]However, it is important to note here that it is not the minimal form of the action, which is needed for anomaly freedom. "Stringy" extra terms of higher order in $(\text{length})^{-1}$ have to be included.

Chapter 3

Generality of Compactification

The general concepts of compactification of higher-dimensional theories according to the construction of the so-called Kaluza-Klein theories is one subject of this chapter. We will apply these concepts to the relevant examples of our further considerations.
These are p-form fields, chiral fermions and non-Abelian gauge fields. As an application of some ideas of the first part we will present the Calabi-Yau compactification of the heterotic string in the second part of this chapter. We will see in detail, how the basic idea of $\mathcal{N} = 1$ supersymmetry in 4D restricts the compact internal manifold as well as the geometry of the four extended dimensions. Finally, we will address the compactification on orbifolds and the principles of the construction of heterotic models in four and six dimensions.

3.1 A Modern Perspective on Kaluza-Klein Theories

The mathematical background for this section is provided by the concepts of differential geometry and the basics of topology. References with emphasis on the physical applications can be found in [37] and [15].

The following presentation is based on the detailed review [14], where almost all relevant further references can be found. A briefer review can be found in [13] with more emphasis on the conceptional ideas instead of outlining all technicalities. A very readable and general description of compactification can be found in chapter 14 of [27].

3.1.1 Concepts of Compactification

First we present the general concept of Kaluza-Klein theories from a modern point of view. We start with D-dimensional spacetime M_D with coordinates z^M labeled by indices $M = 0, \ldots, D-1$. Then, we work according to the following general recipe.

(i) Choosing a theory:

The starting point is the choice of a theory in D-dimensional spacetime M_D. Every theory that we consider here contains also gravity described by the metric g_{MN} with signature $(D-1, 1)$, but also additional matter fields generically denoted by Φ.

(ii) The vacuum state:

Now we have to specify, what is meant by the notion of extended four-dimensional spacetime and additional compact dimensions. The field equations of the underlying theory have to admit vacuum configurations, i.e. ground states $\langle g_{MN}\rangle$ and $\langle\Phi\rangle$, which yield a D-dimensional spacetime of the form $M_D = M_4 \times K$. Here M_4 denotes a four-dimensional[1] non-compact manifold with metric g of signature (3,1) and K a $(D-4)$-dimensional compact Riemannian manifold. Such a case of the vacuum structure is referred to as spontaneously compactified. Then the background metric $\langle g_{MN}\rangle$ is given by

$$\langle g_{MN}(x,y)\rangle = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & G_{mn}(y) \end{pmatrix}, \tag{3.1}$$

where $z = (x^\mu, y^m)$, $\mu = 0,\dots 3$, $m = 1,\dots,D-4$, denote the coordinates of M_4 and K, respectively. A possible warping by the replacement $g_{\mu\nu}(x) \to g_{\mu\nu}(x)f(y)$ is excluded from our discussion. Furthermore, we require maximal symmetry for M_4, that yields de Sitter spacetime (cosmological constant $\Lambda > 0$), flat Minkowski space $\mathbb{M}_4$ $(\Lambda = 0)$ and anti-de-Sitter space $\Lambda < 0)$ for M_4. These are the spaces of constant curvature with curvature tensor

$$R_{\mu\nu\kappa\lambda} = \frac{\Lambda}{3}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right). \tag{3.2}$$

These manifolds are also four-dimensional Einstein spaces as we observe $Ric_{\mu\nu} := g^{\kappa\lambda}R_{\kappa\mu\lambda\nu} = \Lambda g_{\mu\nu}$.

A direct consequences of (3.1) is the splitting of the curvature two-form into a block-diagonal form

$$R_{(\mu\nu)} = R_{(\mu\nu)} \oplus 0 = \begin{pmatrix} R_{(\mu\nu)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_{(mn)} = 0 \oplus \mathcal{R}_{(mn)} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{R}_{(mn)} \end{pmatrix}, \tag{3.3}$$

where we denote the background curvature by $\mathcal{R}$ and the four-dimensional one by R. Hence wedge products of R and $\mathcal{R}$ vanish, what will be applied in concrete calculations in the following chapters.

(iii) Fluctuations and Linearisation:

After the choice of a ground state, we wish to determine the behaviour of the in general non-linear theory, e.g. gravity or non-Abelian gauge theories, near this vacuum. Therefore, we consider small fluctuations of the D-dimensional fields about the vacuum

$$\begin{aligned} g_{MN}(x,y) &= \langle g_{MN}(x,y)\rangle + h_{MN}(x,y), &(3.4)\\ \Phi(x,y) &= \langle\Phi(x,y)\rangle + \phi(x,y) &(3.5)\end{aligned}$$

and keep only terms linear in the fluctuations after the insertion of this ansatz into the D-dimensional equations of motion.

[1] Of course, everything mentioned in this section immediately applies to the case of a six-dimensional external space M_6 as needed for chapter 5 and the following ones.

(iv) Harmonic expansion:

Every field (fluctuation) transforms in an irreducible representation R_D of the D-dimensional tangent space group $\mathrm{SO}(D-1,1)$, i.e. under local Lorentz transformations. These are the tensor- and spinor-representations[2] of $\mathrm{SO}(D-1,1)$. The breaking

$$\mathrm{SO}(D-1,1) \to \mathrm{SO}(3,1) \times \mathrm{SO}(D-4) \tag{3.6}$$

due to compactification implies the branching $R_D \to \oplus_i (R_4^i, R^i)$ into irreducible representations (irreps) R_4^i, R^i of the respective subgroups, labelled by i. This induces the decomposition of the D-dimensional field $\Psi_{R_D}(x,y)$ as $\Psi_{R_D}(x,y) = \sum_i \Psi_{(R_4^i,R^i)}(x,y)$. In the following we will omit labels of the irreps for simplicity. Now one performs the so-called harmonic expansion by the product ansatz

$$\Psi_{(R_4,R)}(x,y) = \sum_l \psi_{R_4}^{(l)}(x) Y_R^{(l)}(y)\,, \tag{3.7}$$

where $\psi_{R_4}^{(l)}$ denote four-dimensional fields in the representation R_4 of $\mathrm{SO}(3,1)$, e.g. forms or spinors. $Y_R^{(l)}$ are the eigenfunctions of some linear operator $\tilde{\Delta}$ on K:

$$\tilde{\Delta}_K Y_R^{(l)}(y) = m_l^2 Y_R^{(l)}(y)\,. \tag{3.8}$$

This operator stems from the D-dimensional equations of motion for the original field Ψ_{R_D} after splitting the occurring wave operator into a four-dimensional $\tilde{\Delta}_4$ and an internal $\tilde{\Delta}_K$. For the examples, that we will consider here, they will commute. Then we understand (3.7) as a simultaneous diagonalisation of $\tilde{\Delta}_4$ and $\tilde{\Delta}_K$. The internal part $\tilde{\Delta}_K$ plays the role of a mass-operator for the resulting four-dimensional equations of motion for $\psi_{R_4}^{(l)}$.

(v) Massless modes and consistency:

Usually, there is a discrete but infinite set of eigenfunctions of $\tilde{\Delta}_K$ in the case of a compact K. Thus, the spectrum of the four-dimensional theory contains an infinite tower of massive fields with discrete masses of order R^{-1}, for R denoting the compactification scale . However, the number of eigenfunctions with vanishing eigenvalue m_l^2 is finite in many cases. Consequently, there is a finite number of massless four-dimensional fields. Due to the in general huge magnitude of the mass scale R^{-1} only the massless fields of the effective four-dimensional theory are possible candidates for the observed elementary particles. Their observed masses have to originate from other effects like breaking of four-dimensional supersymmetry or suppressed loop-corrections at a much lower mass scale. However it is in general not allowed to simply discard all massive modes as the consistency of the theory can

[2] We have to consider $\mathrm{SO}(D-1,1)$, because we are interested in fields transforming under tensor- as well as spinor-representations. However, spinor representations are only possible for $\mathrm{SO}(D-1,1)$ and not for $\mathrm{GL}(D,\mathbb{R})$, generated by coordinate transformations via $\partial z^M/\partial z'^N$ on the tangential space T_pM_D at $p \in M_D$. Fortunately, one can always change between these two groups by means of the vielbein-matrix e_M^A on every Riemannian manifold.

be affected. The massive fields have to be integrated out[3] in the four-dimensional Lagrangian $\mathcal{L}_4$, which is obtained from the D-dimensional $\mathcal{L}_D$ by integration as

$$\mathcal{L}_4 = \int_K d^{(D-4)}y\, \mathcal{L}_D\,. \tag{3.9}$$

3.1.2 Compactification of p-Form Fields

After the introduction of the general recipe we look at the simplest possible example, a bosonic field in the background geometry of (3.1). We consider a k-form $F(x,y)$ as it occurs for example in the higher-dimensional supergravity theories, cf. chapter 2. Furthermore, we suppose $\langle F\rangle = 0$ for simplicity. Now we consider the action governed by the kinetic term

$$S = \int_{M_D} H \wedge *H\,, \tag{3.10}$$

where $H = dF$ denotes the field strength for F. In the gauge $d^*F = 0$, a generalisation of the Lorentz-gauge for one-forms, this yields the equation of motion for F

$$(dd^* + d^*d)\,F = \Delta_D F = (\Delta_4 + \Delta_K)\,F = 0\,, \tag{3.11}$$

with Δ_D, Δ_4 and Δ_K denoting the Hodge-de Rham Laplacians on M_D, M_4 and K, respectively. Noting that the components of F transforms as a k-times antisymmetric tensors $\binom{D}{k}$ of $\mathrm{SO}(D-1,1)$, we make the group theoretical decomposition

$$\binom{D}{k} \to \left(\binom{4}{0},\binom{D-4}{k}\right) \oplus \left(\binom{4}{1},\binom{D-4}{k-1}\right) \oplus \ldots \oplus \left(\binom{4}{4},\binom{D-4}{k-4}\right) \tag{3.12}$$

as described in (iv). This corresponds to a decomposition of F into $(n,k-n)$-forms with n legs on M_4 and $k-n$ legs on K by simply splitting the index M into μ and m. Δ_K acts as a mass operator in (3.11) on four-dimensional fields. According to (3.7) we expand $F_{(n,k-n)}$ into harmonic $(k-n)$-forms $\omega^{(l)}_{k-n} \in \mathrm{Harm}^{k-n}(K)$ on K

$$B_{(n,k-n)}(x,y) = \sum_{i=1}^{b_{k-n}} f^{(i)}_n(x)\,\omega^{(i)}_{k-n}(y), \tag{3.13}$$

with $i = 1,\ldots,b_{k-n}$ and b_{k-n} a Betti number of K. These are precisely the zero-modes of Δ_K. Hence, we obtain b_{k-n} massless four-dimensional fields $f^{(i)}$ for every possible value[4] $k-n$, i.e. b_k scalars, b_{k-1} vectors, etc. . As $\mathrm{Harm}^r(K) \cong H^r(K)$ the number of zero-modes depends only on the topological properties given by the dimension b_r of the cohomology-group $H^r(K)$. Possible applications of this are for example the description of axions from superstring theories, as we will see in section 6.1.3 explicitly.

[3] For example at tree level, this integration means, that one has to minimise the action with respect to the massive fields and then to insert the obtained expressions into it, cf. chapter 17.4 of [51].

[4] It is necessary to dualise the four-dimensional forms to encode the physical degrees of freedom. E.g. a two-form in four dimensions is dual to a pseudo-scalar, that is a possible axion [45, 46].

3.1.3 Chiral Fermions

A further example is the treatment of spinors Ψ on the background (3.1). Hence, K has to exhibit a spin structure. For this purpose one has to consider the decomposition of the spinor representation of $\mathrm{SO}(D-1,1)$ under the breaking into $\mathrm{SO}(3,1)\times SO(D-4)$. The details depend strongly on the concrete value D as different types of spinor representations are possible in different dimensions[5]. Thus, we consider the physically relevant example of $D=10$ and a Dirac spinor $\mathbf{32_{Dirac}}$, which decomposes as

$$\begin{aligned}\mathbf{32_{Dirac}} = \mathbf{16}\oplus\mathbf{16'} \quad\rightarrow\quad & (\mathbf{2},\mathbf{4})\oplus(\mathbf{2'},\mathbf{4'})\oplus(\mathbf{2'},\mathbf{4})\oplus(\mathbf{2},\mathbf{4'}) = (\mathbf{2}\oplus\mathbf{2'},\mathbf{4}\oplus\mathbf{4'}) \\ & = (\mathbf{4_{Dirac}},\mathbf{8_{Dirac}})\end{aligned} \tag{3.14}$$

under $\mathrm{SO}(9,1)\rightarrow\mathrm{SO}(3,1)\times\mathrm{SO}(6)$. Here $\mathbf{2}$, $\mathbf{2'}$ and $\mathbf{4}$, $\mathbf{4'}$ denote the Weyl spinors of both chiralities of $\mathrm{SO}(3,1)$ and $\mathrm{SO}(6)$, respectively, and $\mathbf{4_{Dirac}}$, $\mathbf{8_{Dirac}}$ the corresponding Dirac spinors. We further work with $\langle\Psi\rangle = 0$. Omitting the details the procedure is very similar to our first example. One has to split the 10-dimensional Dirac equation $i\not{D}\Psi(x,y)=0$ into a four-dimensional part $i\not{D}_4$ and an internal $i\not{D}_6$, which acts as a mass operator in four dimensions. Here $i\not{D} := i\Gamma^M\nabla_M$ denotes the 10-dimensional Dirac operator defined using the covariant derivative ∇_M. $i\not{D}_4$ and $i\not{D}_6$ are defined analogously. According to (3.14) we make the expansion (3.7) of the 10-dimensional Dirac spinor into a product of a four-dimensional Dirac spinor $\psi_{\mathbf{4}}^{(k)}$ and zero-modes $\xi_{\mathbf{8}}^{(k)}$ of $i\not{D}_6$:

$$\Psi_{(\mathbf{4},\mathbf{8})}(x,y) = \sum_k \psi_{\mathbf{4}}^{(k)}(x)\otimes\xi_{\mathbf{8}}^{(k)}(y)\,. \tag{3.15}$$

Thus for every zero-mode $\xi^{(k)}$ of $i\not{D}_6$ one obtains a massless four-dimensional spinor. Then one can split the zero modes according to their chirality and count the number of positive and negative chirality states by ν_+ and ν_-, respectively. The so-called chiral asymmetry $\nu_+-\nu_-$ is just the Dirac index and determined by means of the Atiyah-Singer index theorem[6] on K. This also applies for spinors coupled to gauge fields, where the index of the so-called twisted spin complex occurs. We will demonstrate its relevance in the following chapters and how concrete results can be obtained. This is another example, how the interplay between differential geometry and topology occurs in physics.

3.1.4 Gauge Fields with Internal Background

However, the necessity of the complete recipe (i)-(v) is mostly obvious for the non-linear theories like gravity or Yang-Mills theories. Hence, we analyse now D-dimensional Yang-Mills fields $i\mathfrak{A}$ of some gauge group G. To maintain four-dimensional Lorentz-invariance, we have to choose a non-zero vacuum configuration $i\mathcal{A} := \langle i\mathfrak{A}^a T^a\rangle$ on K only, with the generator $T^a\in\mathcal{J}$ for a subgroup $\mathcal{J}\subset G$. Then a linearisation $i\mathfrak{A} = i\mathcal{A}+iA$ leads to charged four-dimensional gauge bosons and scalar fields.

[5]For details on spinors in various dimensions see App. B of [51].

[6]Compare to 12.6.2 of [37]) and the detailed discussion in chapter 14.2 of [27].

Four-dimensional (massless) gauge bosons iA_μ are functions on K and zero-modes of the mass-operator given by the scalar Laplacian $\nabla_m\nabla^m$ built of the (gauge-)covariant derivative ∇_m on K. Thus, they have to correspond to neutral fields under $\mathcal{J}$ as only then the Laplacian has any zero modes. Therefore, the remaining four-dimensional gauge group is the maximal commuting subgroup H, such that $H \times \mathcal{J} \subset G$. This can be seen by the decomposition

$$\mathbf{Adj}(G) \to (\mathbf{Adj}(\mathbf{1}, \mathcal{J})) + (\mathbf{Adj}(H), \mathbf{1}) + \dots \tag{3.16}$$

of the adjoint representation of G under $H \times \mathcal{J}$. The $\mathbf{Adj}(H)$ is always neutral under $\mathcal{J}$, so that these fields occur as the massless gauge bosons in four dimensions. In general, no more singlets of a semi-simple $\mathcal{J}$ are contained in (3.16). Let us note, that for $\mathcal{J}$ containing some U(1)-factors $\mathrm{U}(1) \subset \mathcal{J}$ we have also $\mathrm{U}(1) \in H$ as it a U(1) commutes with itself by definition. The same group U(1) occurs both for the background $i\mathcal{A}$ and for four-dimenional fields iA. This is special for Abelian U(1)-factors in the background bundle $\mathcal{J}$.

The other, omitted representations in (3.16) possibly yield charged matter fields A_m in four dimensions. However, we will not work with gauge bosons but their superpartners, the gauginos, in the following chapters. There we will use the simple formulæ of the Dirac index to calculate the four-dimensional multiplicity of any matter field originating from (3.16).

3.2 Compactification of the Heterotic Strings

The low energy effective theory of the heterotic string is given by the coupled supergravity/ Super-Yang-Mills system in 10D. This is a theory with $\mathcal{N} = 1$ supersymmetry. To obtain a reasonable four-dimensional theory by compactification of this theory[7] on a six-dimensional internal manifold K at most $\mathcal{N} = 1$ supersymmetry should survive in four dimensions. Otherwise there are no chiral fermions as the supermultiplets of theories with more supersymmetry are always non-chiral. Furthermore, there are phenomenological reasons to obtain precisely $\mathcal{N} = 1$ supersymmetry. One example is the attempt to obtain the appealing supersymmetric Grand Unified Theories (GUT) in the context of string compactification. Then supersymmetry has to remain unbroken up to the TeV-scale. Another reason is the necessity of massless scalars after compactification of string theory to provide candidates for the Higgs field with a mass of the order of 100 GeV. Massless bosons emerge naturally due to compactification with remaining unbroken supersymmetry in chiral multiplets as the superpartners of massless fermions.

In this chapter we will present choices for K and gauge backgrounds on it such that $\mathcal{N} = 1$ supersymmetry is unbroken in 4D after compactification of the heterotic string. This was done first in [6]. The allowed smooth spaces K are known as Calabi-Yau manifolds

[7] We only discuss the point-particle, i.e. field theory aspects of this compactification. The chosen approach shall underline, that even for understanding string theory many concepts of point-particle physics are needed, as long as stringy effects like winding of strings are neglected.

that are very special and mathematically difficult objects. The necessary mathematical background material and further details of Calabi-Yau manifolds can be found in [33]. An introduction in the context of string theory can be found in [18] and [28].

After the discussion of Calabi-Yau compactification we will also address compactification on orbifolds [10, 11], which is a much easier compactification scheme. There we will emphasise the model building aspects, whereas we will also describe in principle, how model building on Calabi-Yau manifolds works.

3.2.1 Calabi-Yau Compactification

We consider the heterotic string compactified to 4D on $M_{10} = M_4 \times K$. Again we demand a maximally symmetric M_4. Then we recover AdS , flat Minkowski and dS space as possible choices for M_4, compare section 3.1.1. This restricts the allowed background configurations for the 10-dimensional fields to obey

$$H_{\mu\nu\kappa} = H_{\mu\nu k} = H_{\mu nk} = 0 \qquad \text{and} \qquad \mathfrak{F}_{\mu\nu} = \mathfrak{F}_{\mu n} = 0\,, \tag{3.17}$$

where we use the notation of section 3.1. Here we omitted brackets to indicate the background values of the 10D three-form field strength of the B-field and the 10D Yang-Mills field strength $i\mathfrak{F}$. Additionally, fermionic fields are not allowed to have non-trivial background values as this would break Lorentz invariance explicitly.

In a modern language the identification of an internal space K preserving supersymmetry can be stated equivalently as a vacuum state $|\Omega\rangle$, that is left invariant under a supersymmetry transformation: $Q\,|\Omega\rangle = 0$, where Q denotes the supercharge of unbroken supersymmetry. This is equivalent to the statement that the supersymmetry variation $\delta\Psi := \{Q, \Psi\}$ of all fermionic operators Ψ vanishes in this specific background. The variation of bosonic operators is fermionic and has to vanish identically for Lorentz invariance. Thus, we demand $\langle\Omega|\,\delta\Psi\,|\Omega\rangle = 0$, what reduces at tree level to the same statement

$$\delta\psi = 0 \tag{3.18}$$

for the vacuum field configurations ψ of these operators. The fermionic fields in the low energy effective theory of the heterotic strings are the gravitino $\Psi_M^{(+)}$, the dilatino $\Psi^{(+)}$ and the gauginos $\chi^{(+),a}$, cf. (2.27). We will suppress the chirality label $(+)$ of the Weyl spinors as well as the group index a for convenience. Their supersymmetry transformations are given by

$$\delta\Psi_M \;=\; \nabla_M\epsilon - \frac{1}{4\cdot 4!}H_{MNP}\Gamma^{NP}\epsilon\;, \tag{3.19}$$

$$\delta\Psi \;=\; -\frac{1}{2}\Gamma^M\partial_M\Phi\epsilon + \frac{1}{4}H_{MNP}\Gamma^{MNP}\epsilon\;, \tag{3.20}$$

$$\delta\chi \;=\; -\frac{1}{2}i\mathcal{F}_{MN}\Gamma^{MN}\epsilon\;, \tag{3.21}$$

where we use the conventions of [5] and $\Gamma^{M_1\cdots M_k} := \Gamma^{[M_1}\cdots\Gamma^{M_k]}$. Here we denote the parameter of the local supersymmetry transformations by ϵ and the Yang-Mills field strength by its background value $i\mathcal{F}$ as (3.17) requires that the only non-vanishing components are those tangential to K. Furthermore, we have to include the Bianchi identity of the 10-dimensional field strength H

$$dH = \mathrm{tr}\mathfrak{R}^2 - \mathrm{tr}(i\mathfrak{F})^2 \tag{3.22}$$

as an additional consistency condition of the coupled supergravity/Super-Yang-Mills theory. Here we denote the 10D curvature by $\mathfrak{R}$. We will discuss its origin in the context of section 4.2.

The Calabi-Yau Condition

In the following we assume a vanishing background of H and a constant dilaton Φ. Then the variation of the gravitino Ψ_M reads as

$$\delta\Psi_M = \nabla_M \epsilon \stackrel{!}{=} 0\,. \tag{3.23}$$

This states that ϵ has to be a Killing spinor, i.e. covariantly constant. We decompose it according to section 3.1.3 into

$$\epsilon(x,y) = \xi(x) \otimes \eta(y)\,, \tag{3.24}$$

with a four-dimensional spinor $\xi(y)$ and an internal $\eta(y)$. We demand that ϵ contains only one 4D spinor ξ because we are interested in $\mathcal{N}=1$ after compactification. Hence, the completeness of the expansion (3.24), i.e. the absence of more 4D spinors $\xi_i(x)$, is the condition for a supersymmetric compactification with $\mathcal{N}=1$. The requirement of $\nabla_M\epsilon = 0$ translates into

$$\nabla_\mu \xi = 0 \quad \text{and} \quad \nabla_m \eta(y) = 0\,, \tag{3.25}$$

where the second condition states, that K has to allow only for one covariantly constant[8] spinor η. This is a severe constraint for K. Let us now analyse the consequences of (3.25).

- The existence of a covariantly constant spinor ξ on M_4 implies that it has to be flat Minkowski space $\mathbb{M}_4$ as can be seen by

$$[\nabla_\mu, \nabla_\nu]\,\xi = \frac{1}{4} R_{\mu\nu\kappa\lambda}\Gamma^{\kappa\lambda}\xi = 0 \tag{3.26}$$

  and insertion of (3.2) for the curvature. Here we have used the covariant derivative ∇ for spinor fields. This implies a vanishing cosmological constant $\Lambda = 0$. Then we obtain $\nabla_\mu = \partial_\mu$ and $\partial_\mu \xi = 0$. Thus ξ is a constant spinor $\xi(x) \equiv \xi$. Hence, it is the generator of rigid supersymmetry in 4D.

[8] According to section 3.1.3 the spinor η is a zero-mode of the internal Dirac operator $i\not{D}_6$, hence, ξ is a massless four-dimensional field. We will see that it has to be constant.

- For the internal manifold K we obtain also

$$[\nabla_m, \nabla_n]\,\eta = \frac{1}{4}\mathcal{R}_{mnkl}\Gamma^{kl}\eta = 0\,, \tag{3.27}$$

  that implies a vanishing Ricci tensor $Ric_{mn} = 0$. Here we again denote the internal curvature be $\mathcal{R}$. Furthermore, we observe a connection to the holonomy group of K. A covariantly constant spinor η transforms trivially under holonomy. However, a generic spinor transforms under the holonomy group $Spin(6) = \mathrm{SU}(4)$ as $\mathbf{4}$ of SU(4). Thus, (3.25) implies that we have to find a subgroup $H \subset \mathrm{SU}(4)$ such that the single covariantly constant η is a singlet under H. The maximal subgroup[9] is SU(3) that yields $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$. We identify η with the one singlet and demand SU(3) holonomy for K.

  One consequence of this is, that one can use the covariantly constant η to define an integrable complex structure J, i.e. the Nijenhuis tensor field N vanishes. Furthermore, one can define a form ω with $d\omega = 0$, which we recover as the Kähler form. Thus, the manifold K is also Kähler. Another consequence is the possibility to construct a nowhere vanishing $(3,0)$-form Ω with holomorphic coefficient functions Ω_{mnp}, the so-called holomorphic three-form. It is a non-trivial representative of the unique (3,0) cohomology class of K. The existence of Ω implies a trivial first Chern class c_1, what is another way of stating $Ric = 0$ for a compact Kähler manifold.

  Let us summarise the properties of K. The compact manifold K is a Ricci-flat Kähler manifold or equivalently a Kähler manifold of SU(3) holonomy. This is precisely the definition of a Calabi-Yau manifold.

Six-dimensional Calabi-Yau manifolds are very complicated manifolds. An analytic construction of a Ricci-flat metric in case of a compact Calabi-Yau is not yet known, hence, one can only deal with its topological structure. Thus, the definition of a Calabi-Yau manifold in terms of a vanishing first Chern class c_1 is more convenient than stating that it exhibits a Ricci-flat metric. Many examples of Calabi-Yau manifolds constructed as submanifolds of $\mathbb{CP}^n$ are known but it is not known whether there is a finite number of so-called Calabi-Yau three-folds, i.e. six-dimensional Calabi-Yau manifolds.

Supersymmetric Gauge Backgrounds

Let us now turn to the analysis of the supersymmetry transformation (3.21) of the gauginos. This restricts the allowed gauge background $i\mathcal{F}$ that preserve supersymmetry. A vanishing variation (3.21) requires $g := i\mathcal{F}_{mn}\Gamma^{mn}$ to be a generator of SU(3) rotations, too. This implies g to have a vanishing trace and a transformation under SU(3) as $\mathbf{8} \subset \mathbf{3}\otimes\overline{\mathbf{3}}$

$$\mathrm{tr}\,(\Gamma^{mn}i\mathcal{F}_{mn}) = G^{A\bar{B}}i\mathcal{F}_{A\bar{B}} = 0\,, \qquad i\mathcal{F}_{AB} = i\mathcal{F}_{\bar{A}\bar{B}} = 0\,. \tag{3.28}$$

[9] The U(1) of SU(4) → SU(3)×U(1) is suppressed as it is irrelevant for holonomy transformations on a Ricci-flat Kähler manifold K. The Ricci tensor Ric is precisely the U(1)-part of the holonomy transformation matrix and vanishes here. Thus, this is another argument for K having SU(3)-holonomy.

These conditions are called the hermitian Yang-Mills equations. Here we introduced complex coordinates labelled by indices $A, \bar{B}$ transforming in the fundamental and anti-fundamental of SU(3). $i\mathcal{F}_{AB}$ and $i\mathcal{F}_{\bar{A}\bar{B}}$ are just the coefficients of the generators Γ^{AB} and $\Gamma^{\bar{A}\bar{B}}$ transforming as $\mathbf{3} \otimes \mathbf{3}$ and $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}}$. These have to vanish for $g = \Gamma^{mn} F_{mn}$ to be a SU(3)-generator in the representation $\mathbf{Adj}(\mathrm{SU}(3)) = \mathbf{8}$. These two conditions imply that the $(2,0)$- and $(0,2)$-part of the field strength $i\mathcal{F}$ vanish. Thus, the vector bundle is holomorphic, i.e. all transition functions can be chosen to be holomorphic. The first equation is the statement, that $i\mathcal{F}$ is primitive. It is interesting to note, that (3.28) imply, that $i\mathcal{F}$ automatically obeys its classical field equations[10] $G^{A\bar{B}} \nabla_A \mathcal{F}_{\bar{B}C} = 0$.

However, we have also to include the Bianchi identity (3.22) that is very restrictive. It requires to identify the background $i\mathcal{F}$ with parts or even the full curvature tensor $\mathcal{R}$. This implies the group theoretical embedding $\mathrm{SU}(3) \subset \mathrm{SO}(6) \subset \mathrm{E}_8$ or SO(32) and the identification of the gauge connection with the spin connection. It is interesting to note at this point, that consistency of string compactification requires the use of a non-zero gauge background. Thus, the 10-dimensional gauge group is broken according to section 3.1.4 to the maximally commuting subgroup of the background. Indeed, the choice of a background bundle is the parameter of Calabi-Yau model building, of course besides the choice of the Calabi-Yau K itself. The choice of a background bundle provides the opportunity to obtain phenomenologically interesting four-dimensional gauge groups. Furthermore, chiral four-dimensional matter occurs due to the presence of a background bundle. Its topological properties, encoded in the Dirac index of the total space of the bundle, determine the multiplicity of the matter representations in the four-dimensional theory. We will see all this explicitly in the following chapters where we construct and classify U(1)-bundles over non-compact Calabi-Yau manifolds and consider their phenomenological consequences and applications to compact Calabi-Yau models.

The 6D Compactification

Let us briefly discuss six-dimensional compactification. The supersymmetry requirement in the case of six-dimensional Minkowski space works analogous to the 4D case. One requires $\mathcal{N} = 1$ supersymmetry, that is $\mathcal{N} = 2$ supersymmetry from the 4D perspective. Then we are dealing with a four-dimensional manifold K that has to be Kähler and Ricci-flat again, i.e. a so-called Calabi-Yau two-fold. Consequently, we consider a manifold of SU(2) holonomy, that admits one covariantly constant spinor. It can be shown that there are only two distinct compact Kähler manifolds admitting a covariantly constant spinor, namely T^4 and $K3$. However, T^4 has trivial holonomy as it is flat, thus, it is not really a manifold of SU(2)-holonomy, but a trivial subgroup of it. Therefore, $K3$ remains as the only four-dimensional Calabi-Yau. As the compactification space K is now unique it is interesting to study six-dimensional compactification as a toy model for four-dimensional, that is much more complicated because of the vast number of distinct Calabi-Yau spaces.

[10]This relies on the fact that unbroken (rigid) supersymmetry is an easy way to obtain solutions of the equations of motion. As we are working at tree-level we reproduce the classical equations of motion.

Furthermore, one can hope to use these 6D models also for the analysis of four-dimensional physics by compactifying further on T^2 or the ansatz $(T^2 \times K3)/\mathbb{Z}_2$.

3.2.2 Orbifolds and Heterotic Models

We have seen in the last section that supersymmetric compactification of the heterotic string requires the internal manifold K to be Calabi-Yau. As these spaces are very complicated and difficult to handle one is interested in easier compactifications that also yield $\mathcal{N} = 1$ in 4D. Therefore, the notion of orbifolds was invented [10, 11]. These spaces provide the opportunity to obtain phenomenologically interesting supersymmetric four-dimensional theories by explicitly quantising the string on these spaces[11]. However, they are spaces with singular points, i.e. points of curvature singularities, whereas they are flat everywhere else. However, we will see in detail in the following chapters, that orbifolds can be understood in the context of Calabi-Yau spaces as singular limits of these smooth manifolds. Some cycles can shrink to zero size and an orbifold singularity is recovered. Hence, orbifolds are attractive both from the viewpoint of Calabi-Yau manifolds and of four-dimensional model building.

Let us start our discussion of orbifolds with the general construction scheme of the geometry. Then we will proceed with the presentation of the heterotic string on orbifolds and show, what are the general model building parameters. We will also address six-dimensional compactifications, i.e. four-dimensional orbifolds.
Here we will follow the discussion of [54], where also detailed calculations can be found. A more general review can be found e.g. in [4].

The Geometric Construction

We consider the six-dimensional torus $T^6 = \mathbb{R}^6/\Lambda$ defined by a generic six-dimensional lattice Λ. Compactifiaction of the heterotic string on this compact space yields a 4D theory with $\mathcal{N} = 4$ supersymmetry. No supersymmetry of the 10-dimensional theory is projected out. Hence, torus compactification is not appropriate for a reasonable four-dimensional theory, that should at least allow for chiral fermions.

A possible ansatz to overcome this problem is to identify points on T^6 that are related by a discrete symmetry P of the lattice Λ, called the point group containing the so-called orbifold twists. We will restrict to Abelian symmetries P. Thus, we are searching for Abelian automorphisms of the lattice. There are two possible types of point groups P allowing for $\mathcal{N} = 1$ supersymmetry, which are completely classified.

- One choice for P is the <u>cyclic group</u> $\mathbb{Z}_N$ of order N, that is generated by one element θ as θ^k, $k \in \mathbb{Z}$ with $\theta^N = e$, where we denote the identity by e. This can be represented on the lattice Λ by rotations by an angle of $2\pi/N$.

[11] The CFT on an orbifold is well understood and completely solved. This is a huge advantage compared to Calabi-Yau compactifications, where an adequate CFT-description is not yet known or perhaps impossible.

- Another possibility is to consider the direct product $\mathbb{Z}_N \times \mathbb{Z}_M$ that consists of $\theta_1^k \circ \theta_2^l$, with $k, l \in \mathbb{Z}$ and $\theta_1^N = \theta_2^M = e$.

The orbifold $\mathcal{O}$ can then be expressed as the quotient

$$\mathcal{O} := T^6/P = \mathbb{R}^6/S\,, \tag{3.29}$$

where we have defined the space group S by

$$S := \{g = (\theta, n_i e_i)\, | \theta \in P, n_i \in \mathbb{Z}\}\ . \tag{3.30}$$

Here we denote the basis vectors of the lattice Λ by e_i, $i = 1, \ldots, 6$. The action on $x \in \mathbb{R}^6$ is defined by $gx := \theta x + n_i e_i$ and the equivalence relation for (3.29) as

$$x \sim y \quad \Leftrightarrow \quad x = gy\,. \tag{3.31}$$

However, the action of P on T^6 is not free. There are fixed points x_f satisfying $\theta^k x_f = x_f + n_i e_i$ for some $n_i \in \mathbb{Z}$. Thus, the point x is mapped to its own equivalence class on T^6 and is fixed under θ^k. It is convenient to label a given fixed point x_f by the space group element g under which it is mapped to itself

$$\theta^k x_f + n_i e_i = x_f \quad \Leftrightarrow \quad g = (\theta^k, n_i e_i)\,. \tag{3.32}$$

Simple Examples

Let us now consider specific six- and four-dimensional orbifold geometries. Here we choose a factorisable torus T^6, which means that we can write it as a direct product of three orthogonal two-dimensional tori T^2, i.e. as $T^6 = T^2 \times T^2 \times T^2$. This refers simply to a lattice Λ with three pairs of basis vectors (e_1, e_2), (e_3, e_4) and (e_5, e_6), where all pairs are orthogonal to each other.

A six-dimensional example for an orbifold with $P = \mathbb{Z}_3$ is given for the lattice $\Lambda = (L\,(su(3)))^3$, where $L\,(su(3))$ denotes the root lattice[12] of SU(3), see Fig. 3.1. This lattice is invariant under rotations by $2\pi/3$, thus, it has a $\mathbb{Z}_3$ symmetry. We can represent the basis vectors e_i in each of the three orthogonal planes by complex numbers

$$e_x^a = \sqrt{2}\,, \quad e_y^a = \sqrt{2} e^{2\pi i/3}\,, \tag{3.33}$$

where we label the directions in one plane by x, y and the planes by $a = 1, 2, 3$. Then we can introduce complex coordinates in these planes by

$$z^1 = x^1 + ix^2\,, \quad z^2 = x^3 + ix^4\,, \quad z^3 = x^5 + ix^6\,. \tag{3.34}$$

This is a convenient choice of coordinates, because we can now represent the $\mathbb{Z}_3$ action by an exponential and a so-called twist vector $v := \frac{1}{3}(1, 1, -2)$ as

$$\theta^k : \ z^a \mapsto e^{2\pi i k v^a} z^a. \tag{3.35}$$

[12]Compare with chapter 6 of [19] for this notation.

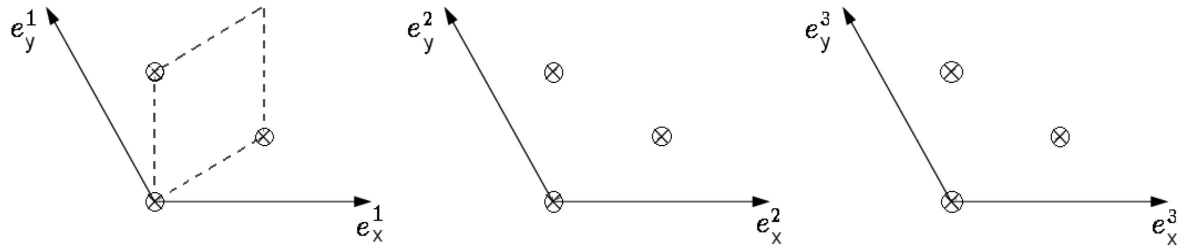

Figure 3.1: The orbifold $T^6/\mathbb{Z}_3$ with its $3 \times 3 \times 3 = 27$ fixed points. A string can wind around the corner of the orbifold, that a fixed point. The dashed line indicates the so-called fundamental region that is left over after the $\mathbb{Z}_3$ action.

We denote the components[13] of v by v^a and allow for $k = 0, 1, 2$. The transformations (3.35) have to be applied simultaneously in all three planes and are rotations by $120°$ and $-240° = 120°$. The orbifold that is obtained after all identifications due to $P = \mathbb{Z}_3$ is referred to as the $T^6/\mathbb{Z}_3$ orbifold. Its fixed points under the action of θ^1 on T^6 are given by $\left(z_f^1, z_f^2, z_f^3\right)$ for any combination of

$$z_f^a = 0\,, \quad z_f^a = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{6}}\,, \quad z_f^a = i\sqrt{\frac{2}{3}}\,. \tag{3.36}$$

Thus, there are $3 \times 3 \times 3 = 27$ different fixed points of the θ^1-action, compare to Fig. 3.1. These fixed points are the loci of curvature singularities as can be seen by a holonomy transformation circuiting one fixed point: A vector is rotated by $\mathbb{Z}_3$, i.e. rotated by $2\pi/3$. However, there is no rotation for every path that does not enclose the fixed point. Therefore, the curvature tensor has to be singular at the fixed point to generate this non-trivial holonomy rotation.

As another example we consider the orbifold $T^4/\mathbb{Z}_2$. We choose a lattice with orthogonal basis e_i given by $e_x^a = 1$, $e_y^a = i$ for $a = 1, 2$. It has an obvious $\mathbb{Z}_2$ rotation symmetry, see Fig. 3.2. Again we use complex coordinates $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$ such that the orbifold twist can be represented by the vector $v = \frac{1}{2}(1, -1)$ and the action

$$\theta^k : z^a \mapsto e^{2\pi i k v^a} z^a \tag{3.37}$$

for $k = 0, 1$. This orbifold has 16 different fixed points $\left(z_f^1, z_f^2\right)$ for any of the $4 \times 4 = 16$ combinations

$$z_f^a = 0\,, \quad z_f^a = \frac{1}{2}\,, \quad z_f^a = i\frac{1}{2}\,, \quad z_f^a = \frac{1}{2}(1+i)\,. \tag{3.38}$$

[13]We note that it is common in the literature to put the factor $1/3$ into the definition of v. However, we will break with this convention in later chapters for calculational convenience. Furthermore, one sometimes includes a component $v^0 = 0$ for the uncompactified directions.

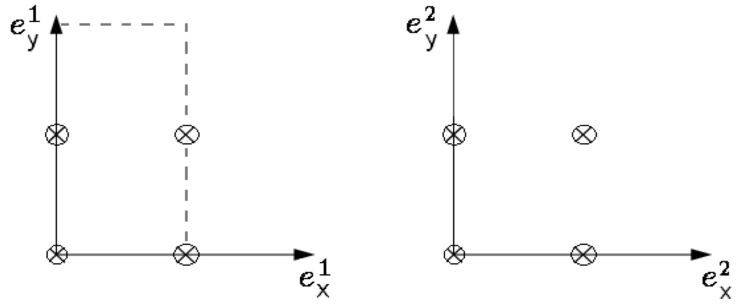

Figure 3.2: The orbifold $T^4/\mathbb{Z}_2$ with its $4 \times 4 = 16$ fixed points. The dashed line indicates the fundamental region that is left over after the $\mathbb{Z}_2$ action.

Supersymmetry Condition

We want to use these two orbifolds $T^6/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_2$ for string compactification. Hence, we demand again a vacuum of $\mathcal{N} = 1$ in four or six dimensions. This translates into a further condition for the orbifold twist v. Let us summarise the results for 4D. We have seen in the context of Calabi-Yau compactification in section 3.2.1 that the existence of the holomorphic three-form Ω is equivalent to the requirement of $\mathcal{N} = 1$ supersymmetry (on a Kähler manifold). This transfers to the situation of orbifold compactification by demanding, that its holomorphic three-form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ is invariant under the orbifold action defined by v. This is equivalent to $v^1 + v^2 + v^3 = 0 \mod 1$. We will always choose $v^1 + v^2 + v^3 = 0$. Furthermore, there should be no fixed torus under the action of P as too much supersymmetry would be present in 4D, namely $\mathcal{N} = 2$. Of course, all this applies analogously for of $\mathcal{N} = 1$ supersymmetry in six dimenisons where we choose $v^1 + v^2 = 0$.

The Heterotic String on Orbifolds

In the case of Calabi-Yau compactification there are also conditions for the supersymmetric gauge backgrounds. These conditions yield a breaking of the 10-dimensional gauge group, where geometrical quantities of the Calabi-Yau, namely the curvature tensor, have to be embedded into the gauge background. We have argued, that there are some similarities between orbifolds and Calabi-Yau manifolds. Hence, we may ask, whether there is an analogous mechanism for breaking the 10D gauge symmetry on the orbifold, too?

It is known, that modular invariance of the one-loop string amplitude requires an embedding of the orbifold action into the gauge sector of the heterotic string, what induces a breaking of gauge symmetry analogous to the Calabi Yau case. This is expressed by a homomorphism $S \hookrightarrow G$ into a subgroup G of the automorphisms of the algebra of SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$. G is called the gauge twisting group. However, we restrict to the so-called shift embedding where the homomorphism is expressed in terms of shift vectors in the respective weight lattice. The embedding can be written as

$$\mathbb{Z}_N : \quad g = (kv, n_i e_i) \quad \hookrightarrow \quad \tilde{g} := (kV, n_i A_i) \tag{3.39}$$

in the case of $\mathbb{Z}_N$ orbifolds[14] we are interested in. Here we have used the identification $\theta^k \equiv kv$. The orbifold twist vector v is embedded by the shift vector V and the torus cycle e_i by the so-called Wilson lines A_i for each possible direction i. The parameters V and A_i 16-dimensional vectors and subject to a couple of consistency conditions, that we address below. V and A_i will enter as parameters in the definition of a 4D model, as we will see below. The combination of all these groups constitutes the so-called orbifold group $(kv, n_i e_; kV, n_i A_i) \in S \times G$.

Now we can define the action of the embedded orbifold spacegroup on the gauge degrees of freedom of the heterotic string. According to section 2.2.2 they are encoded in the 16 internal bosonic left-mover X^I_{L}. We define the orbifold action by

$$\tilde{g} X^I_{\mathrm{L}} := X^I_{\mathrm{L}} + 2\pi \left(kV^I + n_i A^I_i \right) . \tag{3.40}$$

This action implies a transformation of the states of the heterotic string.

- Using the Vertex operator $: e^{ip_{\mathrm{L}} \cdot X_{\mathrm{L}}} :$ we can write

$$|p_{\mathrm{L}}\rangle =: e^{ip_{\mathrm{L}} \cdot X_{\mathrm{L}}} : |0\rangle_{\mathrm{L}} \tag{3.41}$$

  for a state with internal momentum p_{L} as the gauge bosons with $p^2_{\mathrm{L}} = 2$. Then the orbifold action implies a transformation phase

$$|p_{\mathrm{L}}\rangle \mapsto e^{2\pi i p_{\mathrm{L}} \cdot (kV + n_i A_i)} |p_{\mathrm{L}}\rangle \tag{3.42}$$

  due to the action (3.40).
- Simultaneously, the right-movers transform according to their representation under SO(8). Therefore, it is convenient to denote the vacua of the right-movers $|+\rangle_{\mathrm{R}}$ and $b^i_{-1/2} |0\rangle_{\mathrm{NS}}$ of the R- and NS-sector, respectively, by their SO(8)-weight vector

$$|+\rangle_{\mathrm{R}} \equiv \left| (\pm 1/2)^4 \right\rangle , \quad b^i_{-1/2} |0\rangle_{\mathrm{NS}} \equiv \left| \left(\underline{\pm 1, 0^3} \right) \right\rangle . \tag{3.43}$$

  We will label this collectively by $|q\rangle_{\mathrm{r}}$ where q is the SO(8) weight. The two choices for q, $\left(\pm\frac{1}{2}\right)^4$ and $\left(\underline{\pm 1, 0^3}\right)$, are the representations $\mathbf{8}_{\mathrm{S}}$ and $\mathbf{8}_{\mathrm{V}}$, respectively. $\left(\pm\frac{1}{2}\right)^4$ denotes four components $\pm 1/2$ with all choices of signs with an even number of plus signs and $\left(\underline{\pm 1, 0^3}\right)$ all permutations and sign combinations, where $(0^3) = (0, 0, 0)$. Finally, the phase under the orbifold action is given by

$$|q\rangle_{\mathrm{r}} \mapsto e^{2\pi i q \cdot (kv)} |q\rangle_{\mathrm{r}} . \tag{3.44}$$

[14]Let us just note that one has to set $kV_2 + lV_2$ to generalise this description for $\theta^k_1 \circ \theta^l_2$ of $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds.

However, consistency of the heterotic string theory on the orbifold geometry requires a trivial transformation of all states of the spectrum. Of course one has to consider the combined transformation of left- and right-movers. Hence, states with a non-trivial phase are projected out. Let us summarise the consequences of this projection conditions for the massless spectrum in Tab. 2.4 of the heterotic string.

- The gauge bosons of the theory are partially projected out. The right-mover for the four-dimensional[15] gauge bosons is given by $q = (\pm 1, 0^3)$ and $q = (\pm(\frac{1}{2}^4))$ for the gauginos. Hence, we have $q \cdot v = 0$ and recover a 4D $\mathcal{N} = 1$-vector multiplet. Only those charged gauge bosons and gauginos with

$$p_{\mathrm{L}} \cdot (kV) = 0 \mod 1 \quad \text{and} \quad p_{\mathrm{L}} \cdot (n_i A_i) = 0 \mod 1 \tag{3.45}$$

  survive the projection condition and are present in the four-dimensional unbroken gauge group H. However, the rank of the 10D gauge group is unbroken as all gauge bosons with Cartan generators transform trivially. Thus, we obtain a four-dimensional gauge group H of rank 16.

  Consequently, the adjoint representation of the 10D gauge group G is branched into representations L_i of the 4D unbroken gauge group H

$$\mathbf{Adj}(G) \to \sum_i L_i\,, \tag{3.46}$$

  that are the adjoint of H but also some other representations. Of these other representations only those with

$$q \cdot (kv) - p_{\mathrm{L}} \cdot (kV + n_i A_i) = 0 \mod 1 \tag{3.47}$$

  for all k and n_i survive the orbifold action. This can be rewritten as

$$q \cdot v - p_{\mathrm{L}} \cdot V = 0 \mod 1 \quad \text{and} \quad p_{\mathrm{L}} \cdot A_i = 0 \mod 1\,. \tag{3.48}$$

  These states will occur as charged matter in the four-dimensional theory.

- The analysis of the supergravity multiplet shows that precisely one gravitino $q = (\pm(\frac{1}{2}^4))$ is present in 4D after applying the projection conditions. This verifies the $\mathcal{N} = 1$ requirement explicitly in the case of orbifold compactification.

Twisted Strings

On the geometry of an orbifold there are new possible boundary conditions to obtain a closed string. Hence, there are new types of closed strings, that have to be quantised.

[15]Here we discuss only compactification to four dimensions explicitly. In principle, all statements also apply to compactification to 6D and $\mathcal{N} = 1$.

- Untwisted strings are just the strings, that are already closed on the plane. They obey the boundary conditions (2.3), (2.4) of the closed superstring. Only the orbifold projection conditions on the states as given above have to be taken into account.
- Strings that are closed due to the torus cycles e_i are called winding strings. They are already present in case of torus compactification. However, they are in general massive due to the energy that is necessary to wind the string. Hence, we do not consider them as they are irrelevant for the massless spectrum and the 4D theory.
- There are closed strings, which are only present on the orbifold geometry. They are called twisted strings and obey the boundary conditions

$$X^i(\tau, \sigma + \pi) = \left(\theta^k X\right)^i (\tau, \sigma) + (n_l e_l)^i \ , \tag{3.49}$$

$$\Psi^i_-(\tau, \sigma + \pi) = \pm \left(\theta^k \Psi_-\right)^i (\tau, \sigma) \ , \tag{3.50}$$

  for $i = 1, \ldots, 6$. This boundary conditions imply a localisation of the center of mass of the string at the fixed point labelled by the space group element $g = \left(\theta^k, n_i e_i\right)$. Therefore, this is called the k-th twisted sector according to θ^k. These new types of closed strings will contribute new states to the theory. Again it is convenient to change to complex coordinates and then quantise the string. The twisting θ^k changes the labels of the mode expansion to be shifted and fractional. This induces a shift of the SO(8) weight q to

$$q^a_{\mathrm{sh}} := q^a + kv^a \ , \quad a = 0, \ldots, 3 \tag{3.51}$$

  with $v^0 := 0$. Consequently, the states in the right-moving sector are $|q_{\mathrm{sh}}\rangle_{\mathrm{r}}$. The mass formula is changed due to the presence of a zero point energy and the massless states are determined by the solutions of $M^2_{\mathrm{R}} = 0$.
  Furthermore, due to the embedding of the orbifold action into the gauge degrees also there "twisted strings" occur. The new boundary condition reads as

$$X^I_{\mathrm{L}}(\tau, \sigma + \pi) = X^I_{\mathrm{L}}(\tau, \sigma) + 2\pi(kV^I + n_i A^I_i + \lambda^I) \tag{3.52}$$

  for $\lambda \in \Lambda$, denoting the weight lattice of SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$. This induces a shifted

$$p^I_{\mathrm{sh,L}} := p^I_{\mathrm{L}} + (kV^I + n_i A^I_i) \tag{3.53}$$

  and a new mass equation taking into account this shift. Massless states are given by the solutions of $M^2_{\mathrm{L}} = 0$, where internal oscillators may occur as well.

Again there are projection conditions on the states of the twisted sector. They transform in an analogous way as the untwisted string states after replacing $q \to q_{\mathrm{sh}}$ and $p_L \to p_{\mathrm{sh},L}$. However, the construction of a Hilbert space invariant under $S \times G$ is more involved and we just refer to [54] for the details.

Conditions on Model Building Parameters

In the following we address the conditions on the possible choices for V and A_i, that are the parameters of every four-dimensional and six-dimensional model.

- First we discuss the consequences of the homomorphism $S \hookrightarrow G$. On a $\mathbb{Z}_N$-orbifold $g^N = (\theta, n_i e_i)^N = (\mathbb{1}, k_i e_i)$ is only a lattice shift, i.e. a trivial transformation. This has to apply for the corresponding gauge twist $\tilde{g}$, too. This implies

$$NV,\, NA_i \in \Lambda, \quad \forall i\,, \tag{3.54}$$

where Λ denotes the weight lattice of SO(32) or $E_8 \times E_8$. This can be summarised by stating, that V and A are equal to one third of lattice vector.
Another implication of the homomorphism property can be demonstrated for $T^6/\mathbb{Z}_3$. There the basis vectors e_{2i-1} and e_{2i} are related by the orbifold twist θ, e.g. $\theta e_1 = e_2$. Thus, we have to embed them identically into the gauge degrees of freedom. Consequently, there can be only three distinct Wilson line parameters A_1, A_3 and A_5 as A_2, A_4 and A_6 have to be identical, respectively.
- Modular invariance restricts the allowed gauge embedding V in case of no Wilson lines by

$$N\left(V^2 - v^2\right) = 0 \mod 2\,. \tag{3.55}$$

We have $v^2 = \frac{2}{3}$ for $T^6/\mathbb{Z}_3$ and $v^2 = \frac{1}{2}$ for $T^4/\mathbb{Z}_2$ yielding

$$T^6/\mathbb{Z}_3: \quad 3V^2 = 0 \mod 2\,, \qquad T^4/\mathbb{Z}_2: \quad 2V = 1 \mod 2\,. \tag{3.56}$$

This condition is similar to the Bianchi identity (3.22) in the case of Calabi-Yau compactification and we will encounter an analogous relation to (3.55) in the context of later chapters and U(1)-bundles over Calabi-Yau spaces.
For the inclusion of Wilson lines A_i (3.55) changes and one obtains

$$N\left[(kV + n_i A_i)^2 - (kv)^2\right] = 0 \mod 2\,, \quad \forall k = 0, \ldots, N-1\,. \tag{3.57}$$

However, we refer to [49] for a detailed discussion of the modular invariance condition in the context of Wilson lines.

These conditions imply that there are only a finite number of inequivalent models without Wilson lines in the case of e.g. $T^6/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_2$. There are only five inequivalent $E_8 \times E_8$ models on $T^6/\mathbb{Z}_3$ and just three SO(32) models on $T^4/\mathbb{Z}_2$. Two models are equivalent if their shifts differ by a weight vector or if the signs of two entries are flipped pairwisely. This goes back to so-called Weyl reflections of the roots.

However, the situation is different for the inclusion of Wilson lines. As most of these conditions are only " mod "-conditions there is a huge number of inequivalent orbifold models with Wilson lines and they are not yet classified in the case of all possible orbifold types.

Influence of Wilson Lines

It is important for the analysis of chapter 8 to understand the influence of Wilson lines A_i compared to a model without any. The changes due to the inclusion of Wilson lines can be summarised in two statements:

1. The projection conditions (3.45) of the gauge bosons show explicitly the influence of Wilson lines A_i on the 4D theory: The only change compared to a model without Wilson lines is that additional gauge bosons are projected out according to $p_{\mathrm{L}} \cdot (n_i A_i) = 0 \mod 1$. The same applies for untwisted as well as twisted matter states. This can be seen explicitly in (3.48), where the second projection condition is not present in the case of no Wilson line A_i.

2. Wilson lines A_i distinguish between fixed points. As every fixed point can be labelled by a space group element $g = \left(\theta^k, n_i e_i\right)$ that leaves it invariant, its embedding into the gauge degrees of freedom by $\tilde{g} = (kV, n_i A_i)$ depends on the lattice vectors $n_i e_i$. Wilson lines enters into the mass equation of the twisted string localised at the fixed point g. Therefore, the massless twisted spectrum at different fixed points, i.e. different n_i, also differs.

In the case of $T^6/\mathbb{Z}_3$ we have the fixed points $g_1 = (\theta, 0)$, $g_2 = (\theta, e_1)$ and $g_3 = (\theta, e_1+e_2)$ in the first twisted sector and the first complex plane. The corresponding gauge embeddings are $\tilde{g}_1 = (V, 0)$, $\tilde{g}_2 = (V, A_1)$ and $\tilde{g}_2 = (V, 2A_1)$ leading to different mass equations and twisted matter localised at g_1, g_2 and g_3. This is due to the fact that the center of mass of the string is located at the respective fixed point.
This is the general idea and motivation to introduce the notion of a local picture of the orbifold. The action of $\tilde{g}_i$ on the gauge sector is given by $V^1 := V$, $V^2 := V + A$ and $V^3 := V + 2A$, respectively. In the following chapters we will use these different V^i as so-called local shift vector to calculate the local gauge group[16] and spectrum at a fixed point g_i. In this picture, the knowledge of the global orbifold does not matter and the term of Wilson lines disappears completely into the definition of V^i. However, we will see how to reconstruct the global picture by taking into account all the local informations V^i of the different fixed points.

[16] In general, the local gauge group is bigger than the gauge group of the complete, "global" orbifold. The is due to the fact, that the local gauge group is again calculated , see (3.45), by $p \cdot V^i = 0 \mod 1$. The second projection due to A_i disappears as the knowledge of Wilson lines is not present in the local picture.

Chapter 4

Anomalies and Their Cancellation

Anomalies seem to be a topic of quantum field theory and not ad hoc of string theory. However, string theory is described as two-dimensional conformal (quantum) field theory. Therefore, consistency of string theory means consistency of this two-dimensional quantum field theory, which leads for example to the condition of a vanishing Weyl anomaly, that is a field theory notion. Furthermore, string theory should give rise to a low-energy effective field theory in 10 dimensions, which should be consistent as a quantum field theory, i.e. be free of anomalies. Last but not least it should be possible to deduce the four-dimensional standard model or its supersymmetric extensions like the MSSM from this 10-dimensional theory by compactification. Also these four-dimensional theories should be reasonable, i.e. anomaly-free quantum field theories. Hence, the notion of anomalies and their cancellation plays an important role for understanding string theory and its phenomenological aspects.

Furthermore, the characterisation of anomalies takes into account the fascinating area of characteristic classes occurring in differential geometry and topology. This will provide the appropriate framework to discover the hidden relationships between the different types of anomalies and further mathematics.

Therefore, we will address this topic referring to physics as well as mathematics. Physically motivated introductions can be found in chapter 19 of [47], chapter 13 of [27] or e.g. [30]. A more mathematical description can be found in chapter 13 of [37] or chapter 12 and 13 of [24].

4.1 Description of Anomalies

For the whole section we work in even-dimensional spacetime M_D with $D = 2l$. The breakdown of symmetries of the classical theory during quantization is referred to as anomaly. This is most easily summarised by stating that the measure of the path integral has not all the symmetries of the classical action. However, anomalies can be observed explicitly in some one-loop Feynman diagrams with massless chiral fermions inside the loop and $l + 1$ external gauge bosons or gravitons. These graphs contribute to some matrix elements of the operators corresponding to classically conserved Noether currents j^μ. These contri-

butions induce a violation of the pendant of the classical conservation law $\partial_\mu j^\mu = 0$. In other words, diagrams with conserved currents of the same or different type attached, i.e. cut external legs, give non-trivial contributions to the current conservation laws. It is not possible to evaluate these anomalous diagrams using a regulator that simultaneously preserves the conservation of all the currents, i.e. all physical principles like Bose symmetry, gauge invariance, local Lorentz invariance, etc. . Thus, there is no preferred regulator and one can make a choice, that respects a chosen symmetry and violates others. Hence, the anomaly can alter with this choice, e.g. gauge invariance can be maintained but chiral symmetry be violated and vice versa.
One can also state that it is not possible to cancel these anomalous diagrams by adding local counter terms to the action, that respect all its classical symmetries.
To analyse anomalies of a given theory it would in principle be necessary to consider all loop diagrams, that might violate classical conservation laws. Fortunately, anomalies are one-loop exact. But even the lengthy task to consider all possible one-loop diagrams is not necessary, because anomalies can only occur in parity violating theories, i.e. theories with parity violating interactions. Amplitudes with parity preserving gauge couplings can always be regularised without spoiling local classical symmetries. This is equivalent to restrict to diagrams with massless chiral fermions propagating inside the loop, i.e. theories with different couplings of the left-handed from the right-handed fermions. Note that chirality is no parity invariant notion as it is reversed under a parity transformation. Consequently, there are no anomalies in odd-dimensional spacetime, because it is not possible to define the notion of chirality, i.e. chiral fermions, cf. App. B of [51].

Before analysing anomalies in detail we mention the different types of anomalies first:

- Anomalies of local symmetries like gauge symmetries, so-called gauge anomalies, spoil the consistency of the theory as gauge transformations can no longer be used to decouple unphysical degrees of freedom, i.e. longitudinally polarised states from the theory. Unitarity of the S-matrix is no more guaranteed, because precisely these anomalous diagrams do not vanish for longitudinal polarisations of the external gauge boson legs[1]. As briefly mentioned above also gravity as a gauge theory of local Lorentz symmetry can be anomalous. These gravitational anomalies correspond to a breakdown of general covariance. One simply has to attach gravitons coupling to the conserved current, i.e. the energy momentum tensor of the fields propagating inside the loop, to the loop-diagram. There might be even mixed gauge/gravitational anomalies for external gauge bosons and gravitons.

- The anomalous breakdown of global symmetries does not affect the consistency of a theory. It is even phenomenological necessary to explain some observed processes like the famous $\pi^0 \to \gamma\gamma$, which corresponds to the anomalous breaking of a global chiral symmetry due to the presence of the U(1)-gauge field of electromagnetism.

[1]This would otherwise be guaranteed precisely by the current conservation law $p^\mu j_\mu = 0$ in momentum space.

It is convenient to consider the quantum effective action Γ defined by the path integral for the analysis of anomalies as it encodes the relevant properties of the quantum theory. Its variation under gauge transformations should be zero for the absence of anomalies, but is not for the presence of anomalies, what is then called an anomalous variation. This is due to the fact that the measure of the path integral is not invariant under the considered symmetry. To follow this prescription for the manifold field theories in different dimensionalities of spacetime, including gravity couplings, it is absolutely necessary to develop a rigorous and simple mathematical framework. This is given by the so-called characteristic classes. To explore this connection to mathematics, it is reasonable to start with an explicit example in $D = 2l$ dimensions to generalise to more complicated situations.

4.1.1 Fujikawa's Method: The Chiral Anomaly

Consider a theory of fermions Ψ coupled to an external gauge field[2] $\mathcal{A}$, i.e. a field that is not integrated in the path integral. Define the quantum effective action Γ of the theory by

$$e^{-\Gamma[\mathcal{A}]} = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}e^{-\mathcal{S}}. \tag{4.1}$$

$i\not{D} = i\Gamma^\mu\left(\partial_\mu + i\mathcal{A}_\mu\right)$ denotes the Dirac operator using the covariant derivative and $\mathcal{S} = \int dx\overline{\Psi}\not{D}\Psi$ the classical action.

This is invariant under the global chiral transformation

$$\Psi \to e^{i\Gamma_d\alpha}\Psi, \;\; \overline{\Psi} \to \overline{\Psi}e^{i\Gamma_d\alpha}, \tag{4.2}$$

which is a symmetry of the classical action.

The breakdown of this classical symmetry during quantization is the so-called chiral or Abelian anomaly. It was analysed by Fujikawa [20] by means of the path integral expression (4.1) using methods known to mathematicians from the proof of the Atiyah-Singer index theorem using the heat kernel. One expands the field as

$$\Psi = \sum_i a_i\psi_i\,, \quad \overline{\Psi} = \sum_i \overline{b}_i\Psi_i^\dagger \tag{4.3}$$

in eigenfunctions $i\not{D}\psi_i = \lambda_i\psi_i$ of the Dirac operator. The a_i denote Grassmann-valued expansion coefficients. The quintessence of the calculation is the non-trivial transformation of the path integral measure given by

$$\prod_i da_i d\overline{b}_i \to \prod_i da_i' d\overline{b}_i' \exp\left(-2i\int \alpha(x)\sum_n \psi_n^\dagger\Gamma_D\psi_n\, dx\right)\,, \tag{4.4}$$

combined with the transformation of the classical action $\mathcal{S}$

$$\delta_{\alpha(x)}\mathcal{S} = \int \alpha(x)\partial_\mu j_D^\mu\, dx\,. \tag{4.5}$$

[2]We use $\mathcal{A}$ instead of A to emphasise that it is external, i.e. not quantised with the caveat not to confuse it with the $\mathcal{A}$ appearing in the context of compactification.

a'_i and b'_i denote the expansion coefficients of the transformed Ψ' and $\overline{\Psi}'$. The conserved axial current $j^\mu_D(x) = \overline{\Psi}\Gamma^\mu\Gamma_D\Psi$ occurs in $\delta_{\alpha(x)}\mathcal{S}$ as usual for $\alpha \to \alpha(x)$ applied as a calculational trick. The quantity in the exponent of (4.4) is evaluated inserting a Gaussian regulator with cut-off M

$$\int \mathbf{A}(x)\, dx := \sum_i \langle\psi_i|\exp\left[-(i\not{D}/M)^2\right]\Gamma_D|\psi_i\rangle\,, \tag{4.6}$$

which is precisely the heat-kernel $h(t)\ |_{t=1/M^2}= \exp(t\Delta)\ |_{t=1/M^2}$. As in the proof of the index theorem, only states $|\psi_{0,i}\rangle$ with $\lambda_i = 0$ contribute. For every $|\psi_i\rangle$ with $\lambda_i \neq 0$ there is a state $\Gamma_D|\psi_i\rangle$ with eigenvalue $-\lambda_i$. Hence, their contribution to (4.6) cancels as different eigenstates $\Gamma_D|\psi_i\rangle$ and $|\psi_i\rangle$ are orthogonal: $\langle\psi_i|\Gamma_D|\psi_i\rangle = 0$. After splitting $Ker\left(i\not{D}\right)$ according to the chirality, i.e. $\Gamma_d|\psi_{0,i}\rangle_\pm = \pm|\psi_{0,i}\rangle_\pm$, we obtain

$$\int \mathbf{A}(x)\, dx = \sum_i \langle\psi_{0,i}|\Gamma_D|\psi_{0,i}\rangle = \nu_+ - \nu_- =: \mathrm{ind}\left(i\not{D}\right). \tag{4.7}$$

Calculating $\mathbf{A}(x)$ differently, what is a more or less rigorous proof of the index theorem in this context, yields by combining with (4.5)

$$\partial_\mu j^\mu_D = \left(\frac{i}{4\pi}\right)^l \frac{1}{l!}\epsilon^{\mu_1\ldots\mu_{2n}}\mathrm{tr}\left(\mathcal{F}_{\mu_1\mu_2}\cdots\mathcal{F}_{\mu_{2n-1}\mu_{2n}}\right)\,, \tag{4.8}$$

$$\int \partial_\mu j^\mu_D\, dx = \nu_+ - \nu_- = \int \mathrm{ch}_l(\mathcal{F}) = \int \frac{1}{l!}\mathrm{tr}\left(\frac{i\mathcal{F}}{2\pi}\right)^l \tag{4.9}$$

for $\mathcal{F}$ denoting the field strength of $\mathcal{A}$ and tr the trace in the fundamental representation. $\mathrm{ch}_l(\mathcal{F})$ is the l-th Chern character. This operator relation is the anomalous conservation law for the axial current j^μ_D, also called Adler-Bell-Jackiw-anomaly. In four dimensions, this can be checked by calculation of the matrix element $\int e^{-iqx}\langle p,k|\, j^\mu_5\,|0\rangle$ for the creation of two photons recovering the anomalous contribution of the triangle diagram mentioned above, compare section 19.2 of [47].

The relation (4.9) agrees with the D-dimensional Atiyah-Singer index theorem for the twisted spin complex

$$\Delta(M_D)^+ \otimes E \underset{D^\dagger}{\overset{D}{\rightleftharpoons}} \Delta(M_D)^- \otimes E \tag{4.10}$$

of a D-dimensional manifold with trivial $\hat{A}$-genus $\hat{A}(\mathcal{R}) = 1$. Here $\Delta(M_D) := \Gamma(M_D, S(M_D))$ denoting the section of the spinor bundle $S(M_D)$ splits into the chirality eigenspaces $\Delta(M_D)^\pm$ with $\Gamma_D|_{\Delta(M_D)^\pm} = \pm\mathbb{1}$. Furthermore, we have

$$i\not{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \Rightarrow D = i\not{D}P^+ \tag{4.11}$$

for the projection operator $P^+ = 1/2\,(1+\Gamma_D)$. Inclusion of gravity to the theory yields the complete index for the twisted spin complex

$$\int \partial_\mu j^\mu_D\, dx = \nu_+ - \nu_- = \int \mathrm{ch}_l(\mathcal{F})\,\hat{A}(\mathcal{R})\,. \tag{4.12}$$

4.1.2 Connection with Non-Abelian Anomalies

Now we turn to the more interesting case of anomalous non-Abelian gauge symmetries, also referred to as non-Abelian anomalies. We consider a theory of a left-handed Weyl fermion Ψ in a representation R of some gauge group G with the (external) gauge field $\mathcal{A}$ in $D = 2l$ dimensions. This is a parity violating gauge coupling in the sense mentioned above.

The Direct Calculation

Consider the quantum effective action

$$e^{-\Gamma[\mathcal{A}]} = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi} \exp\left(-\int dx\overline{\Psi}i\not{D}^+\Psi\right) \tag{4.13}$$

for $i\not{D}^+ = i\Gamma^\mu\left(\partial_\mu + i\mathcal{A}_\mu\right)P_+$. The generators of the gauge group are denoted by T_a. Then the classically conserved non-Abelian gauge current is given by

$$j_a^\mu = i\overline{\Psi}\Gamma_d^\mu T_a P_+\Psi\,. \tag{4.14}$$

Considering the variation of the quantum effective action under the infinitesimal gauge transformation $\mathcal{A}_\mu \mapsto \mathcal{A}_\mu - D_\mu\lambda(x)$ with $D_\mu\lambda = \partial_\mu\lambda + [\mathcal{A}_\mu, \lambda]$, $\lambda = \lambda^a T_a$, yields an anomalous conservation law. Again the path integral measure is not invariant under the above gauge transformation leading to a Chern-Simons like expression. For $D = 4$ (and $\hat{A}\left(\mathcal{R}\right) = 1$) we obtain

$$D_\mu\left\langle j_a^\mu\right\rangle = \frac{1}{24\pi^2}\mathrm{tr}\left[T_a\epsilon^{\kappa\lambda\mu\nu}\partial_\kappa\left(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \frac{1}{2}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right)\right], \tag{4.15}$$

what differs only in the last term from the Chern-Simons three-form $Q_{3,Y}\left(\mathcal{A},\mathcal{F}\right) \supset 2/3\cdot\mathcal{A}^3$ of the Yang-Mills theory. Thus, it differs also from (4.9).

The Descent Equations

A further formalisation is necessary to understand the connection between Abelian and non-Abelian anomalies, namely the descent equations. We have seen, that an anomaly corresponds to a situation with an quantum effective action, which is not invariant under a classical symmetry. This is expressed by

$$\delta_{\Lambda(x)}\Gamma = G\left(\Lambda\right) \equiv \int I_D^1\left(\Lambda\right), \tag{4.16}$$

where $\Lambda(x)$ denotes the parameter of a gauge or general coordinate transformation[3] and I_D a $2l$-form. Such an integral expression for (4.16) is always possible. An example is given

[3] δ_Λ may be a combined gauge and local Lorentz transformation.

above for a non-Abelian gauge transformation yielding

$$\delta_{\lambda(x)}\Gamma\left[\mathcal{A}\right] = \int \lambda^a D_\mu \left\langle j_a^\mu \right\rangle \,. \tag{4.17}$$

The explicit form of I_D^1 may be very complicated and not unique due to ambiguities of adding exact forms and local counterterms.

For (4.16) to be consistent we have to require the so-called Wess-Zumino consistency condition

$$\delta_{\Lambda_1} G\left(\Lambda_2\right) - \delta_{\Lambda_2} G\left(\Lambda_1\right) = \left[\delta_{\Lambda_1}, \delta_{\Lambda_2}\right]\Gamma \stackrel{!}{=} G\left(\left[\Lambda_1, \Lambda_2\right]\right) \,, \tag{4.18}$$

which is a simple consequence of $\left[\delta_{\Lambda_1}, \delta_{\Lambda_2}\right] = \delta_{[\Lambda_1,\Lambda_2]}$.

The solution of this expression can be constructed very elegantly by means of a formal $(D+2)$-form I_{D+2}, which is a gauge invariant $\delta_\Lambda I_{D+2} = 0$ and closed $dI_{D+2} = 0$ polynomial in $\mathrm{tr}\left(\mathcal{F}^k\right)$ and $\mathrm{tr}\left(\mathcal{R}^l\right)$. Here $\mathcal{F}$ and $\mathcal{R}$ denote the field strength and curvature tensor of the D-dimensional theory. Therefore, I_{D+2} fulfills precisely the definition of a characteristic class and defines a cohomology class. This is known as the anomaly polynomial. It is unique and in a convenient form, that allows advanced calculations. Once I_{D+2} is known one is able to deduce a corresponding G fulfilling (4.18) using the descent equations:

$$\begin{aligned} I_{D+2} = dI_{D+1} \quad &\text{with} \quad \delta_\Lambda I_{D+1} = dI_D^1 \\ \Longrightarrow\ I_D^1 \,:\ & G = \textstyle\int I_D^1 \,. \end{aligned} \tag{4.19}$$

Here we utilise the fact, that $dI_{D+2} = 0$ to obtain a locally defined $(d+1)$-form by $I_{D+2} = dI_{D+1}$. This form is not gauge-invariant but its gauge variation is closed again. Hence, we are able to define a d-form by $\delta_\Lambda I_{D+1} = dI_D^1$, which is a possible anomalous variation of the effective action (4.16). We will not prove, that I_D^1 constructed using the descent equations (4.19) obeys the Wess-Zumino condition (4.18), but only refer to the more rigorous treatment of the descent equations in the literature using BRS-cohomolgy, cf. chapter 5 of [48]. Instead, the descent equations will be applied backwards in section 4.2 to obtain I_{D+2} for a concrete example.

Now we are ready to clarify the relation between the Abelian anomaly (4.9) and the non-Abelian anomaly (4.15):

$$I_{D+2} = \mathrm{ch}_{l+1}\left(\mathcal{F}\right)\hat{A}\left(\mathcal{R}\right) \,. \tag{4.20}$$

This $(D+2)$-dimensional chiral anomaly yields precisely the D-dimensional non-Abelian anomaly by applying the descent equations. Comparing to the initial description of anomalies in terms of Feynman diagrams this can be understood qualitatively. As mentioned above, anomalies in $D = 2l$ dimensions occur in diagrams with $l+1$ external legs given by gauge bosons and gravitons. Hence, these diagrams depend on $l+1$ gauge fields $\mathcal{A}$ and gravitons. Also the terms of lowest order in I_{D+2} contain only $l+1$ fields, because $\mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ and $\mathcal{R} = d\omega + \omega\wedge\omega$ using the spin connection ω. Note that even the diagrams with more external fields are determined by the form of I_{D+2} due to the terms $\mathcal{A}\wedge\mathcal{A}$ and $\omega\wedge\omega$ in $\mathcal{F}$ and $\mathcal{R}$.

4.1.3 Anomaly Formulæ

In principle it is now possible to determine $I_{D+2}^{(a)}$ for every field in a possible physical theory. The total anomalyI_{D+2} of any theory is then characterised by simply adding up all the contributions

$$I_{D+2} = \sum_a I_{D+2}^{(a)} \,. \tag{4.21}$$

This yields the complete anomaly polynomial of a theory including different types of interacting fields. A theory is said to be free of anomalies, if the whole anomaly polynomial I_{D+2} vanishes identically.

The detailed calculations for all those possible anomaly polynomials can be found in [1]. We just outline the results without commenting on any details of the calculations. All formulæ given below apply also to right-handed Weyl-fermions after reversing the sign. This will guarantee the anomaly freedom of parity conserving theories as mentioned above. For convenience we will use here and in all following chapters the notation $\hat{I}_{2l+2} := (2\pi i)^l \, I_{2l+2}$ to absorb the factors of $2\pi i$ into the definition.

- The anomaly polynomial of a left-handed Weyl fermion in a representation R of a gauge group G was deduced above:

  $$\hat{I}_{D+2}^{(1/2)} (\mathcal{R}, \mathcal{F}) = \mathrm{tr}_R \left(e^{i\mathcal{F}} \right) \hat{A} \left(\mathcal{R} \right) \Big|_{D+2} \,. \tag{4.22}$$

  Here $(\ldots)|_{D+2}$ denotes the $(D+2)$-form part of the complete characteristic class, that contains sums of forms of different degrees. $\mathrm{ch} \left(\mathcal{F} \right) := \mathrm{tr}_R \left(e^{i\mathcal{F}} \right)$ is the total Chern character. The anomaly of a Majorana-Weyl fermion is given by one half of (4.22). There are several charged as well as uncharged Weyl-fermions in coupled supergravity/Super-Yang-Mills theories, where (4.22) applies, also for $i\mathcal{F} = 0$.

- Considering supergravity theories it is essential to include also the anomaly polynomial of gravitinos. The anomaly of a left-handed gravitino is most easily given by

  $$\hat{I}_{D+2}^{(3/2)} \left(\mathcal{R} \right) = \left(\sum_i 2 \cosh x_i - 1 \right) \hat{A} \left(\mathcal{R} \right) \Bigg|_{D+2} \tag{4.23}$$

  after diagonalising $i\mathcal{F} \mapsto \mathrm{diag} \left(x_1, \ldots \right)$ where x_i denote the eigenvalue two-forms.

- There are also several self-dual p-form fields present in the spectrum of supergravity theories. They contribute

  $$\hat{I}_{D+2}^{(A)} \left(\mathcal{R} \right) = \frac{-1}{8} L \left(\mathcal{R} \right) \Bigg|_{D+2} \tag{4.24}$$

  using the Hirzebruch L-function.

This concludes our general discussion of anomalies. We are now prepared for the anomaly analysis of the low-energy effective theories of the string theories, the supergravity and Super-Yang-Mills theories in $D = 10$.

4.2 Anomalies in Superstring Theories

The discussion of anomalies in the context of superstrings refers to the field theoretical anomalies of the low-energy supergravity and Super-Yang-Mills theories in $D = 10$. The anomaly in 10 dimensions is governed by hexagonal diagrams. To analyse their anomalous behaviour we have to calculate the anomaly polynomial I_{12} for the fields given in (2.20), (2.21) and (2.27) for the Type II and heterotic strings, respectively. The explicit formulæ for the 12-form part of (4.22), (4.23) and (4.24) can be found e.g. in chapter 12.2 of [51].

We just note that Type IIA is non-chiral, i.e. parity conserving, consequently anomaly-free. Type IIB theory is chiral, hence anomaly cancellation is a non-trivial consistency check. But the contributions of the various fields to I_{12} cancel and one obtains $I_{12} \equiv 0$.

4.2.1 The Green-Schwarz Mechanism

Let us now address the case of the heterotic strings. For the moment we forget about the two gauge groups SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$ obtained by the construction of the heterotic string. We show, how anomaly cancellation fixes an ansatz G for the gauge group of the Super-Yang-Mills sector. This yields (beside two other) the two gauge groups, that appear naturally in the construction of the heterotic string.

The field content of (2.27) gives rise to

$$\hat{I}_{12} = \hat{I}_{12}^{(3/2)}(R) - \hat{I}_{12}^{(1/2)}(R) + \hat{I}_{12}^{(1/2)}(F, R)\,, \tag{4.25}$$

where we denote the field strength and curvature tensor by F and R for notational convenience. The first term $\hat{I}_{12}^{(3/2)}$ is the contribution of the left-handed Majorana-Weyl gravitino $(\mathbf{56}_{\mathrm{S}}, \mathbf{1})$, the second summand $\hat{I}_{12}^{(1/2)}$ due to the right-handed Majorana-Weyl-dilatino $(\mathbf{8}_{\mathrm{C}}, \mathbf{1})$ and the third summand $\hat{I}_{12}^{(1/2)}$ contributed by the left-handed Majorana-Weyl gauginos in $(\mathbf{8}_{\mathrm{S}}, \mathbf{Adj}(G))$, where we show also the representations under the gauge group G.

Extracting the 12-form parts from the general formulæ (4.22) and (4.23) we obtain an expression $\hat{I}_{12}$, that contains all possible terms of order six, $\mathrm{Tr}(iF)^{2k}\mathrm{tr}R^{6-2k}$ for $k = 1, 2, 3$ as well as $\left(\mathrm{tr}R^2\right)^3$, $\mathrm{tr}(iF)^2\left(\mathrm{tr}R^2\right)^2$ and $\mathrm{tr}R^2\mathrm{tr}R^4$. Note, that we use Tr to denote the trace in the adjoint representation of the gauge group G. All these pure gauge, gravitational and mixed anomalies are non-vanishing for all values of $n \equiv \dim(G)$.

The Anomaly Cancellation

It is possible to cancel this anomaly by a tree-diagram of the massless two-form B of the supergravity multiplet coupled to gauge bosons and gravitons in an appropriate way [25]. The possibility to cancel the hexagonal diagram, that is one-loop, by a tree-graph is caused by the peculiar properties of the coupled supergravity/Super-Yang-Mills theory and the massive fields of the underlying string theory generating the right counterterms. For a consistent coupling of these two multiplets, i.e. requiring local supersymmetry, the field strength $H = dB$ has to be changed [7, 8] to

$$H' = dB - Q_{3,Y}\,(A, F) + Q_{3,L}\,(\omega, R) \tag{4.26}$$

in already convenient units. The occurrence of $Q_{3,Y}$ was originally found in the analysis of a minimal coupling these two multiplets. As the Chern-Simons three-form $Q_{3,L}$ of the local Lorentz symmetry depending on the spin connection ω and the curvature tensor R is of higher order in $(\text{length})^{-1}$ it was not found directly. H' appears in the kinetic term for the B field in the action of the considered theory. Therefore, it is a crucial point to require H' in (4.26) to be gauge and locally Lorentz invariant. Hence, one has to demand a non-trivial transformation law for B, which compensates for the gauge transformation of $Q_{3,Y}$ and $Q_{3,L}$. This is possible, because the transformations of the Chern-Simons forms are again exact, $\delta_\Lambda Q_{3,X} = dQ^1_{2,X}$, $X = Y, L$ with a given two-form[4] Q^1_2. Hence one has to demand

$$\delta_\Lambda B = Q^1_{2,Y} - Q^1_{2,L} \tag{4.27}$$

to obtain an invariant H' in (4.26).

Exactly this non-trivial transformation law for B was utilised by Green and Schwarz [25] to cancel all anomalies of the form

$$\hat{I}_{12} = X_8 \cdot X_4 \qquad \text{for} \qquad X_4 = \left(\mathrm{tr}(iF)^2 - \mathrm{tr}R^2\right)\,, \tag{4.28}$$

with a gauge invariant and closed eight-form X_8. This is known as the Green-Schwarz mechanism. One has to introduce the local GS-counterterm

$$S_{\mathrm{GS}} = \int\limits_{M^{10}} B \cdot X_8\,, \tag{4.29}$$

that arises after integrating out massive fields of the underlying string theory in one-loop graphs. Reversing the steps of the descent equations (4.19) yields

$$I^1_{10} = \delta_\Lambda\,(BX_8) = \left(Q^1_{2,Y} - Q^1_{2,L}\right) X_8 \;\Rightarrow\; dI^1_{10} = \left(\delta_\Lambda\,(Q_{3,Y} - Q_{3,L})\right) X_8 \overset{!}{=} \delta_\Lambda I_{11} \tag{4.30}$$

$$I_{11} = (Q_{3,Y} - Q_{3,L})\, X_8 \;\Rightarrow\; I_{12} = dI_{11} = (\mathrm{tr}F^2 - \mathrm{tr}R^2)\, X_8\,. \tag{4.31}$$

Hence, the anomalous variation of the local counterterm (4.29) cancels the factorised anomaly (4.28). This is the usual procedure using the descent equations to obtain the corresponding anomaly polynomial I_{D+2} from terms in the quantum effective action Γ.

Precisely this factorisation (4.28) is possible for the following choices of G

$$\mathrm{SO}(32).\ \mathrm{E}_8 \times \mathrm{E}_8,\ \ \mathrm{E}_8 \times [\mathrm{U}(1)]^{248}\,,\ \ [\mathrm{U}(1)]^{496}\,. \tag{4.32}$$

Detailed calculations can be found e.g in chapter 12.2 of [51]. The key points in the proof are to demand $n = 496$ to cancel some pure gravitational anomalies and several trace

[4] Cf. chapter 11 of [37] for details of their construction.

relations for $\mathrm{Tr}(iF)^6$, $\mathrm{Tr}(iF)^4$, $\mathrm{Tr}(iF)^2$ and the corresponding ones in the fundamental representation tr. The needed group theoretical expressions are precisely obeyed for SO(32) and $E_8 \times E_8$ and the other, phenomenologically uninteresting gauge groups of (4.32). This is the promised determination of the gauge group G by means of anomaly cancellation. For $E_8 \times E_8$ we define tr:$= \frac{1}{30}$Tr for X_4 in (4.28). Then $\hat{I}_{12}$ can be recast into the form (4.28) for

$$X_8 = \frac{1}{96}\left[\frac{1}{24}\mathrm{Tr}(iF)^4 - \frac{1}{7200}(\mathrm{Tr}(iF)^2)^2 - \frac{1}{240}\mathrm{Tr}(iF)^2\mathrm{tr}R^2 + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2\right]. \tag{4.33}$$

Therefore, (4.32) are the only gauge groups G that give rise to a consistent 10-dimensional theory of coupled Super-Yang-Mills/supergravity. This rather mysterious fact in the context of consistent theories of local supersymmetry coupled to matter is clarified by the discovery of the heterotic string theories. It is the consistency and well-defined quantum theory of these string theories, that naturally induces its well-behaved effective theories.

The Bianchi-Identity

Let us investigate the consequences of (4.26). The original gauge invariant field strength $H = dB$ obeys $dH = 0$ trivially. But only H itself is present in the kinetic term in the Lagrangian $H \wedge *H$. Therefore, the theory defined by this action and the subsequent equations of motion $d * H = 0$ do not know of the fact $H = dB$. Hence, one should also state the so-called Bianchi identity $dH = 0$, which implies $H = dB$ at least locally.

For the new field strength H' this simple form of the Bianchi identity does not apply. We obtain instead the non-trivial result

$$dH' = \mathrm{tr}R^2 - \mathrm{tr}(iF)^2\,. \tag{4.34}$$

Integration over any compact, closed four-dimensional submanifold $\mathcal{C}_4$ of 10-dimensional spacetime yields

$$\int_{\mathcal{C}_4} dH' = \int_{\partial\mathcal{C}_4} H' = 0 = \int_{\mathcal{C}_4}\left(\mathrm{tr}R^2 - \mathrm{tr}(iF)^2\right) = 0 \tag{4.35}$$

as a requirement for a globally well-defined H'. Demanding that $\int H'$ makes sense we obtain (4.35) as a necessary condition. We will refer to it as the integrated Bianchi identity, compare to (3.22). It requires, that $\mathrm{tr}R^2$ and $\mathrm{tr}(iF)^2$ define the same cohomology class, which is a severe condition for the construction of allowed gauge bundles on Calabi-Yau manifolds. We encounter this condition again in the context of the construction of consistent U(1)-bundles in section 5.1.2.

Furthermore, we just mention, that the integrated Bianchi identity (4.35) is needed to guarantee vanishing non-Abelian anomalies obtained from the anomaly-free heterotic string by compactification. However, anomalies due to some Abelian U(1)-factors in the gauge group seem to be present in the compactified theories. But the lower-dimensional axions, that are obtained also by compactification solve this problem, as they cancel the remaining anomalies by a lower-dimensional pendant of the Green-Schwarz mechanism. They even generate masses for these anomalous U(1)'s. In 6D compactifications, there are some subtle differences as analysed in [31]. The issue of anomalous U(1)s is also a crucial ingredient for the analysis in section 6 as discussed in section 6.1. There we address the relevant mechanisms, that occur in this context.

Part II

Blowing Up: Orbifolds and their Resolutions

It is known to mathematicians as well as physicists, that orbifolds can emerge as special singular limits of smooth Calabi-Yau manifolds. In this manner the orbifold can be understood as a special point, the so-called orbifold point, on the boundary of a given Calabi-Yau moduli space, i.e. the manifold describing the parameter space of Calabi-Yau manifolds with the same topological data. There are two ways to understand the connection between orbifolds and Calabi-Yau spaces:

- The geometrical picture is quite demonstrative. Starting with a Calabi-Yau one or several cycles of a given smooth Calabi-Yau shrink continuously to zero size and form the orbifold singularity. This is known as the blowing down procedure. The inverse process is referred to as blowing up. Starting with a given orbifold geometry one can resolve one of its singular points to obtain an appropriately constructed smooth space. Proceeding this way for every singular point one can finally obtain a smooth Calabi-Yau, referred to as the resolution: one has moved away from the orbifold point in moduli space.
  Consequently, all four- or six-dimensional models constructed on this blown-up orbifold are models on Calabi-Yau spaces. As this blowing up is a continuous process, the orbifold models should convert into Calabi-Yau models.
- Another point of view is the field theory picture of blowing up. The starting point is already a four- or six-dimensional model obtained after compactification on an orbifold.
  First, we have to introduce the notion of moduli fields: The position in the moduli space is described (at least locally) by the vacuum expectation values $\langle \phi \rangle$ of the so-called moduli fields, collectively denoted by ϕ. These are scalar fields present in the massless four-dimensional spectrum of the heterotic models. They describe fluctuations (of the metric) around the background fixed by the point in moduli space, i.e. $\langle \phi \rangle$. They serve so to say as local coordinates on the moduli space. Unfortunately in general, but fortunately for our discussion, there seems to be no natural and "ad hoc" mechanism in heterotic string theory to fix these moduli by a potential. Therefore, the vacua of string theory are degenerated and the moduli are massless, as already mentioned.
  Next we use this fact to blow up the orbifold model. First of all note that the orbifold is in general no stable vacuum e.g. due to the presence of D-terms. These have to be cancelled by assigning vevs to some states of the model. A possible choice for this is a twisted field Ψ with a flat direction in the potential. It can be interpreted as such a moduli field, the so-called blowing up mode[5] [29, 12, 17]. Along this flat direction arbitrary values of the vev $\langle \Psi \rangle$ are allowed, where the value of the vev corresponds to the radius of curvature of the singular cycle[6]. Hence, utilising this flat direction we can consistently put its vev $\langle \Psi \rangle$ to a non-zero value: we again move away from the orbifold point by resolving the singular cycle and finally obtain a smooth Calabi-Yau. We refer to the model obtained this way as the blown up orbifold.

[5] For a more recent review, see for example [2].
[6] Cf. section 16.4 of [51]

> It is the crucial point of this field theory language, that we do not look explicitly at the geometry. All physics happens due to assigning a non-vanishing vev to a twisted field Ψ of an orbifold model. Possible physical consequences of this vev are a spontaneous breaking of the gauge group including a reduction of its rank, some mass terms for other fields and the cancellation of D-terms. Furthermore, knowledge from orbifold models like the possibility to calculate the superpotential using the orbifold selection rules [29, 12] still applies.

Following this abstract prescription, the two ways should be equivalent blow-up procedures: The construction of a heterotic model on the orbifold and a field theoretical blowing up afterwards should match a specific model constructed directly on a Calabi-Yau. The considered models should explicitly match, concerning all phenomenological aspects like four-dimensional gauge group, matter spectrum, couplings, etc..

In the following chapters we will check this matching explicitly using the field theoretic description to obtain blown up orbifold models that match certain Calabi-Yau models.
As only the local geometry of the orbifold is affected by the geometric blow-up procedure, a local description of orbifolds and Calabi-Yau manifolds is chosen. Thus, we just consider non-compact orbifolds and their resolutions, that are non-compact Calabi-Yau manifolds, as models for the description of the local geometry. We first construct heterotic models on the non-compact versions of the compact orbifold[7] $T^6/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_2$, namely $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_2$, respectively, then classify and explicitly construct four- and six-dimensional Calabi-Yau models with U(1)-bundle background [43]. Then we assign a non-zero vev to a twisted field of the orbifold models , the field theoretic blow up, recover an additional axionic degree of freedom and manipulate the U(1)-charges of all twisted fields [42]. After that, we will match all resolution models in terms of the gauge group, the massless spectrum, the anomalies, the axions and/or B-fields. Furthermore, the calculation of matter multiplicities on the Calabi-Yau will reveal how the notion of localisation of states emerges in this context, i.e. how twisted states of orbifold models might appear in the blowing down procedure. The necessary tools we need are the concepts of local anomalies, local anomalous U(1)'s and axions as discussed in [23, 41].
We will perform the detailed matching for the SO(32) heterotic models on the local space $\mathbb{C}^2/\mathbb{Z}_2$ and its resolution including the analysis of the anomalies, the axions and B-fields. We will then use this local prescription to match also the compact orbifold models on $T^6/\mathbb{Z}_3$ with Wilson lines with the models on compact resolutions obtained by gluing the local, non-compact spaces. We will define this gluing on the level of the gauge background as well as the spectra. Therefore, we will develop appropriate projection rules for both the 4D gauge group and the charged matter. Finally, we will briefly discuss open question and give a general outlook on possibilities for further research.

[7]For an overview of the heterotic models on $T^6/\mathbb{Z}_3$ see [35]. The $T^4/\mathbb{Z}_2$ models are discussed in [32], what we will use as a reference for the gauge group and spectra.

Chapter 5

Resolutions of $\mathbb{C}^n/\mathbb{Z}_n$

In the following we explicitly construct the local geometries both of the orbifold and its resolution. Then we classify all heterotic six- and four-dimensional models obtained by turning on gauge fluxes, namely U(1)-bundles. We will show the details of the six-dimensional models and list also the 4D heterotic models.

5.1 Local Orbifolds and Their Resolutions

First we present a convenient description of the local orbifolds and the gauge backgrounds and then turn to the construction of the resolution and gauge bundles on it. We follow the lines of [43] and apply the results to the relevant cases for our discussion. We also adopt the notation, but always mention the possible changes. Although a more general treatment for all geometries $\mathbb{C}^n/\mathbb{Z}_n$ is possible, we restrict to the cases of $n = 2, 3$ we are interested in.

5.1.1 The Orbifold Geometry and Gauge Backgrounds

For the simplicity of our notation we will only write down the special case $n = 2$. But all results immediately apply to $n = 3$. Furthermore, the description of $n = 3$ can be found in [40], which parallels our discussion.

Local Orbifold Geometry of $T^4/\mathbb{Z}_2$

All singular points, i.e. the fixed points of the orbifold action Θ, are of the same geometrical structure. There is only one type of singularity, that needs to be resolved. Locally, the $T^4/\mathbb{Z}_2$ singularity looks like $\mathbb{C}^2/\mathbb{Z}_2$, that is the non-compact orbifold we mentioned above. Introducing complex coordinates $\tilde{Z}^A$, $A = 1, 2$ the orbifold action Θ is represented by

$$\Theta : \ \tilde{Z}^A \mapsto e^{2\pi i \phi_A/2} \tilde{Z}^A \,, \quad \text{for} \quad \phi = (1, -1) \,, \tag{5.1}$$

what differs from the usual orbifold $\mathbb{Z}_2$-twist of (3.37) by the factor of 2, that is now explicit in the exponential. We introduce the two open sets to cover the orbifold

$$U_{(A)} := \left\{ \tilde{Z} \in \mathbb{C}^2 \middle| \tilde{Z}^A \neq 0\,,\; 0 < \arg\left(\tilde{Z}^A\right) < \pi \right\} \subset \mathbb{C}^2 \tag{5.2}$$

for $\tilde{Z}^A$ having a deficit angle of π. We introduce new coordinates in the chart $U_{(A)}$ defined by

$$z^B := \tilde{Z}^B/\tilde{Z}^A\,,\;\; A \neq B,\;\; x := \left(\tilde{Z}^A\right)^2 \;\Rightarrow\; \left(\tilde{Z}^1, \tilde{Z}^2\right) \mapsto \left(z^B, x\right) \tag{5.3}$$

as an appropriate choice for the construction of the corresponding resolution below. Then the Kähler potential reads as

$$\mathcal{K}_{\text{orbi}} = \sum_{A=1}^{2} \overline{\tilde{Z}}^A \tilde{Z}^A = X^{1/2} \tag{5.4}$$

for $X := \bar{x}\,(1 + \bar{z}z)\,x$. This is simply the Kähler potential of $\mathbb{C}^2$, because the flatness of $\mathbb{C}^2$ is not affected by the identifications due to $\mathbb{Z}_2$, except at the origin.

Gauge Fluxes on the Non-Compact Orbifolds

Now we consider the heterotic string compactified[1] on $M_6 \times \mathbb{C}^2/\mathbb{Z}_2$. To analyse the breaking of the 10D gauge group due to the orbifolding we consider just its low energy effective theory, the coupled supergravity/Super-Yang-Mills system[2] with the field content according to (2.27), on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The relevant part for the discussion in this section is the Super-Yang-Mills sector of the theory, that we address in detail in the following. This includes the Lie-algebra valued one-form field $i\mathfrak{A}$, denoting the gauge field of the 10-dimensional gauge groups $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32) with field strength $i\mathfrak{F}$. The action (5.1) of the orbifold twist Θ induces a non-trivial action on $i\mathfrak{A}$ and boundary conditions for the gauge fields

$$i\mathfrak{A}(z, |x|, \phi + 2\pi) = U i\mathfrak{A}(z, |x|, \phi) U^{-1}\,, \tag{5.5}$$

where $x = |x|e^{i\phi}$ and $U = e^{2\pi i (V_{\text{orb}}^I H_I)/2}$, compare section 3 of [43]. V_{orb}^I, $I = 1, \ldots, 16$ denotes the shift vector of the gauge twisting group, but is here already a lattice vector due to the explicit factor of 2 in (5.5). H_I are the basis elements of the CSA of the respective Lie-algebra. The boundary condition (5.5) induces a gauge symmetry breaking of the form

$$p \cdot V_{\text{orb}} = 0 \bmod 2\,, \tag{5.6}$$

[1] Although it might seem to be a contradiction to use this non-compact space for compactification it makes sense anyhow, because we think of it as the local, though non-compact description of a compact orbifold. Therefore we will refer to this choice of internal space still as compactification.

[2] This low energy description is convenient for the analogy with our discussion of the heterotic models on the smooth resolution $\mathcal{M}^2$. However, we state the caveat that this field theoretic description does not reproduce all aspects of string theory on an orbifold due to the possibility of twisted strings.

compare section 2.1 of [23] for details. Here p is again an $E_8 \times E_8$ or SO(32) root.

For our further discussion in terms of gauge bundles on the resolution manifold of $\mathbb{C}^2/\mathbb{Z}_2$, denoted by $\mathcal{M}^2$, it is necessary to interpret V_{orb} differently. In this context we understand the embedding of the orbifold twist Θ into the gauge degrees of freedom as a localised gauge flux in the orbifold singularity [43]. It will become "visible" on the resolution as a U(1)-gauge bundle.

5.1.2 Resolutions and their Gauge Bundles

The following presentation is again according to the structure and results of [43] and [40], but we emphasise the relevant case of $n = 2$, i.e. the resolution of $\mathbb{C}^2/\mathbb{Z}_2$.

The Resolution Geometry

The smooth space obtained after resolving $\mathbb{C}/\mathbb{Z}_2$ is the resolution $\mathcal{M}^2$. The new coordinates on the orbifold (5.3) are reminiscent of the coordinates of $\mathbb{C}^2\mathbb{P}^2$. It turns out [38], that the resolution is described by

$$\mathcal{M}^2 = \left(\mathbb{C}^2 - \{z_1 = z_2 = 0\}\right)/\mathbb{C}^*, \quad \text{for} \quad (z_1, z_2, x) \sim \left(\lambda z_1, \lambda z_2, \lambda^{-2} x\right) \tag{5.7}$$

defining the $\mathbb{C}^*$-action. The (z_1, z_2, x) can be taken as coordinates of the line bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^2$, described in more detail in [38], using the methods of toric geometry. The fibre coordinate x can be thought of as $x := \left(\tilde{Z}^A\right)^2$ in the chart $U_{(A)}$ to remove the deficit angle of this coordinate. At the locus $x = 0$ the former orbifold singularity is replaced by $\mathbb{CP}^2$ and we recognises the usual definition of this space in (5.7) for $x = 0$. This is a blow-up of a point by a whole manifold in the sense of chapter 2.5 of [33].

The resolution $\mathcal{M}^2$ is already Kähler, but the construction of a Ricci-flat metric is still missing. Following the constructive way of [43] the requirement of Ricci-flatness serves as a non-trivial condition on its Kähler potential $\mathcal{K}_{\mathcal{M}^2}$ yielding a differential equation. Fixing initial data by demanding $\mathcal{K}_{\mathcal{M}^2} = \mathcal{K}_{\text{orbi}}$ in the blow down limit leads to a one-parameter family of Ricci-flat Kähler potentials

$$\mathcal{K}_{\mathcal{M}^n} = \int_1^X \frac{M(X')}{X'}\, dX' \quad \text{for} \quad M(X) = \frac{1}{n}\,(r + X)^{\frac{1}{n}} \tag{5.8}$$

with $n = 2, 3$ for $\mathcal{M}^2$ or $\mathcal{M}^3$ and one parameter $r \in \mathbb{R}$. This formula can be used to calculate the metric G, the vielbeins, the spin connection and the curvature two-form $\mathcal{R}$ of this non-compact Calabi-Yau as presented also in [43]. One phenomenological interesting result, e.g. for the number of generations, is the Euler number of the resolutions

$$\chi\left(\mathcal{M}^2\right) = \frac{1}{2}\int_{\mathcal{M}2} \text{tr}\left(\frac{\mathcal{R}}{2\pi i}\right)^2 = -\frac{3}{2}, \quad \chi\left(\mathcal{M}^3\right) = \frac{1}{3}\int_{\mathcal{M}^3} \text{tr}\left(\frac{\mathcal{R}}{2\pi i}\right)^3 = -\frac{8}{3}, \tag{5.9}$$

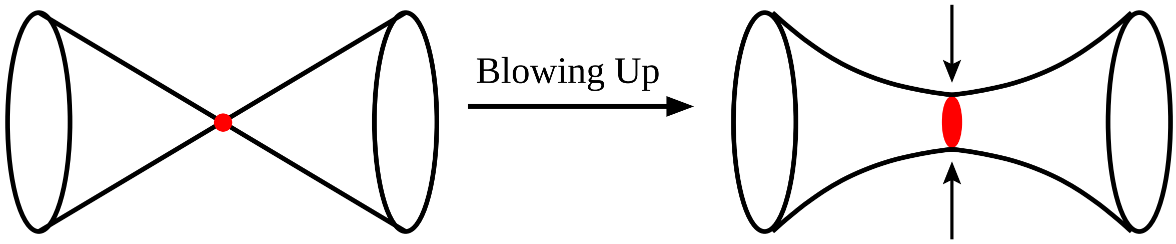

Figure 5.1: The resolution of an orbifold singularity in a local picture. We replace the orbifold singularity by the smooth space $\mathbb{CP}^2$ by continuous variation of r.

where the Euler class $e\left(\mathcal{M}^n\right)$, $n = 2, 3$ occurs as the integrand due to the index theorem for the de Rham complex, cf. section 14.3.4 of [27]. This non-integral numbers occur, because we integrate over a non-compact space, such that usual mathematical theorems for compact manifolds do not apply. We have to interpret this result physically by multiplying it with the number of fixed points, 16 for $T^4/\mathbb{Z}_2$ or 27 for $T^6/\mathbb{Z}_3$, to obtain the reasonable result for the compact resolution, that is of course integral. This can be imagined as gluing the local, non-compact pieces together, a concept, that we will encounter several times in the following discussion.

The limit $r \to 0$ corresponds to the blow-down recovering the orbifold, cf. Fig. 5.1. This blow-down procedure $r \to 0$ can then be visualised as shrinking $\mathbb{CP}^2$, the so-called exceptional divisor, to zero size such that one encounters a curvature singularity at $x = 0$, whereas it vanishes everywhere else. The curvature localises at the orbifold singularity as intuitively expected. We will see, that some states of the spectrum of the model on the resolution can be interpreted as localised at the singularity, see section 5.1.3. This is crucial for the promised matching with orbifold models as one needs to recover the twisted fields of the orbifold models, that are by definition localised.

Construction of Gauge Bundles

The resolutions $\mathcal{M}^2$ and $\mathcal{M}^3$ shall now be used for compcatification of the heterotic string in the same spirit as in section 5.1.1, where we choose the non-compact orbifolds as internal space.

To define reasonable six- or four-dimensional models, that can be matched with some orbifold counterparts, we have to consider gauge bundles on the constructed resolution $\mathcal{M}^2$. These bundles define the background field configuration for the gauge bosons of the Super-Yang-Mills sector of the massless spectrum of the heterotic string in the sense of section 3.1.4. Let us consider the 10-dimensional field strength $i\mathfrak{F}$ with values in $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{SO}(32)$. Thus, we have to expand the 10D field strength $i\mathfrak{F}$ around the internal background $\langle i\mathfrak{F}\rangle =: i\mathcal{F}$ as

$$i\mathfrak{F} = iF + i\mathcal{F}\,, \tag{5.10}$$

for iF denoting the four-dimensional fluctuations with values in the maximally commuting subgroup H, the six-dimensional gauge group. $\mathcal{J}$ denotes again the structure group of the background bundle. It is important to note once more, that a possible Abelian factor $\mathcal{J}^A \subset \mathcal{J}$ is also part of the six-dimensional gauge group $\mathcal{J}^A \subset H$.

The background gauge bundle has to satisfy the consistency conditions of section 3.2.1.

- The condition for $\mathcal{N} = 1$ supersymmetry requires the background field strength to fulfill the hermitian Yang-Mills equations (3.28) as presented in section 3.2.1.
- Another consistency condition is the Bianchi identity that we need in the integrated form (4.35). For the case of $n = 2$ the only four-cycle is the non-compact[3] $\mathcal{M}^2$ itself and for $n = 3$ there are $\mathbb{CP}^2$ and $\mathcal{M}^2$. However, the two four-cycles of $\mathcal{M}^3$ imply only one consistency conditions.

In the following, we consider the easiest solutions to these two conditions.

- The standard embedding introduced in [6] is the easiest solution to (3.28) and (4.35) only taking things into account, that are already present on the Calabi-Yau manifold. That is to put $i\mathcal{F} = \mathcal{R}$, which is a solution because $\mathcal{R}$ interpreted as SU(3) gauge field has vanishing trace, i.e. Ricci tensor. Hence, the resulting unbroken gauge group is $H = \mathrm{E}_6 \times \mathrm{E}_8$.
- It is also possible to construct U(1) gauge bundles by embedding only the U(1)-part of the curvature tensor $\mathcal{R}$. An adequate 1-form $i\mathcal{A}$ obeying the conditions (3.28) can be constructed [43] from the curvature $\mathcal{R}$. Hence, it also localises at the resolved orbifold singularity $x = 0$ and diverges in the blow down limit $r \to 0$ like the curvature $\mathcal{R}$. An embedding into the Cartan subalgebra of $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32) is defined

$$i\mathcal{A}_V := i\mathcal{A}H_V \;\Rightarrow\; i\mathcal{F}_V = i\mathcal{F}H_V \tag{5.11}$$

with $H_V := V^I H_I$. This embedding vector V characterises a specific Calabi-Yau model with U(1)-bundle completely. The integrated Bianchi identity (4.35) can now be obeyed by adequate choices for V as discussed in section 5.1.3.

A further, non-trivial condition for the case of U(1) bundle is given by

$$\frac{1}{2\pi}\int\limits_{\mathbb{CP}^1} i\mathcal{F}_V = V^I H_I \in \mathbb{Z} \tag{5.12}$$

on all considered representations of the models. This is the so-called K-theory constraint for the first Chern class $c_1(\tilde{F}) := \mathrm{tr}\left(i\tilde{F}/2\pi\right)$ of a generic $\tilde{F}$, cf. [31] for further details and literature. Therefore, V has to be a vector of the root lattice of the gauge group, analogous to the requirement on the orbifold side for V_{orb}. This is

[3]Again there appear non-compact spaces violating the compactness condition of various mathematical theorems. It is the physical intuition guided by the transition to global and compact orbifolds and Calabi-Yau manifolds, that are compact.

additional constraint (5.12) is trivial for non-Abelian gauge bundles of simple groups that obey $\mathrm{tr}\tilde{F} = 0$.

For our further discussion, we focus only on these U(1)-bundles as background configurations. The heterotic models thereof are constructed explicitly in section 5.1.3. We just note here that the embedding vector V corresponds to V_{orb} of the corresponding orbifold model with the caveat that this identification works only up to 2λ with λ a vector of the $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32) root lattice. This can be seen by evaluating an integral of $i\mathcal{F}_V$ encircling the resolved orbifold singularity, what gives in the blow down limit just the orbifold shift vector V_{orb} up to 2λ, cf. [43]. Hence, the geometry of the resolution including its U(1)-bundle V matches in its orbifold limit the singular orbifold geometry including the gauge flux V_{orb}.

The four- or six-dimensional gauge group H of a given model defined by V is given by demanding

$$p \cdot V = 0 \tag{5.13}$$

to obtain only the commuting group H. This is a more restrictive condition than the analogous on the orbifold side (5.6) as (5.13) is an exact equation, whereas (5.6) has only to be even. Therefore, we expect the orbifold gauge group to be bigger than the corresponding one on its resolution.

The matter spectrum of a resolution model is given by the representations with $p \cdot V \neq 0$ in the branching

$$\mathbf{Adj}(G) \to \sum_i L_i \otimes Q_i \tag{5.14}$$

of the adjoint of $G = \mathrm{E}_8 \times \mathrm{E}_8$, SO(32) into irreducible representations L_i, Q_i of H, $\mathcal{J}$, compare (3.16). The multiplicity n_{L_i} of a matter multiplet in representation L_i is counted by the number of 6D (or 4D) massless fields occurring from the gauginos $\chi^{(+),a}$ of the 10-dimensional Super-Yang-Mills sector. They transform in the adjoint representation $\mathbf{Adj}(G)$ by supersymmetry and are subject to the branching (5.14). Thus, n_{L_i} is given by the Dirac index of the internal manifold

$$n_{L_i}^{6D} = \frac{1}{2(2\pi i)^2} \int\limits_{\mathcal{M}^2} \left(\mathrm{tr}_{Q_i} (i\mathcal{F})^2 - \frac{n}{24} \mathrm{tr}\mathcal{R}^2 \right) , \tag{5.15}$$

$$n_{L_i}^{4D} = -\frac{1}{6(2\pi i)^3} \int\limits_{\mathcal{M}^3} \left(\mathrm{tr}_{Q_i} (i\mathcal{F})^3 - \frac{1}{4} \mathrm{tr}_{Q_i} (i\mathcal{F}) \mathrm{tr}\mathcal{R}^2 \right)$$

counting the chiral asymmetry $\nu_+ - \nu_-$ of the Weyl-fermions. Due to lower dimensional supersymmetry we observe complete supermultiplets of matter. It is only this chiral asymmetry $\nu_+ - \nu_-$ that is important for the lower dimensional theory as vectorlike matter acquires a mass at a very high scale. It is a very important phenomenological condition to obtain chiral fermions as observed in the standard model. The non-trivial background $i\mathcal{F}$ provides the possibility to obtain chiral fermions in the compactified theory.

	Gauge Group Shift Vector	Untwisted Matter	Twisted Matter
2A	$\mathrm{SO}(28)\times\mathrm{SU}(2)\times\mathrm{SU}(2)'$ $V_{\mathrm{orb}}=(0^{13},1^2,0)$	$\frac{1}{16}(\mathbf{28},\mathbf{2},\mathbf{2})$	$\frac{1}{2}(\mathbf{28},\mathbf{1},\mathbf{2})+2(\mathbf{1},\mathbf{2},\mathbf{1})$
2B	$\mathrm{SO}(20)\times\mathrm{SO}(12)$ $V_{\mathrm{orb}}=(0^{10},1^6)$	$\frac{1}{16}(\mathbf{20},\mathbf{12})$	$\frac{1}{2}(\mathbf{1},\mathbf{32})$
2C	$\mathrm{SU}(16)\times\mathrm{U}(1)$ $V_{\mathrm{orb}}=(\frac{1}{2}^{15},-\frac{3}{2})$	$\frac{1}{8}\mathbf{120}_{(2)}$	$\mathbf{16}_{(-3)}$

Table 5.1: Spectra of the SO(32) heterotic models on $\mathbb{C}^2/\mathbb{Z}_2$. Here we count representations of full hypermultiplets.

5.1.3 Heterotic Models on the Orbifold and its Resolution

We are now in the position to classify and list the various orbifold models defined by V_{orb} and the resolution models with the different U(1) gauge embeddings V. Again, we will do this in detail for the 6D models on $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{M}^2$ and just state the results for $\mathcal{M}^3$, referring to the discussion in [40] and [43].

The Orbifold Models

Here we list the SO(32)-models on $\mathbb{C}^2/\mathbb{Z}_2$ and the $\mathrm{E}_8\times\mathrm{E}_8$-models on $\mathbb{C}^3/\mathbb{Z}_3$ as they are needed for the later matching procedure. Starting with the inequivalent models on compact orbifolds constructed according to the rules and methods described in section 3.2.2 one obtains a finite number of three and five models, respectively. However, we have to divide the given matter multiplicities on the compact $T^4/\mathbb{Z}^2$ [32] and $T^6/\mathbb{Z}_3$ [35] by the number of fixed points, i.e. 16 and 27, to obtain the local spectrum[4]. This yields the right result as all fixed points are identical with respect to their geometry as well as their local gauge embedding V_{orb}. Hence, we obtain the gauge group and spectra given in Tab. 5.1 and Tab. 5.2 where model **2A** and **3A** are just the respective standard embeddings. The $\mathbb{C}^3/\mathbb{Z}_3$ model with trivial shift V_{orb} is omitted as it cannot be used to match a resolution model. Let us now address the notion of localisation of fields guided by the consideration of local orbifolds. The untwisted states are distributed over the whole orbifold, i.e. they are free to move and not localised. Hence, a untwisted field, that has a multiplicity of 1 on the compact orbifold can only occur with the multiplicity of 1/16 at one of the 16 equivalent fixed points. So to say, the "probability" to find it at one fixed point during a "measurement" is 1/16. Twisted fields are localised at one fixed point. They are not free to move. Consequently they are identified with the states with integral multiplicity. This explanation of fractional multiplicities will also be a crucial point for a meaningful analysis of the spectra of the local resolution models, where an easy explanation in terms of twisted fields is not possible. In general, states with integral multiplicity will be interpreted as

[4]We just give the states originating from the 10D vector multiplet. There are additional gauge singlets obtained from the supergravity multiplet. However, they are not relevant for our discussion and omitted.

	Gauge Group Shift Vector	Untwisted Matter Twisted Matter
3A	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8'$ $V_{\text{orb}} = (-2, 1^2, 0^5)(0^8)$	$\frac{1}{9}(\mathbf{27}, \mathbf{3}; \mathbf{1})$ $(\mathbf{27}, \mathbf{1}; \mathbf{1}) + 3(\mathbf{1}, \overline{\mathbf{3}}; \mathbf{1})$
3B	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_6' \times \mathrm{SU}(3)'$ $V_{\text{orb}} = (-2, 1^2, 0^5)(-2, 1^2, 0^5)$	$\frac{1}{9}\left[(\mathbf{27}, \overline{\mathbf{3}}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{27}, \overline{\mathbf{3}})\right]$ $(\mathbf{1}, \mathbf{3}; \mathbf{1}, \mathbf{3})$
3C	$\mathrm{E}_7 \times \mathrm{SO}(14)' \times \mathrm{U}(1) \times \mathrm{U}(1)'$ $V_{\text{orb}} = (1^2, 0^6)(-2, 0^7)$	$\frac{1}{9}\left[(\mathbf{56}; \mathbf{1})_{(2,2)} + (\mathbf{1}; \mathbf{64})_{(-1,2)} + (\mathbf{1}; \mathbf{14})_{(2,-4)} + (\mathbf{1}; \mathbf{1})_{(-4,-4)}\right]$ $(\mathbf{1}; \mathbf{14})_{(2,0)} + (\mathbf{1}; \mathbf{1})_{(-4,0)} + 3(\mathbf{1}; \mathbf{1})_{(0,4)}$
3D	$\mathrm{SU}(9) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$ $V_{\text{orb}} = (-2, 1^4, 0^3)(-2, 0^7)$	$\frac{1}{9}\left[(\mathbf{84}; \mathbf{1})_{(0)} + (\mathbf{1}; \mathbf{64})_{(-1)} + (\mathbf{1}; \mathbf{14})_{(2)}\right]$ $(\overline{\mathbf{9}}; \mathbf{1})_{(-4/3)}$

Table 5.2: Spectra of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic models on $\mathbb{C}^3/\mathbb{Z}_3$. The two U(1)-generators of **3C** are chosen to be $Q = (2^2, 0^6)\,(2, 0^7)$, $Q' = (2^2, 0^6)\,(-4, 0^7)$. Q already equals the embedding vector V of resolution model **3C I** for later convenience. The other U(1)-generators are fixed by the breaking of the 10D gauge group.

localised states and those with fractional multiplicity as bulk states.

Note however that we count $\mathcal{N} = 1$ hypermultiplets in $D = 6$. Hence, a factor of 1/2 might occur even for localised states like the twisted states in model **2A** and **2B** of $\mathbb{C}^2/\mathbb{Z}_2$ that are in real representations. This is due to the fact that these states form only half-hypermultiplets, that become full hypermultiplets if another fixed point is taken into account.

A hypermultiplet in 6D, $\mathcal{N} = 1$ in the representation R has the field content ψ^+, ϕ_1, ϕ_2 where ϕ_i complex scalars. The spectrum of states for all choices R, even real ones, is no irreducible representation of the supersymmetry algebra. It is the sum of two irreducible ones of the same kind, the half-hyper that is itself in general not CPT invariant for generic R. It is CPT-invariant only for R denoting a real representation. One half-hyper contains the particle states in representation R, the other one the antiparticle states in representation $\bar{R}$. In $D = 6$, both fermions and antifermions are of the same chirality. Thus, the states of equal spin are in a real representation $R \oplus \bar{R}$. Compare App. B of [51] for more details. However, we will always denote the states in hypermultiplets only by the representation R. Hence, we always have the freedom to change between R and $\overline{R}$ wherever it is needed. Thus, we could denote the states of Tab. 5.1 and 5.3 either by R or its complex conjugate representation $\bar{R}$ by counting one time particles, the other time antiparticles. As they are of the same chirality, both a left-handed fermion in R and also the left-handed antifermion in $\bar{R}$ give a positive contribution to the chiral asymmetry.

Classification of U(1)-Bundles and Models

The construction of a consistent resolution model is constraint by the Bianchi identity (4.35) and the K-theory constraint (5.12). These are severe restrictions reducing the number of models to a finite number. Let us analyse the integrated Bianchi identity for the U(1)-background $\mathcal{F}_V$. Applying it to $\mathcal{M}^2$ and $\mathcal{M}^3$ we obtain

$$\mathcal{M}^2 \,:\, V^2 = 6, \quad \mathcal{M}^3 \,:\, V^2 = 12, \tag{5.16}$$

using the necessary integrals of App. B of [43] for the relevant 4-cycles of $\mathcal{M}^2$ and $\mathcal{M}^3$. This enormously restricts the allowed choices for V to a finite number. (5.16) can be interpreted as the counterpart of the condition for modular invariance (3.55) of the orbifold. The (inequivalent) solutions of (5.16) including the corresponding spectra and gauge groups are given in Tab. 5.3 and Tab. 5.4. Additionally, the standard embedding is listed as the first model to complete our list. It will not be considered further. One of the U(1)-generators of the gauge group is identical to the embedding vector V as the background U(1) is always part of the 4D or 6D gauge group. Hence, we denote it by $\mathrm{U}(1)_V$ and its generator by Q_V. The multiplicity of all four- and six-dimensional matter is calculated by means of the Dirac index (5.15) on the resolution. According to the explicit formulæ and calculations of [43] one can express the multiplicity for the U(1)-bundles in terms of operators

$$\mathcal{M}^2 \,: \quad N_V := \frac{1}{4}\left(H_V^2 - \frac{1}{2}\right)\,, \qquad \mathcal{M}^3 \,: \quad N_V := \frac{1}{6}\left(-H_V^3 + H_V\right)\,, \tag{5.17}$$

where we use again $H_V = V^I H_I$. The value H_V is simply the charge under $\mathrm{U}(1)_V$ and can be read off from Tab. 5.3 and Tab. 5.4.

At a first glance the fractional numbers for the multiplicity seem odd. However, we have to remember that we are working in a local patch of a global manifold. Analogous to the orbifold we observe only one sixteenth (or one twenty-seventh) of the matter on the global resolution manifold. Hence, we interpret matter with integral multiplicity as localised matter that is distributed only in our local patch. Matter with fractional multiplicity is again bulk matter, of which we observe only the sixteenth (or twenty-seventh) part. However, a multiplicity of 7/8 is more complicated. It is interpreted as a localised state with multiplicity 1 and two bulk states of opposite chirality and multiplicity $2 \cdot \frac{1}{16} = \frac{1}{8}$. We will see in the considerations of the explicit matching of section 7.1 how this can be understood physically. Analogously, the 17/8 are considered as 2 local and two bulk field $2 \cdot \frac{1}{16} = \frac{1}{8}$.

5.2 Details of the 6D Orbifold Models

Here we present the details of the orbifold models. We will show the root vectors of the gauge groups as well as the weight vectors of all matter representations[5]. We have checked all our results with a computer-based program [52]. We also used it to obtain the shifted weight vectors of the twisted states. The results are given in Tab. 5.5.

[5]For a introduction to group theory, algebras and their representations see e.g. [19], where in chapter 7 the convenient basis for the weight lattice as used here is introduced.

	Gauge Group Shift Vector	Matter
St. Emb.	SO(28) × SU(2) SU(2) bundle	$\frac{5}{8}\,(\mathbf{28},\mathbf{2}) + \frac{45}{16}\,(\mathbf{1},\mathbf{1})$
2A I	$\mathrm{SO}(26)\times\mathrm{SU}(2)\times\mathrm{U}(1)_V\times\mathrm{U}(1)$ $V=(0^{13},1^2,2)$	$\frac{1}{8}\left[(\mathbf{26},\mathbf{2})_{(1,1)} + (\mathbf{1},\mathbf{2})_{(1,-2)} + (\mathbf{1},\mathbf{2})_{(3,0)}\right]$ $+\,2(\mathbf{1},\mathbf{2})_{(3,0)} + \frac{7}{8}(\mathbf{1},\mathbf{1})_{(2,2)} + \frac{7}{8}(\mathbf{26},\mathbf{1})_{(2,-1)}$
2B I	$\mathrm{SO}(20)\times\mathrm{SU}(6)\times\mathrm{U}(1)_V$ $V=(0^{10},1^6)$	$\frac{1}{8}(\mathbf{20},\mathbf{6})_{(1)} + \frac{7}{8}(\mathbf{1},\mathbf{15})_{(2)}$
2C I	$\mathrm{SU}(15)\times\mathrm{U}(1)_V\times\mathrm{U}(1)$ $V=(\frac{1}{2}^{15},-\frac{3}{2})$	$\frac{1}{8}\left[\mathbf{15}_{(-1,6)} + \frac{7}{8}\mathbf{105}_{(1,2)}\right] + \mathbf{15}_{(2,-4)}$

Table 5.3: Spectra of the SO(32) heterotic models on $\mathcal{M}^2$. The first model is the standard embedding that is given just for completeness. $\mathrm{U}(1)_V$ denotes the U(1) that is chosen to equal the direction of the U(1)-background over $\mathcal{M}^2$.

	Gauge Group Shift Vector	Untwisted Matter Twisted Matter
3A I	$\mathrm{E}_6\times\mathrm{E}_8$ SU(3) bundle	$\frac{12}{9}(\mathbf{27};\mathbf{1})$
3A II	$\mathrm{SO}(10)\times\mathrm{U}(3)\times\mathrm{E}_8$ $V=(3,1^3,0^4)(0^8)$	$\frac{1}{9}\left[(\mathbf{16},\mathbf{3};\mathbf{1})_{-1} + (\mathbf{10},\mathbf{3};\mathbf{1})_{(2)} + (\mathbf{1},\mathbf{3};\mathbf{1})_{(-4)}\right]$ $+(\mathbf{16},\mathbf{1};\mathbf{1})_{(3)} + 3(\mathbf{1},\overline{\mathbf{3}};\mathbf{1})_{(4)}$
3B I	$\mathrm{E}_6\times\mathrm{U}(2)\times\mathrm{E}_6\times\mathrm{U}(2)'$ $V=(1^6,0^2)(2,1^2,0^5)$	$\frac{1}{9}\left[(\mathbf{27},\mathbf{2};\mathbf{1},\mathbf{1})_{(-1,-1)} + (\mathbf{1},\mathbf{1};\mathbf{27},\mathbf{2})_{(-1,1)}\right.$ $\left.+(\mathbf{27},\mathbf{1};\mathbf{1},\mathbf{1})_{(2,2)} + (\mathbf{1},\mathbf{1};\mathbf{27},\mathbf{1})_{(2,-2)}\right]$ $+(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(3,3)} + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{(3,-3)}$
3C I	$\mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)\times\mathrm{U}(1)'$ $V=(2^2,0^6)(2,0^7)$	$\frac{1}{9}\left[(\mathbf{56};\mathbf{1})_{(2,2)} + (\mathbf{1};\mathbf{64})_{(-1,2)} + (\mathbf{1};\mathbf{14})_{(2,-4)}\right.$ $\left.+(\mathbf{1};\mathbf{1})_{(-4,-4)}\right] + 3(\mathbf{1};\mathbf{1})_{(4,4)}$
3C II	$\mathrm{E}_7\times\mathrm{SO}(12)\times\mathrm{U}(1)\times\mathrm{U}(1)'\times\mathrm{U}(1)''$ $V=(1^2,0^6)(3,1,0^6)$	$\frac{1}{9}\left[(\mathbf{56};\mathbf{1})_{(-1,-1,-2)} + (\mathbf{1};\mathbf{32})_{(-1,-1,-2)}\right.$ $+(\mathbf{1};\overline{\mathbf{32}})_{(2,-1,0)} + (\mathbf{1};\mathbf{12})_{(-1,2,-2)}$ $\left.+(\mathbf{1};\mathbf{1})_{(2,2,-4)} + (\mathbf{1};\mathbf{1})_{(-4,2,0)} + (\mathbf{1};\mathbf{1})_{(2,2,4)}\right]$ $+(\mathbf{1};\mathbf{12})_{(3,0,-2)} + 3(\mathbf{1};\mathbf{1})_{(4,-2,0)}$
3D I	$\mathrm{SU}(8)\times\mathrm{SO}(14)\times\mathrm{U}(1)\times\mathrm{U}(1)'$ $V=((2,1^4,0^3)(2,0^7)$	$\frac{1}{9}\left[(\mathbf{56};\mathbf{1})_{(-1,-1)} + (\mathbf{1};\mathbf{64})_{(-1,2)}\right.$ $\left.+(\mathbf{28};\mathbf{1})_{(2,2)} + (\mathbf{1};\mathbf{14})_{(2,-4)}\right] + (\overline{\mathbf{8}};\mathbf{1})_{(3,3)}$

Table 5.4: Spectra of the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic models on $\mathcal{M}^3$. The many equivalent shifts V yielding the same model are omitted here.

Let us introduce our notation. Each weight or root is represented by a 16-dimensional vector. We denote consecutive k zeros by 0^k. Furthermore, we underline parts of the vector to denote that all permutations of the underlined components have to be taken into account. It is also possible that there is more than one line indicating permutations within one vector. Then we use different types of lines to identify the relevant components. Additionally, we use "$\pm z$" for components z that occur with both signs. However, in the case of more than one "$\pm$" in one vector they have to be interpreted as independent. Otherwise, we will denote coupled plus signs with additional brackets $(\ldots)$. In the case of a spinor representation we use $(\pm\frac{1}{2})^k$ to denote k components with "$\pm$" signs. This is always restricted by the requirement of an even or odd number of plus signs corresponding to the chirality of the spinor representation. One can check the validity of this table by calculating the Dynkin labels Λ for each representation. It is given by the vector Λ with components λ_i, defined by

$$\lambda_i = p_i \cdot W, \tag{5.18}$$

where p_i denotes the i-th simple root and W the highest weight of a given representation R. We also list the U(1)-generators Q that are part of the 16-dimensional Cartan subalgebra (CSA) of the full gauge group. They are again represented by vectors with the interpretation $H_Q = H^I Q_I$ for the standard basis H^I, $I = 1, \ldots, 16$ of the CSA. H_Q defines the direction of the respective U(1) in the CSA. The U(1)-charges q of the various representations R with highest weight W of the non-Abelian gauge factors are determined by

$$H_Q\,|W\rangle = (Q \cdot W)\,|W\rangle \;\;\Rightarrow\;\; q = (Q \cdot W)\,. \tag{5.19}$$

This charge is the same for all states of the representation R as they are obtained from the highest weight by descending with the lowering operators E_p of the roots p. This is due to the fact that $[H_Q, E_p] = (Q \cdot p)E_p = 0$ as the U(1) has to be orthogonal to the rest of the gauge group.

Model	Representation	Weights	#(Vectors)
	$\mathbf{Adj}(\mathrm{SO}(28))$	$(\underline{\pm1,\pm1,0^{11}},0,0,\underline{0})$	$364\,(+14=378)$
	$\mathbf{Adj}(\mathrm{SU}(2))$	$(0^{13},\pm(1,-1),0)$	$2\,(+1=3)$
2A	$\mathbf{Adj}(\mathrm{SU}(2)\text{'})$	$(0^{13},\pm(1,1),0)$	$2\,(+1=3)$
$V_{\mathrm{orb}}=(0^{13},1^2,0)$	$(\mathbf{28},\mathbf{2},\mathbf{2})$	$(\underline{\pm1,0^{12}},\pm(\utilde{1,0}),\underline{0})$	$28\cdot4=112$
	$(\mathbf{28},\mathbf{1},\mathbf{2})$	$(\underline{\pm1,0^{12}},\pm(\tfrac{1}{2},\tfrac{1}{2}),\underline{0})$	$28\cdot2=56$
	$(\mathbf{1},\mathbf{2},\mathbf{1})$	$(0^{13},\pm(\tfrac{1}{2},-\tfrac{1}{2}),0)$	2
	$\mathbf{Adj}(\mathrm{SO}(20))$	$(\underline{\pm1,\pm1,0^8},0^6)$	$180\,(+10=190)$
2B	$\mathbf{Adj}(\mathrm{SO}(12))$	$(0^{10},\underline{\pm1,\pm1,0^4})$	$60\,(+6=66)$
$V_{\mathrm{orb}}=(0^{10},1^6)$	$(\mathbf{20},\mathbf{12})$	$(\underline{\pm1,0^9},\utilde{\pm1,0^5})$	$20\cdot12=240$
	$(\mathbf{1},\mathbf{32})$	$(0^{10},(\pm\tfrac{1}{2})^6)$	32
	$\mathbf{Adj}(\mathrm{SU}(16))$	$\pm(\underline{1,0^{14},-1})$	$240\,(+15=255)$
2C	$\mathrm{U}(1)$	(1^{16})	1
$V_{\mathrm{orb}}=({\tfrac{1}{2}}^{15},-\tfrac{3}{2})$	$\mathbf{120}_{(2)}$	$(\underline{1,1,0^{14}})$	120
	$\mathbf{16}_{(-3)}$	$(\underline{\tfrac{3}{4},(-\tfrac{1}{4})^{15}})$	16

Table 5.5: Orbifold models **2A**, **2B** and **2C**. The vectors for the adjoint cover only the non-zero roots, not the elements of the Cartan subalgebra (CSA). The last column gives the number of vectors in each row, that is in the case of the adjoint representation enhanced by the CSA vectors with (0^{16}). $(\pm1/2)^6$ has an even number of plus signs.

Chapter 6

The Matching Recipe

After presenting the local models on the non-compact orbifolds and their geometrical resolutions $\mathcal{M}^n$ we address now the general prescription for a matching of them on the level of gauge group and spectrum. As mentioned above, this will in principle be achieved by assigning a non-vanishing vev to some twisted field Ψ of the orbifold models of Tab. 5.1 and Tab. 5.2.

However, we will encounter several difficulties to match the U(1)-charges of the twisted (local) matter and the anomaly polynomials of the two sides. At first glance, this seems to prohibit a matching of the models. Some additional physical input is necessary, the notion of a non-universal axion a^T [42] appearing in the superfield parametrising the perturbations around the vev of the twisted field Ψ. We will first encounter this subtle issue by comparing the structure of the anomaly polynomials of the orbifold and resolution models in sections 6.1 and 6.1.2. On the resolution there will be one additional term not present on the orbifold, that corresponds to one additional anomalous U(1), for $\mathcal{M}^2$ as well as $\mathcal{M}^3$. Following the analysis of anomaly cancellation in section 6.1.3 an additional localised non-universal axion is necessary to cancel this new term. Of course, this axion is not present on the orbifold. Finally, in section 6.2 we will solve this problem by describing the field theoretic blowing up of the orbifold models in general, introducing the axion a^T and perform anomalous field redefinitions according to [42]. There we will also present the right relation between the anomalies of the orbifold and resolution side and the consequences for the axions.

6.1 Multiple Anomalous U(1)'s and Localised Axions

The following discussion is devoted to the anomaly analysis of the orbifold and resolution models and their cancellation. As the original 10D theory is free of anomalies also the compactified theories have to be anomaly-free. Hence, we will also see how the notion of lower dimensional versions of the Green-Schwarz mechanism and axions occur. Again, we will present the details only for $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{M}^2$ and just cite the results for $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathcal{M}^3$.

6.1.1 The Anomaly of the Orbifold Models

The 6D case for $\mathbb{C}^2/\mathbb{Z}_2$

We start with the discussion of six dimensional anomalies [16] for the $\mathbb{C}^2/\mathbb{Z}_2$ models. We will calculate the anomaly polynomials explicitly in section 6.3.1. But let us just outline their general structure, following our calculations and [32]. There the anomalies for all orbifold models of Tab. 5.1 are given in the factorised form

$$\hat{I}_8^{\text{het}} = X_{0,4} \cdot X_4^{\text{het}} \tag{6.1}$$

for $X_{0,4} := \text{tr}R^2 - \text{tr}(iF)^2$ with $\text{tr} \equiv \text{tr}_{\mathbf{32}} = \sum_i \text{tr}_{R_i}$ for the appropriate branching of $\mathbf{32}$ into the fundamental and anti-fundamental representations R_i of the group factors of the 6D gauge group. As there are no Wilson-lines, that might distinguish between the various fixed point, the expression for the local model is obtained by dividing the global one by 16. We collectively refer to the anomalies of the orbifold models as the heterotic anomalies.

The anomaly (6.1) can precisely be cancelled by the anomalous variation of the 6D B-field h_2 as discussed in section 6.1.3. This is just the 6D analogue of the 10D Green-Schwarz mechanism with the GS-counterterm

$$S_{\text{GS}} = \int_{\mathbb{M}_6} h_2 \cdot X_4^{\text{het}}. \tag{6.2}$$

Anomalies for $\mathbb{C}^3/\mathbb{Z}_3$ and their Cancellation

As already analysed in [41] for compact orbifolds there is at most one anomalous U(1) present at each fixed point. All non-Abelian anomalies have to vanish yielding again a factorisable local anomaly

$$\hat{I}_6^{\text{het}} = X_2 \cdot X_4^{\text{het}}, \tag{6.3}$$

where X_2 is proportional to the field strength iF of the anomalous U(1)-direction. The global one is obtained as the sum of the local terms, that is simply 27 times the local anomaly for the absence of Wilson-lines. For our further discussion we will refer to this kind of anomaly as the heterotic anomaly of the orbifold.

It was also shown that this local anomalies are cancelled by the anomalous variation of a 4D axion a^{het}, the heterotic axion, localised at the fixed point [23]. This is the 4D remnant of the Green-Schwarz mechanism as the 4D B-field is dual to a pseudo-scalar a^{het}. The GS-counterterm reads as

$$S_{\text{GS}} = \int_{\mathbb{M}_4} a^{\text{het}} \cdot X_4^{\text{het}}\,. \tag{6.4}$$

6.1.2 Structure of the Anomaly on the Resolution

In contrast to the orbifold models with a complete factorisation (6.1) in 6D and a maximum of one anomalous U(1) in 4D at every fixed point, (6.3), we encounter on the respective local resolutions one possible additional anomalous U(1) in 6D as well as in 4D. This issue seems at first glance to be a contradiction to our statements about the matching of the models. However, this problem is analysed and completely solved in [42], which will be used for the matching prescription in 6.2. But let us first deduce the factorisation of the 6D anomalies. The calculations in four dimensions can be found in detail in [40] and we will just give the results here.

6D Anomaly for $\mathcal{M}^2$

The form of the six-dimensional anomaly on $\mathcal{M}^2$ is completely fixed by the underlying 10D theory, the coupled supergravity/Super-Yang-Mills theory. Start with the anomaly $\hat{I}_{12}$ in the already factorised form of $\hat{I}_{12} = X_4 \cdot X_8$ of section 4.2 with $X_4 = \mathrm{tr}\mathfrak{R}^2 - \mathrm{tr}\,(i\mathfrak{F})^2$ and

$$X_8 = \frac{1}{96}\left[\frac{1}{24}\mathrm{Tr}(i\mathfrak{F})^4 - \frac{1}{7200}(\mathrm{Tr}(i\mathfrak{F})^2)^2 - \frac{1}{240}\mathrm{Tr}(i\mathfrak{F})^2\mathrm{tr}\mathfrak{R}^2 + \frac{1}{8}\mathrm{tr}\mathfrak{R}^4 + \frac{1}{32}(\mathrm{tr}\mathfrak{R}^2)^2\right]\,. \quad (6.5)$$

The expansion $i\mathfrak{F} = i\mathcal{F}_V + iF$ and $\mathfrak{R} = \mathcal{R} + R$ in terms of fluctuations iF, iR around the background $i\mathcal{F}$, $\mathcal{R}$ yields

$$\hat{I}_{12} = \left(\mathrm{tr}R^2 - \mathrm{tr}(iF)^2\right) X_{4,4} + 2\left[\mathrm{tr}\,(R\mathcal{R}) - \mathrm{tr}\,((iF)\,(i\mathcal{F}_V))\right] X_{2,6} + \left(\mathrm{tr}\mathcal{R}^2 - \mathrm{tr}\,(i\mathcal{F}_V)^2\right) X_{8,0}, \quad (6.6)$$

where the first (second) index refers to the number of legs in 4D internal (6D Minkowski) space of the various forms $X_{a,b}$ with $a + b = 8$. Note, that the term $\mathrm{tr}\,(R\mathcal{R})$ vanishes as R and $\mathcal{R}$ can be thought of as 10×10-matrices, that are block-diagonal in the upper 4×4- and lower 6×6-block, respectively, cf. 3.1.1.

Integration over the internal space $\mathcal{M}^2$ yields the six-dimensional anomaly polynomial

$$\hat{I}_8^{\mathrm{res}} = (2\pi i)^3\, I_8^{\mathrm{res}} \equiv (2\pi i)^3 \int_{\mathcal{M}^2} I_{12} = \frac{1}{(2\pi i)^2}\int_{\mathcal{M}^2} X_4 \cdot X_8\,. \quad (6.7)$$

Plugging in the expansion (6.6) we immediately see, that the last term drops out as it is simply the integrated Bianchi identity (4.35), that our U(1)-backgrounds fulfill by construction. Therefore, we only have to consider the remaining two terms. Note that we will drop the i in the expressions of the various field strengths iF, $i\mathcal{F}$ and $i\mathfrak{F}$ in the steps of the calculations. They will appear again in the final results.

<u>1. $X_{4,4}$:</u> To extract the $(4,4)$-part of X_8 we need the following results

$$\mathrm{Tr}(\mathfrak{F})^4 = \mathrm{Tr}\,(F + \mathcal{F}_V)^4 \supset \binom{4}{2}\mathrm{Tr}\left(F^2\mathcal{F}_V^2\right), \quad \left(\mathrm{Tr}\mathfrak{F}^2\right)^2 \supset 2\mathrm{Tr}F^2\mathrm{Tr}\mathcal{F}_V^2 + 4\left(\mathrm{Tr}\,(F\mathcal{F}_V)\right)^2, \quad (6.8)$$

$$\mathrm{Tr}\mathfrak{F}^2\mathrm{tr}\mathfrak{R}^2 \supset \mathrm{Tr}\mathcal{F}_V^2\mathrm{tr}R^2 + \mathrm{Tr}F^2\mathrm{tr}\mathcal{R}^2, \quad \mathrm{tr}\mathfrak{R}^4 \supset \binom{4}{2}\mathrm{tr}\left(R^2\mathcal{R}^2\right) = 0, \quad \left(\mathrm{tr}\mathfrak{R}^2\right)^2 \supset 2\mathrm{tr}R^2\mathrm{tr}\mathcal{R}^2, \quad (6.9)$$

where again the terms $R\cdot\mathcal{R}$ vanish and $\supset$ denotes, that some terms are omitted. Inserting into (6.5) yields

$$\begin{aligned}
96\cdot X_{4,4} &= \frac{1}{4}\mathrm{Tr}\left(F^2\mathcal{F}_V^2\right) - \frac{1}{3600}\mathrm{Tr}F^2\mathrm{Tr}\mathcal{F}_V^2 - \frac{1}{1800}\left(\mathrm{Tr}\left(F\mathcal{F}_V\right)\right)^2 - \frac{1}{240}\mathrm{Tr}\mathcal{F}_V^2\mathrm{tr}R^2 && (6.10)\\
&\quad - \frac{1}{120}\mathrm{Tr}F^2\mathrm{tr}_{\mathbf{2}}\mathcal{R}^2 + \frac{1}{8}\mathrm{tr}_{\mathbf{2}}\mathcal{R}^2\mathrm{tr}R^2 && (6.11)\\
&= \mathcal{F}^2\left[\frac{1}{4}\mathrm{Tr}\left(F^2H_V^2\right) - \frac{1}{3600}\mathrm{Tr}F^2\mathrm{Tr}H_V^2 - \frac{1}{1800}\left(\mathrm{Tr}\left(FH_V\right)\right)^2 - \frac{1}{240}\mathrm{Tr}H_V^2\mathrm{tr}R^2\right] && (6.12)\\
&\quad + \mathrm{tr}_{\mathbf{2}}\mathcal{R}^2\left[\frac{1}{8}\mathrm{tr}R^2 - \frac{1}{120}\mathrm{Tr}F^2\right], && (6.13)
\end{aligned}$$

where $\mathcal{F}_V = \mathcal{F}H_V$ of (5.11) and $\mathrm{tr}\mathcal{R} \equiv \mathrm{tr}_{\mathbf{10}}\mathcal{R} = 2\mathrm{tr}_{\mathbf{2}}\mathcal{R}$ is used. This is due to the breaking of the local Lorentz group $\mathrm{SO}(9,1) \to \mathrm{SO}(5,1)\times\mathrm{SO}(4)$, the further $\mathrm{SO}(4) \to \mathrm{SU}(2)\times\mathrm{U}(1)$ and $\mathbf{10} \to (\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{4})$ and the branching $(\mathbf{1},\mathbf{4}) \to \mathbf{2}_{(1)}\oplus\mathbf{2}_{(-1)}$ using $\overline{\mathbf{2}} = \mathbf{2}$. This induces $\mathrm{tr}_{\mathbf{10}}\mathcal{R} = 2\mathrm{tr}_{\mathbf{2}}\mathcal{R}$ as $\mathcal{R} \equiv 0\oplus\mathcal{R}$ on $(\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{4})$ and because of the vanishing U(1)-part of the internal curvature $\mathcal{R}$, compare section 3.2.1.
After re-expressing this trace we can apply the integrals $\frac{1}{(2\pi i)^2}\int\limits_{\mathcal{M}^2}(i\mathcal{F})^2 = -1/2$ and $\frac{1}{(2\pi i)^2}\int\limits_{\mathcal{M}^2}\mathrm{tr}\mathcal{R}_{\mathbf{2}}^2 = -3$ of App. B of [43] to evaluate the Bianchi identity to obtain

$$\int\limits_{\mathcal{M}^2}\left(2\mathrm{tr}_{\mathbf{2}}\mathcal{R}^2 - \mathrm{tr}(i\mathcal{F}_V)^2\right) = 0 \quad\Rightarrow\quad \mathrm{Tr}H_V^2 = 30\cdot\mathrm{tr}H_V^2 = 360. \tag{6.14}$$

Finally, we use the same integrals to obtain the expression for the four-form part

$$X_4^{\mathrm{uni}} := \int\limits_{\mathcal{M}^2}\frac{X_{4,4}}{(2\pi i)^2} = -\frac{1}{96}\left\{\mathrm{Tr}\left[\left(\frac{1}{8}H_V^2 - \frac{3}{40}\right)(iF)^2\right] - \frac{1}{4\cdot 30^2}\left(\mathrm{Tr}\left[H_V(iF)\right]\right)^2 - \frac{3}{8}\mathrm{tr}R^2\right\}. \tag{6.15}$$

<u>**2.** $X_{2,6}$</u>: The $(2,6)$-part of X_8 is obtained from

$$\mathrm{Tr}\mathfrak{F}^4 \supset \binom{4}{3}\mathrm{Tr}\left(F^3\mathcal{F}_V\right),\quad \left(\mathrm{Tr}\mathfrak{F}^2\right)^2 \supset \binom{4}{3}\mathrm{Tr}F^2\mathrm{Tr}\left(\mathcal{F}_VF\right), \tag{6.16}$$

$$\mathrm{Tr}\mathfrak{F}^2\mathrm{tr}\mathfrak{R}^2 \supset 2\mathrm{Tr}\left(F\mathcal{F}_V\right)\mathrm{tr}R^2,\quad \mathrm{tr}\mathfrak{R}^4 \supset \binom{4}{3}\mathrm{tr}\left(R\mathcal{R}^3\right) = 0,\quad \left(\mathrm{tr}\mathfrak{R}^2\right)^2 \supset 0. \tag{6.17}$$

Applying all formulæ mentioned above, we finally deduce

$$X_{2,6} = \frac{\mathcal{F}}{96}\left[\frac{1}{6}\mathrm{Tr}\left(H_VF^3\right) - \frac{1}{1800}\mathrm{Tr}\left(H_VF\right)\mathrm{Tr}F^2 - \frac{1}{120}\mathrm{tr}R^2\mathrm{Tr}\left(H_VF\right)\right] \Longrightarrow \tag{6.18}$$

$$X_6^{\mathrm{non}} := \int\limits_{\mathcal{M}^2}\frac{\mathcal{F}X_{2,6}}{(2\pi i)^3} = -\frac{1}{192}\left\{\frac{1}{6}\mathrm{Tr}\left(H_V(iF)^3\right) - \frac{1}{120}\mathrm{Tr}\left(H_V(iF)\right)\left[\frac{1}{15}\mathrm{Tr}\left(iF\right)^2 + \mathrm{tr}R^2\right]\right\}, \tag{6.19}$$

where $-2\mathrm{tr}\,(\mathcal{F}_V F) = -2\mathrm{tr}\,(H_V F)\,\mathcal{F}$ in (6.6) was used to obtain the additional $\mathcal{F}$.

We obtain the following structure for the anomaly $\hat{I}_8^{\mathrm{res}}$ in (6.6)

$$\begin{gathered}\hat{I}_8^{\mathrm{res}} = \hat{I}_8^{\mathrm{uni}} + \hat{I}_8^{\mathrm{non}}\,,\\ \hat{I}_8^{\mathrm{uni}} := X_{0,4}\cdot X_4^{\mathrm{uni}}\,,\quad \hat{I}_8^{\mathrm{non}} := X_2^{\mathrm{non}}\cdot X_6^{\mathrm{non}}\end{gathered} \tag{6.20}$$

using the formulæ (6.15), (6.19) and the notation

$$X_{0,4} := \mathrm{tr}R^2 - \mathrm{tr}\,(iF)^2\,,\ \ X_2^{\mathrm{non}} := -2\mathrm{tr}\,[H_V\,(iF)]\,. \tag{6.21}$$

This is a very clear structure, completely determined by compactification. The two-form X_2^{non} is precisely the anomaly part corresponding to the additional anomalous U(1). It is in general (even for a generic $\mathcal{F}\neq\mathcal{F}_V$) only non-vanishing for at least one U(1)-factor in the background and gauge group.

The necessary anomaly cancellation is again achieved by the residual Green-Schwarz mechanism, that now, contrary to the orbifold models, includes the B-field b_2 and an additional model-dependent axion a^{md}, compare section 6.1.3. The latter is not present on the orbifold.

Structure of the 4D Anomaly

The calculations in four dimensions can be found in detail in [40]. The analogous formulæ for $\mathcal{M}^3$ are

$$(2\pi i)^2 I_6^{\mathrm{res}} = X_2^{\mathrm{uni}}\cdot X_{0,4} + X_2^{\mathrm{non}}\cdot X_4^{\mathrm{non}} = (2\pi i)^2\left(I_6^{\mathrm{uni}} + I_6^{\mathrm{non}}\right)\,, \tag{6.22}$$

with

$$\begin{array}{rclcrcl} X_2^{\mathrm{uni}} & := & \frac{1}{(2\pi i)^3}\int\limits_{\mathcal{M}^3} X_{6,2}\,, & \qquad & X_{0,4} & = & \mathrm{tr}R^2 - \mathrm{tr}(iF)^2\,,\\ X_2^{\mathrm{non}} & := & -2\,\mathrm{tr}[H_V iF]\,, & & X_4^{\mathrm{non}} & := & \frac{1}{(2\pi i)^3}\int\limits_{\mathcal{M}^3} i\mathcal{F}X_{4,4}\,. \end{array} \tag{6.23}$$

Integration using again the formulæ of App. B of [43] yields

$$X_2^{\mathrm{uni}} = -\tfrac{1}{96}\mathrm{Tr}\left[\left(\tfrac{1}{18}H_V^3 - \tfrac{1}{5}H_V\right)(iF)\right]\,, \tag{6.24}$$

$$X_4^{non} = -\tfrac{1}{192}\left[\mathrm{Tr}\left[\left(\tfrac{1}{6}H_V^2 - \tfrac{1}{5}\right)(iF)^2\right] - \tfrac{1}{3\cdot 30^2}\left(\mathrm{Tr}[H_V(iF)]\right)^2 - \mathrm{tr}R^2\right],. \tag{6.25}$$

Contrary to the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ the anomaly is now more factorisable: There are two terms containing the two-forms, X_2^{uni} and X_2^{non}. These are just proportional to the field strengths along the directions of the two anomalous U(1)'s mentioned above. Again, these two-forms vanishes unless there are U(1)-factors in the background and gauge group.

Remarks on the Structure of the Anomalies

It is interesting to note, that the Dirac index of the resolution manifold is implicit in the anomaly formulæ on the resolution. This can be seen in the abstract form of the anomaly of a Weyl fermion $\hat{I}_{12}^{(1/2)}$ in (4.22). Insertion of the expansion around the internal background[1] for $i\mathfrak{F}$ and $\mathfrak{R}$ yields

$$\hat{I}_{12}^{(1/2)} = \mathrm{tr}_{Q_i}\left(e^{i\mathcal{F}}\right)\hat{A}\left(\mathcal{R}\right)\Big|_4 \cdot \mathrm{tr}_{L_i}\left(e^{iF}\right)\hat{A}\left(R\right)\Big|_6 . \tag{6.26}$$

Thus, its Dirac index $\mathrm{ch}(i\mathcal{F})\hat{A}(\mathcal{R})$ enters as a coefficient for all relevant representations L_i and Q_i of the 6D gauge group H and the internal background $\mathcal{J}$ in (5.14), cf. [43]. Hence, we only have to know all occurring representations L_i and plug them into the anomaly formula (6.20). The Dirac index takes automatically care for the right multiplicity. An analogous statement is possible for the compactification to 4D.

6.1.3 Axions and Anomaly Cancellation

In the following we describe the anomaly cancellation on the resolutions $\mathcal{M}^2$ and $\mathcal{M}^3$. We will find the six- and four-dimensional version of the Green-Schwarz mechanism of section 4.2 and the notion of axions in string compactification[2]. We will also note below, how the right couplings of the axion breaking the shift symmetry $a \to a + c$ [45], [46] occur in principle.

The 6D Anomaly Cancellation

The anomaly freedom of the 10D theory transfers also to the compactified theory. To prove this, we have to consider the 10D Green-Schwarz mechanism. Therefore, we use the complete expansion [42] of the 10D B-field

$$B(x, y) = b_2(x) + i\mathcal{F}(y)a^{\mathrm{md}}(x) + \omega_2(y)B_0(x) \tag{6.27}$$

into 6D fluctuations denoted by b_2 and the scalars a^{md} and B_0 according to the general recipe (3.13). ω_2 denotes the Kähler form on $\mathcal{M}^2$ and $\mathcal{F}$ the background field strength, that are both harmonic. However, only the first two terms are relevant for the 6D Green-Schwarz mechanism as B_0 has no anomalous variation. This can be deduced by plugging in the above expansion into the expression (4.26) for H', compare [42].

[1] This is equivalent to the consideration of the structure of the vectorbundles in which the fermions take values. Abstractly, this is $E \otimes \tilde{E}$. Furthermore we have the splitting of the tangential bundle into $T\mathbb{M}_6 \oplus T\mathcal{M}^2$. Then we apply the rules $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(\tilde{E})$ and $\hat{A}(T\mathbb{M}_6 \oplus T\mathcal{M}^2) = \hat{A}(T\mathbb{M}_6) \wedge \hat{A}(T\mathcal{M}^2)$, that reproduce (6.26).

[2] See e.g. [53] for a modern review of axions from string theory and their applications.

Inserting B in the 10D GS-counterterm (4.29) we obtain the 6D couplings

$$S_{\mathrm{GS}} = \tfrac{1}{(2\pi i)^5} \int\limits_{\mathbb{M}_6\times\mathcal{M}^2} B X_8 = \tfrac{1}{(2\pi i)^5} \int\limits_{\mathbb{M}_6\times\mathcal{M}^2} \left(b_2 X_{4,4} + i\mathcal{F} a^{\mathrm{md}} X_{2,6}\right) \tag{6.28}$$

$$= \tfrac{1}{(2\pi i)^3} \int\limits_{\mathbb{M}_6} \left(b_2 X_4^{\mathrm{uni}} + a^{\mathrm{md}} X_6^{\mathrm{non}}\right) , \tag{6.29}$$

with precisely the forms X_4^{uni} and X_6^{non} of (6.15) and (6.19). As X_6^{non} contains the usual axion coupling $\mathrm{tr}(iF)^3$ we identify a^{md} with the so-called model-dependent (md) axion. It depends on the properties of the internal space. This axion $a^{\mathrm{md}}(x)$ is localised on the internal space $\mathcal{M}^2$ at $x = 0$ as $i\mathcal{F}(y)$ is and becomes strongly peaked in the blow-down limit recovering the orbifold. Hence, it should be interpreted as a twisted state from the orbifold point of view. b_2 is model-independent as it is present for every compactification, i.e. independent of the properties of the internal space.

The anomalous variations of b_2, a^{md} and the couplings (6.29) cancel the whole anomaly (6.20) using the usual prescription of the descent equations. The obtained factors on the level of the anomaly polynomial are

$$b_2 \nearrow X_{0,4}, \quad a^{\mathrm{md}} \nearrow X_2^{\mathrm{non}} \tag{6.30}$$

of (6.21), where X_2^{non} is proportional to the field strength iF of the anomalous U(1)-direction in the space of U(1)-generators. This axion can cancel at most one additional anomalous U(1) not already encoded in $X_{0,4}X_4^{\mathrm{uni}}$.

Hence, the issue of matching the anomalies of the orbifold and its resolution model beyond spectrum and gauge group is extended to the question how to obtain the additional term $X_2^{\mathrm{non}}X_6^{\mathrm{non}}$ on the resolution from the blown up orbifold model and how to find an additional axion analogous to a^{md}.

Anomaly Cancellation in 4D

The situation four the 4D models is quite similar to the 6D case. We obtain the same expansion for B by just taking the same two-forms on $\mathcal{M}^3$. The details can be found in [40] and the final result for the 4D GS-counterterm reads as

$$S_{\mathrm{GS}} = \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{M}_4} \left(b_2 X_2^{\mathrm{uni}} + a^{\mathrm{md}} X_4^{\mathrm{non}}\right) \tag{6.31}$$

with the forms X_2^{uni} and X_4^{non} as in (6.23).

However, there is no 4D B-field as it is dual to a 4D-scalar $*(db_2) =: da^{\mathrm{mi}}$. The couplings of a^{mi} can be obtained either from the kinetic term for H', compare [31] for the recipe, or by using the 4D Bianchi identity $dH'_{0,4} = \mathrm{tr}R^2 - \mathrm{tr}F^2$ for $H'_{0,4}$ denoting the B-field in 4D, cf. section 14.3.2 of [27]. The coupling is given by

$$S_{\mathrm{GS}} = \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{M}_4} a^{\mathrm{mi}} X_{0,4} , \tag{6.32}$$

that contains the usual axion couplings $(iF)(i\tilde{F})$ for $i\tilde{F}$ denoting the dual field strength. Hence, we call this scalar the model-independent axion (mi) a^{mi}. The anomalous variation of a^{mi} and a^{md} lead to the following factors in the anomaly polynomial by means of the descent equations

$$a^{\mathrm{mi}} \nearrow X_2^{\mathrm{uni}}, \quad a^{\mathrm{md}} \nearrow X_2^{\mathrm{non}}. \tag{6.33}$$

Thus, S_{GS} cancels the 4D anomaly on the resolution $\mathcal{M}^3$. According to the number of axions on the resolution, there can be at most two anomalous U(1)'s. Each of them is cancelled by one axion with gauge variation leading to X_2^{uni} or X_2^{non}, that are proportional to the field strengths of the respective anomalous U(1)-directions.

Comparing one more time with the anomaly on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, we have to solve the problem of obtaining two anomalous U(1)'s and two axions a^{mi}, a^{md} starting from an orbifold model with only one anomalous U(1) and one axion a^{het}, cf. section 6.1.1.

6.2 Blowing Up Orbifold Models

The following solution along the lines of [42] to the problems mentioned above applies to the six- and four-dimensional case without changes. Hence the first part of this section is valid for both $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/\mathbb{Z}_3$. However, the second part about axions deserves closer attention concerning the 6D case.

6.2.1 The Blowing Up Description

The blowing up of orbifolds in the field theoretic picture means that one assigns a non-vanishing vev to a twisted field Ψ, the blow-up mode [29, 12, 17]. The non-zero vev spontaneously breaks the orbifold gauge group, i.e. reduces its rank. This is just the usual Higgs effect, where a twisted field is identified as the Higgs of this symmetry breaking. One of its components is precisely the blow-up mode Ψ with non-vanishing vev, that will govern the breaking of the gauge group. Furthermore, we will see that the broken gauge bosons acquire a mass by eating parts of this Higgs. However, there is still a mismatch. For the case of the blown up $\mathbb{C}^2/\mathbb{Z}_2$- and $\mathbb{C}^3/\mathbb{Z}_3$-models the U(1)-charges of the various twisted fields under the vev induced broken $\mathrm{U}(1)_V$ with generator $H_V = V^I H_I$ do not match the charges of the localised fields on the corresponding resolutions $\mathcal{M}^2$ and $\mathcal{M}^3$, respectively. This is solved by the following procedure.

Let us represent the blow-up mode Ψ_q, because it carries in general some U(1)-charges $q = (q_1, \ldots, q_l)$, as an exponential with a new degree of freedom T parametrising the perturbations around the vev v. The field T has the following gauge transformation properties

$$\Psi_q = e^T v, \quad T \mapsto T + iq \cdot \Phi, \tag{6.34}$$

where $\Phi = (\Phi_1, \ldots, \Phi_l)$ denotes the gauge parameters of the U(1)-gauge transformations. Because of these transformation property we can identify the imaginary part of the (scalar component of the) superfield T with an axion. This axion can further be used for the anomalous field redefinition

$$\Phi_Q = e^{rT}\Phi'_{Q'}, \quad \Phi'_{Q'} \mapsto e^{i(Q-rq)\cdot\Phi}\Phi'_{Q'} \tag{6.35}$$

of another twisted field Ψ_Q with U(1)-charges Q. Using this method we can achieve, that all twisted fields have the U(1)-charges of the localised states on the resolution model.

However, these field redefinitions are anomalous, i.e. the effective action Γ is not invariant under the transformations. Thus, (6.35) induce anomalous couplings to the axion a^T, that contribute to the anomaly with a piece

$$S_{\mathrm{a}} = \int a^T X^{\mathrm{red}} \quad \Rightarrow \quad \hat{I}^{\mathrm{red}} =: q_i i F^i X^{\mathrm{red}}, \tag{6.36}$$

where this factorisation occurs as only T is involved in the field redefinitions and by the use of the descent equations for the transformation (6.34). Hence, we do have to compare the anomaly including this new term with the anomaly on the resolution. The spectra are now identical, consequently the anomalies will match in the following way

$$\hat{I}^{\mathrm{het}} + \hat{I}^{\mathrm{red}} =: \hat{I}^{\mathrm{blow}} \overset{!}{=} \hat{I}^{\mathrm{res}} = \hat{I}^{\mathrm{uni}} + \hat{I}^{\mathrm{non}}, \tag{6.37}$$

where an index denoting the dimensionality of spacetime was omitted.

Finally, let us not that there is a surprising regularity of the U(1)-charges of the blow-up mode under the broken $\mathrm{U}(1)_V$, namely

$$\mathbb{C}^2/\mathbb{Z}_2 \to \mathcal{M}^2 \quad : \quad \Psi_q \equiv (\mathbf{1}, \dots, \mathbf{1})_{(\pm 3, 0, \dots, 0)} \tag{6.38}$$

$$\mathbb{C}^3/\mathbb{Z}_3 \to \mathcal{M}^3 \quad : \quad \Psi_q \equiv (\mathbf{1}, \dots, \mathbf{1})_{(\pm 4, 0, \dots, 0)} \,. \tag{6.39}$$

for all orbifold models matching a resolution model with U(1)-background given by $\mathrm{U}(1)_V$.

6.2.2 Relating the Axions

After all the above manipulations the anomaly of the blown up orbifold matches the one of the resolution model, see (6.37). Following the discussion of anomaly cancellation on the orbifold and the resolution in sections 6.1.1 and 6.1.3 we are able to write down all axion and B-field couplings. The new ingredient now is the coupling (6.36). As all these axions a^T, a^{md} and/or B-fields h_2, b_2 on the orbifold and resolution have to cancel the same anomaly $\hat{I}^{\mathrm{blow}} = \hat{I}^{\mathrm{res}}$ we can write down the relations

$$\mathbb{C}^2/\mathbb{Z}_2 \to \mathcal{M}^2 \quad : \quad h_2^{\mathrm{het}} X_4^{\mathrm{uni}} + a^T X_6^{\mathrm{red}} = b_2 X_4^{\mathrm{het}} + a^{\mathrm{md}} X_6^{\mathrm{non}} \,, \tag{6.40}$$

$$\mathbb{C}^3/\mathbb{Z}_3 \to \mathcal{M}^3 \quad : \quad a^{\mathrm{het}} X_4^{\mathrm{uni}} + a^T X_4^{\mathrm{red}} = a^{\mathrm{md}} X_4^{\mathrm{het}} + a^{\mathrm{md}} X_4^{\mathrm{non}} \,. \tag{6.41}$$

For a given model all forms are determined. Hence, in 4D this is a system of linear equations, that is in general overdetermined. But it has a unique solution in terms of model-dependent coefficients α, β as

$$a^{\mathrm{mi}} = a^{\mathrm{het}} + \alpha a^T, \;\; a^{\mathrm{md}} = \beta a^T. \tag{6.42}$$

Although α is model-independent, see Tab. A.1, $\beta = -\frac{1}{16}$ is a model-independent statement. However, the model-independent axion a^{mi} is in general a superposition of the heterotic axion a^{het} and the axion a^T.

Unfortunately, the situation in 6D is not as simple as in four dimensions. The axions a^T, a^{md} and the B-fields h_2^{het}, b_2 are forms of different degree, namely 0-forms and two-forms. Hence, one would expect relations only between forms of the same degree, i.e. $a^T \sim a^{\text{md}}$ and $h_2^{\text{het}} \sim b_2$. However, this requires also $X_4^{\text{uni}} \sim X_4^{\text{het}}$ and $X_6^{\text{red}} \sim X_4^{\text{non}}$, what is in general not true, see Tab. 7.2 and section 7.3. Only for model **2B I** this condition is fulfilled. Both X_4^{uni}, X_4^{het} and X_6^{red}, X_4^{non} are not proportional for model 2A I and 2C I. More tedious work is needed to relate the axions and B-fields in the 6D case. We will address this issue in the context of concrete models in the next section.

6.3 Anomaly Calculations of the 6D Models

Let us now use the general formulæ of section 6.1 to calculate the anomalies for all non-compact 6D orbifold and resolution models with U(1)-bundles. Again we will suppress the factor i of the occurring gauge field strengths for convenience during all calculations. However, we will write it explicitly in all final results. The same applies for all traces of gauge field strengths tr_R in a representation R of some gauge factor.

6.3.1 Anomaly Formulae of the Orbifold Models

First we present the calculations for the 6D orbifold models Tab. 5.1. We will use the formula for the anomaly of a 6D Majorana-Weyl fermion

$$\hat{I}_8^{(1/2)} \supset \frac{1}{24}\text{Tr}(iF)^4 - \frac{1}{96}\text{Tr}(iF)^2\text{tr}R^2 \tag{6.43}$$

to calculate the anomaly polynomial omitting the purely gravitational part. This is sufficient as the purely gravitational part is fixed as in (6.15) to $-1/256\,(\text{tr}R^2)^2$. Then we can obtain the factorisation in $X_4 \cdot X_4'$ and reproduce the results of [32].

Orbifold Model 2A - SO(28)×SU(2)×SU(2)'

Besides the massless matter $\frac{1}{16}(\mathbf{28},\mathbf{2},\mathbf{2}) + \frac{1}{2}(\mathbf{28},\mathbf{1},\mathbf{2}) + 2(\mathbf{1},\mathbf{2},\mathbf{1})$ we have to take into account the gauginos in $\frac{1}{16}(\mathbf{378},\mathbf{1},\mathbf{1})$, $\frac{1}{16}(\mathbf{1},\mathbf{3},\mathbf{1})$ and $\frac{1}{16}(\mathbf{1},\mathbf{1},\mathbf{3})$. They are of opposite chirality to the matter fermions. Here, $F_{\mathbf{28}}$, $F_{\mathbf{2}}$ and $F'_{\mathbf{2}}$ denote the field strengths of the respective gauge group factors.

1. TrF^4:

$$\begin{aligned}
\frac{1}{16}\mathrm{tr}_{(\mathbf{28},\mathbf{2},\mathbf{2})}F^4 &= \frac{1}{16}\big(4\cdot F^4_{\mathbf{28}} + 2\cdot 6F^2_{\mathbf{28}}F^2_{\mathbf{2}} + 2\cdot 6F^2_{\mathbf{28}}F'^2_{\mathbf{2}} + 2\cdot 28F^4_{\mathbf{2}} \\
&\quad +28\cdot 6F^2_{\mathbf{2}}F'^2_{\mathbf{2}} + 2\cdot 28F'^4_{\mathbf{2}}\big) && (6.44)\\
\frac{1}{2}\mathrm{tr}_{(\mathbf{28},\mathbf{1},\mathbf{2})}F^4 &= F^4_{\mathbf{28}} + 3F^2_{\mathbf{28}}F'^2_{\mathbf{2}} + 14F'^4_{\mathbf{2}} && (6.45)\\
2\mathrm{tr}_{(\mathbf{1},\mathbf{2},\mathbf{1})}F^4 &= 2F^4_{\mathbf{2}} && (6.46)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{378},\mathbf{1},\mathbf{1})}F^4 &= -\frac{5}{4}F^4_{\mathbf{28}} - \frac{3}{16}(F^2_{\mathbf{28}})^2 && (6.47)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{3},\mathbf{1})}F^4 &= -\frac{1}{4}F^4_{\mathbf{2}} - \frac{3}{8}(F^2_{\mathbf{2}})^2 && (6.48)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{1},\mathbf{3})}F^4 &= -\frac{1}{4}F'^4_{\mathbf{2}} - \frac{3}{8}(F'^2_{\mathbf{2}})^2 && (6.49)
\end{aligned}$$

Here we apply the trace relations of App. B for the relations between the adjoint representation and the fundamental. Furthermore, we use $F^4_{\mathbf{2}} = \frac{1}{2}(F^2_{\mathbf{2}})^2$.
1. TrF^2:

$$\begin{aligned}
\frac{1}{16}\mathrm{tr}_{(\mathbf{28},\mathbf{2},\mathbf{2})}F^2 &= \frac{1}{16}\big(4F^2_{\mathbf{28}} + 2\cdot 28F^2_{\mathbf{2}} + 2\cdot 28F'^2_{\mathbf{2}}\big) && (6.50)\\
\frac{1}{2}\mathrm{tr}_{(\mathbf{28},\mathbf{1},\mathbf{2})}F^2 &= F^2_{\mathbf{28}} + 14F'^2_{\mathbf{2}} && (6.51)\\
2\mathrm{tr}_{(\mathbf{1},\mathbf{2},\mathbf{1})}F^2 &= 2F^2_{\mathbf{2}} && (6.52)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{378},\mathbf{1},\mathbf{1})}F^2 &= -\frac{13}{8}F^2_{\mathbf{28}} && (6.53)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{3},\mathbf{1})}F^2 &= -\frac{1}{4}F^2_{\mathbf{2}} && (6.54)\\
-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{1},\mathbf{3})}F^2 &= -\frac{1}{4}F'^2_{\mathbf{2}} && (6.55)
\end{aligned}$$

Again we apply appropriate trace relations.

This yields a part of the anomaly polynomial

$$\begin{aligned}
\hat{I}_8^{\mathrm{het}} \supset\; & \frac{3}{32}(F^2_{\mathbf{2}})^2 + \frac{11}{32}(F'^2_{\mathbf{2}})^2 + \frac{1}{32}F^2_{\mathbf{28}}F^2_{\mathbf{2}} + \frac{5}{32}F^2_{\mathbf{28}}F'^2_{\mathbf{2}} - \frac{1}{128}(F^2_{\mathbf{28}})^2 \\
& -\mathrm{tr}R^2\left(-\frac{1}{256}F^2_{\mathbf{28}} + \frac{7}{128}F^2_{\mathbf{2}} + \frac{23}{128}F'^2_{\mathbf{2}}\right) . \qquad (6.56)
\end{aligned}$$

Using $\mathbf{32} \to (\mathbf{28},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2})$ we obtain

$$\mathrm{tr}_{\mathbf{32}}F^2 - \mathrm{tr}R^2 = F^2_{\mathbf{28}} + 2F^2_{\mathbf{2}} + 2F'^2_{\mathbf{2}} - \mathrm{tr}R^2, \qquad (6.57)$$

what we use to factorise (6.56) to obtain

$$\begin{aligned}\hat{I}_8^{\rm het} &= \left(\mathrm{tr}_{\mathbf{28}}(iF_{\mathbf{28}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF'_{\mathbf{2}})^2 - \mathrm{tr}R^2\right) \\ &\times \left(\frac{3}{64}\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + \frac{11}{64}\mathrm{tr}_{\mathbf{2}}(iF'_{\mathbf{2}})^2 - \frac{1}{128}\mathrm{tr}_{\mathbf{28}}(iF_{\mathbf{28}})^2 - \frac{1}{256}\mathrm{tr}R^2\right) .\end{aligned} \tag{6.58}$$

The requirement of this factorisation fixes the coefficient of $\mathrm{tr}R^2$ to $-1/256$ and those of the F_R^2-terms are fixed by comparison with the coefficients of the $(F_R^2)^2$-terms in (6.56).

Orbifold Model 2B - SO(20)×SO(12)

The massless matter $\frac{1}{16}(\mathbf{20},\mathbf{12}) + \frac{1}{2}(\mathbf{1},\mathbf{32})$ and the gauginos in the adjoint $\frac{1}{16}(\mathbf{190},\mathbf{1})$ and $\frac{1}{16}(\mathbf{1},\mathbf{66})$ have to be taken into account. Here, $F_{\mathbf{20}}$ and $F_{\mathbf{12}}$ denote the field strengths of the respective gauge group factors.

<u>**1**. TrF^4:</u>

$$\frac{1}{16}\mathrm{tr}_{(\mathbf{20},\mathbf{12})}F^4 = \frac{1}{16}\left(12F_{\mathbf{20}}^4 + 6F_{\mathbf{20}}^2F_{\mathbf{12}}^2 + 20F_{\mathbf{12}}^4\right) \tag{6.59}$$

$$\frac{1}{2}\mathrm{tr}_{(\mathbf{1},\mathbf{32})}F^4 = -F_{\mathbf{12}}^4 + \frac{3}{4}(F_{\mathbf{12}}^2)^2 \tag{6.60}$$

$$-\frac{1}{16}\mathrm{tr}_{(\mathbf{190},\mathbf{1})}F^4 = -\frac{3}{4}F_{\mathbf{20}}^4 - \frac{3}{16}(F_{\mathbf{20}}^2)^2 \tag{6.61}$$

$$-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{66})}F^4 = -\frac{1}{4}F_{\mathbf{12}}^4 - \frac{3}{16}(F_{\mathbf{12}}^2)^2 \tag{6.62}$$

We have applied here the trace relations for the spinor **32** given in App. B.

<u>**1**. TrF^2:</u>

$$\frac{1}{16}\mathrm{tr}_{(\mathbf{20},\mathbf{12})}F^2 = \frac{1}{16}\left(12F_{\mathbf{20}}^2 + 20F_{\mathbf{12}}^2\right) \tag{6.63}$$

$$\frac{1}{2}\mathrm{tr}_{(\mathbf{1},\mathbf{32})}F^2 = 2F_{\mathbf{12}}^2 \tag{6.64}$$

$$-\frac{1}{16}\mathrm{tr}_{(\mathbf{190},\mathbf{1})}F^2 = -\frac{9}{8}F_{\mathbf{20}}^2 \tag{6.65}$$

$$-\frac{1}{16}\mathrm{tr}_{(\mathbf{1},\mathbf{12})}F^2 = -\frac{5}{8}F_{\mathbf{12}}^2 \tag{6.66}$$

Again we apply appropriate trace relations.

Then we obtain the factorised form by the same arguments as given above

$$\begin{aligned}\hat{I}_8^{\rm het} &= \left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + \mathrm{tr}_{\mathbf{12}}(iF_{\mathbf{12}})^2 - \mathrm{tr}R^2\right) \\ &\times \left(\frac{3}{128}\mathrm{tr}_{\mathbf{12}}(iF_{\mathbf{12}})^2 - \frac{1}{128}\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 - \frac{1}{256}\mathrm{tr}R^2\right) ,\end{aligned} \tag{6.67}$$

where we used $\mathbf{32} \to (\mathbf{20}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{12})$ to obtain

$$\mathrm{tr}_{\mathbf{32}} F^2 - \mathrm{tr} R^2 = F_{\mathbf{20}}^2 + F_{\mathbf{12}}^2 - \mathrm{tr} R^2. \tag{6.68}$$

Orbifold Model 2C - SU(16)×U(1)

The massless spectrum is given by $\frac{1}{8}\mathbf{120}_{(2)} + \mathbf{16}_{(-3)}$ and the adjoint $\frac{1}{16}\mathbf{255}_{(0)}$ and $\frac{1}{16}\mathbf{1}_{(0)}$. Here, $F_{\mathbf{16}}$ and F denote the field strengths of the respective gauge group factors.

1. TrF^4:

$$\begin{aligned}
\frac{1}{8}\mathrm{tr}_{\mathbf{120}_{(2)}} F^4 &= F_{\mathbf{16}}^4 + \frac{3}{8}(F_{\mathbf{16}}^2)^2 + 12F_{\mathbf{16}}^3 F + 42F_{\mathbf{16}}^2 F^2 + 240F^4 && (6.69)\\
\mathrm{tr}_{\mathbf{16}_{(-3)}} F^4 &= F_{\mathbf{16}}^4 - 12F_{\mathbf{16}}^3 F + 54F_{\mathbf{16}}^2 F^2 + 16 \cdot 81F^4 && (6.70)\\
-\frac{1}{16}\mathrm{tr}_{\mathbf{255}_{(0)}} F^4 &= -2F_{\mathbf{16}}^4 - \frac{3}{8}(F_{\mathbf{16}}^2)^2 && (6.71)\\
-\frac{1}{16}\mathrm{tr}_{\mathbf{1}_{(0)}} F^4 &= 0 && (6.72)
\end{aligned}$$

Here we have used the trace relations for the two-times antisymmetric tensor representation **120** given in App. B.
1. TrF^2:

$$\begin{aligned}
\frac{1}{8}\mathrm{tr}_{\mathbf{120}_{(2)}} F^2 &= \frac{7}{4}F_{\mathbf{16}}^2 + 60F^2 && (6.73)\\
\mathrm{tr}_{\mathbf{16}_{(-3)}} F^2 &= F_{\mathbf{16}}^2 + 16 \cdot 9F^2 && (6.74)\\
-\frac{1}{16}\mathrm{tr}_{\mathbf{255}_{(0)}} F^2 &= -2F_{\mathbf{16}}^2 && (6.75)
\end{aligned}$$

Again we used appropriate trace relations.

This fixes the anomaly to

$$\hat{I}_8^{\mathrm{het}} = \left(2\mathrm{tr}_{\mathbf{16}}(iF_{\mathbf{16}})^2 + 32(iF)^2 - \mathrm{tr}R^2\right)\left(2(iF)^2 - \frac{1}{256}\mathrm{tr}R^2\right), \tag{6.76}$$

where $\mathbf{32} \to \mathbf{16}_{(1)} \oplus \overline{\mathbf{16}}_{(-1)}$ was used for

$$\mathrm{tr}_{\mathbf{32}} F^2 - \mathrm{tr} R^2 = 2F_{\mathbf{16}}^2 + 32F - \mathrm{tr} R^2. \tag{6.77}$$

6.3.2 Anomaly Polynomials of the Resolution Models

Using the abstract form (6.20)

$$\begin{aligned}\hat{I}_8^{\mathrm{res}} = \;& \left(\mathrm{tr}\,(iF)^2 - \mathrm{tr}R^2\right) \\ & \times\frac{1}{96}\left\{\mathrm{Tr}\left[\left(\frac{1}{8}H_V^2 - \frac{3}{40}\right)(iF)^2\right] - \frac{1}{4\cdot 30^2}\left(\mathrm{Tr}\left[H_V(iF)\right]\right)^2 - \frac{3}{8}\mathrm{tr}R^2\right\} \\ & +\frac{1}{96}\mathrm{tr}\left[H_V\,(iF)\right]\left\{\mathrm{Tr}\left(H_V(iF)^3\right) - \frac{1}{120}\mathrm{Tr}\left(H_V(iF)\right)\left[\frac{1}{15}\mathrm{Tr}\,(iF)^2 + \mathrm{tr}R^2\right]\right\}\end{aligned} \tag{6.78}$$

we will calculate the anomalies for each 6D resolution model separately. We just note that there are quantities in the polynomial that are model independent and fixed by the geometry of $\mathcal{M}^2$ and the U(1)-bundle. Hence, we will obtain the same values for these terms for every model.

Resolution Model 2A I - $\mathbf{SO(26)\times SU(2)\times U(1)^2}$

The relevant representations are $(\mathbf{26},\mathbf{2})_{(1,1)}$, $(\underline{\mathbf{1}},\mathbf{2})_{(-1,2)}$, $(\mathbf{1},\mathbf{1})_{(2,2)}$, $(\mathbf{26},\mathbf{1})_{(2,-1)}$, $(\mathbf{1},\mathbf{2})_{(3,0)}$ and their complex conjugate ones, denoted by $\overline{R}$ generically. Furthermore, there are also the adjoint representations $(\mathbf{325},\mathbf{1})_{(0,0)}$, $(\mathbf{1},\mathbf{3})_{(0,0)}$ and the two singlets of the U(1)-factors. All these originate from the breaking of the 10-dimensional gauge group SO(32), i.e. the **496**, due to the U(1)-bundle. The multiplicity of all representations is counted by the four-dimensional Dirac index of $\mathcal{M}^2$ according to (5.17), that is implicit in all anomaly formulæ for the resolution models. Thus, the opposite chirality of the gauginos is automatically respected. Again $F_{\mathbf{26}}$, $F_{\mathbf{2}}$, F and F' denote the field strengths of the various gauge factors. The U(1)-factors are chosen such that the first U(1) is along the direction of V defining the background bundle.

<u>**1**. First summand:</u>

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{26},\mathbf{2})_{(1,1)}}F^2 = \frac{1}{48}F_{\mathbf{26}}^2 + \frac{13}{48}F_{\mathbf{2}}^2 + \frac{13}{24}F^2 + \frac{13}{12}FF' + \frac{13}{24}F'^2 \tag{6.79}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(-1,2)}}F^2 = \frac{1}{96}F_{\mathbf{2}}^2 + \frac{1}{48}F^2 - \frac{1}{12}FF' + \frac{1}{12}F'^2 \tag{6.80}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{1})_{(2,2)}}F^2 = \frac{1}{24}F^2 + \frac{1}{12}FF' + \frac{1}{24}F'^2 \tag{6.81}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{26},\mathbf{1})_{(2,-1)}}F^2 = \frac{1}{96}F_{\mathbf{26}}^2 + \frac{13}{12}F^2 - \frac{13}{12}FF' + \frac{13}{48}F'^2 \tag{6.82}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(3,0)}}F^2 = \frac{1}{96}F_{\mathbf{2}}^2 + \frac{3}{16}F^2 \tag{6.83}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{325},\mathbf{1})_{(0,0)}}F^2 = \frac{1}{4}F_{\mathbf{26}}^2 \tag{6.84}$$

$$\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{3})_{(0,0)}}F^2 = \frac{1}{24}F_{\mathbf{2}}^2 \tag{6.85}$$

The multiplicity factor $\frac{1}{8}H_V^2 - \frac{3}{40}$ is calculated by noting that the value of H_V on a representation is given by the first U(1)-charge. This yields the following results:

Representations	$\frac{1}{8}H_V^2 - \frac{3}{40}$
$(\mathbf{26},\mathbf{2})_{(1,1)}$, $(\mathbf{1},\mathbf{2})_{(-1,2)}$	1/20
$(\mathbf{1},\mathbf{1})_{(2,2)}$, $(\mathbf{26},\mathbf{1})_{(2,-1)}$	17/40
$(\mathbf{1},\mathbf{2})_{(3,0)}$	21/20
$(\mathbf{325},\mathbf{1})_{(0,0)}$, $(\mathbf{1},\mathbf{3})_{(0,0)}$	−3/40

Furthermore, we have to use $\mathbf{32} \to (\mathbf{26},\mathbf{1})_{(0,0)} \oplus (\mathbf{1},\mathbf{2})_{(1,1)} \oplus (\mathbf{1},\mathbf{2})_{(-1,-1)} \oplus (\mathbf{1},\mathbf{1})_{(2,-1)} \oplus (\mathbf{1},\mathbf{1})_{(-2,1)}$ to obtain

$$X_{0,4} = \mathrm{tr}_{\mathbf{32}}F^2 - \mathrm{tr}R^2 = F_{\mathbf{26}}^2 + 2F_{\mathbf{2}}^2 + 12F^2 + 6F'^2 - \mathrm{tr}R^2. \tag{6.86}$$

Additionally, we need $\mathrm{tr}_{(R_1,R_2)_{(q_1,q_2)}}H_V F = |R_1||R_2|q_1\left(q_1 F + q_2 F'\right)$ where $|R_i|$ denotes the dimension of the representation R_i.

2. Second Summand:

Here we use the fact that $\mathrm{tr}\left(T_R^a\right)^3 = 0$ for real R as $\mathrm{tr}\left(T_R^a\right)^3 = \mathrm{tr}\left(T_{\overline{R}}^a\right)^3 = -\mathrm{tr}\left(T_R^a\right)^3$. **26**, **2**, **325** and **3** are real:

$$\begin{aligned}
\frac{1}{6}\mathrm{tr}_{(\mathbf{26},\mathbf{2})_{(1,1)}}F^3 &= F_{\mathbf{26}}^2F + F_{\mathbf{26}}^2F' + 13F_{\mathbf{2}}^2F + 13F_{\mathbf{26}}^2F' + \frac{26}{3}F^3 + 26F^2F' && (6.87)\\
&\quad +26FF'^2 + \frac{26}{3}F'^3 &&\\
\frac{1}{6}\mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(-1,2)}}F^3 &= -\frac{1}{2}F_{\mathbf{2}}^2F + F_{\mathbf{2}}^2F' - \frac{1}{3}F^3 + 2F^2F' - 4FF'^2 + \frac{8}{3}F'^3 && (6.88)\\
\frac{1}{6}\mathrm{tr}_{(\mathbf{1},\mathbf{1})_{(2,2)}}F^3 &= \frac{4}{3}F^3 + 4F^2F' + 4FF'^2 + \frac{4}{3}F'^3 && (6.89)\\
\frac{1}{6}\mathrm{tr}_{(\mathbf{26},\mathbf{1})_{(2,-1)}}F^3 &= F_{\mathbf{26}}^2F - \frac{1}{2}F_{\mathbf{26}}^2F' + \frac{104}{3}F^3 - 52F^2F' + 26FF'^2 - \frac{13}{3}F'^3 && (6.90)\\
\frac{1}{6}\mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(3,0)}}F^3 &= 9F^3 + \frac{3}{2}F_{\mathbf{2}}^2F && (6.91)
\end{aligned}$$

Applying the branching of **32** as given above once again we deduce

$$\begin{aligned}
X_2^{\mathrm{non}} &= -2\mathrm{tr}_{\mathbf{32}}\left(H_V F\right) && (6.92)\\
&= -2\left(\mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(1,1)}} + \mathrm{tr}_{(\mathbf{1},\mathbf{2})_{(-1,-1)}} + \mathrm{tr}_{(\mathbf{1},\mathbf{1})_{(2,-1)}} + \mathrm{tr}_{(\mathbf{1},\mathbf{1})_{(-2,1)}}\right)\left(H_V F\right) && (6.93)\\
&= -24F. && (6.94)
\end{aligned}$$

Finally we use $\mathrm{Tr}\left(H_V F\right) = 30\mathrm{tr}\left(H_V F\right) = 360F$. Now we can apply these results to obtain the full anomaly. As the anomaly $\hat{I}_8^{\mathrm{res}}$ is a polynomial of even degree in F and R we do not have to calculate the complex conjugate representations $\overline{R}$ separately. They yield the same anomaly contribution as R because $T^a_{\overline{R}} := -\left(T^a_R\right)^* = -\left(T^a_R\right)^T$ for all generators T^a and $\mathrm{tr}(T^a)^T = \mathrm{tr}T^a$. Hence, we take twice the contribution of all representations R that come along with their complex conjugate $\overline{R}$. Of course, the adjoints occur only once. First we give the results

$$\begin{aligned}
\frac{1}{96}\mathrm{Tr}\left(\left(\frac{1}{8}H_V^2 - \frac{3}{40}\right)F^2\right) &= -\frac{1}{128}F_{\mathbf{26}}^2 + \frac{3}{64}F_{\mathbf{2}}^2 + \frac{45}{32}F^2 - \frac{3}{4}FF' + \frac{21}{64}F'^2, \quad (6.95)\\
\frac{1}{6}\mathrm{Tr}\left(H_V F^3\right) &= 24F_{\mathbf{2}}^2F' + 6F_{\mathbf{26}}^2F + 36F_{\mathbf{2}}^2F + 216F^3 + 180FF'^2 \quad (6.96)\\
&\quad -144F^2F',\\
\frac{1}{15}\mathrm{Tr}F^2 &= 2\mathrm{tr}F^2 = 2F_{\mathbf{26}}^2 + 4F_{\mathbf{2}}^2 + 24F^2 + 12F'^2. \quad (6.97)
\end{aligned}$$

The final result for the anomaly reads as

$$\begin{aligned}
\hat{I}_8^{\mathrm{res}} &= \left(\mathrm{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 12(iF)^2 + 6F'^2 - \mathrm{tr}R^2\right) \quad (6.98)\\
&\quad\times\left(-\frac{1}{128}\mathrm{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + \frac{33}{32}(iF)^2 - \frac{3}{4}(iF)(iF') + \frac{21}{64}(iF')^2 - \frac{1}{256}\mathrm{tr}R^2\right)\\
&\quad+\frac{1}{8}(iF)\left[144(iF)^3 + 24\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2(iF) + 24\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2(iF') - 144(iF)^2(iF')\right.\\
&\quad\left.+144(iF)(iF')^2 - 3(iF)\mathrm{tr}R^2\right].
\end{aligned}$$

Resolution Model 2B I - SO(20)×SU(6)×U(1)

We have to consider the representations $(\mathbf{20},\mathbf{6})_{(1)}$, $(\mathbf{1},\mathbf{15})_{(2)}$ and their complex conjugate ones. Furthermore, there are the adjoint representations $(\mathbf{190},\mathbf{1})_{(0)}$, $(\mathbf{1},\mathbf{35})_{(0)}$ and the singlet of the U(1)-factor. Again all these originate from the breaking of the 10-dimensional gauge group SO(32), i.e. the **496**, due to the U(1)-bundle. Here $F_{\mathbf{20}}$, $F_{\mathbf{6}}$ and F denote the field strengths of the various gauge factors.

1. First summand:

$$\begin{aligned}
\frac{1}{96}\mathrm{tr}_{(\mathbf{20},\mathbf{6})_{(1)}}F^2 &= \frac{1}{16}F_{\mathbf{20}}^2 + \frac{5}{24}F_{\mathbf{6}}^2 + \frac{5}{4}F^2 \quad (6.99)\\
\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{15})_{(2)}}F^2 &= \frac{1}{24}F_{\mathbf{6}}^2 + \frac{5}{8}F^2 \quad (6.100)\\
\frac{1}{96}\mathrm{tr}_{(\mathbf{190},\mathbf{1})_{(0)}}F^2 &= \frac{3}{16}F_{\mathbf{20}}^2 \quad (6.101)\\
\frac{1}{96}\mathrm{tr}_{(\mathbf{1},\mathbf{35})_{(0,0)}}F^2 &= \frac{1}{8}F_{\mathbf{6}}^2 \quad (6.102)
\end{aligned}$$

Here we use App. B to relate the antisymmetric tensor **15** to **6**. We calculate the multiplicity factor $\frac{1}{8}H_V^2 - \frac{3}{40}$ by using H_V on a representation. The value is just the U(1)-charge of the respective representation. This yields

Representations	$\frac{1}{8}H_V^2 - \frac{3}{40}$
$(\mathbf{20},\mathbf{6})_{(1)}$	$1/20$
$(\mathbf{1},\mathbf{15})_{(2)}$	$17/40$
$(\mathbf{190},\mathbf{1})_{(0)}$, $(\mathbf{1},\mathbf{35})_{(0)}$	$-3/40$

Additionally we have to use $\mathbf{32} \to (\mathbf{20},\mathbf{1})_{(0)} \oplus (\mathbf{1},\mathbf{6})_{(1)} \oplus (\mathbf{1},\bar{\mathbf{6}})_{(-1)}$ to obtain

$$X_{0,4} = \mathrm{tr}_{\mathbf{32}}F^2 - \mathrm{tr}R^2 = F_{\mathbf{20}}^2 + 2F_{\mathbf{6}}^2 + 12F^2 - \mathrm{tr}R^2. \tag{6.103}$$

Again we need $\mathrm{tr}_{(R_1,R_2)_{(q_1,q_2)}} H_V F = |R_1||R_2|q_1\,(q_1F + q_2F')$.

<u>**2**. Second Summand:</u>

Then we use the fact that $\mathrm{tr}F^3 = 0$ for the real representations **20**, **190** and **35**:

$$\frac{1}{6}\mathrm{tr}_{(\mathbf{20},\mathbf{6})_{(1)}}F^3 = 3F_{\mathbf{20}}^2F + 10F_{\mathbf{6}}^2F + \frac{10}{3}F_{\mathbf{6}}^3 + 20F^3 \tag{6.104}$$

$$\frac{1}{6}\mathrm{tr}_{(\mathbf{1},\mathbf{15})_{(2)}}F^3 = \frac{1}{3}F_{\mathbf{6}}^3 + 4F_{\mathbf{6}}^2F + 20F^3 \tag{6.105}$$

$$\tag{6.106}$$

We apply now the branching of **32** as given above once again to deduce

$$\begin{aligned} X_2^{\mathrm{non}} &= -2\mathrm{tr}_{\mathbf{32}}(H_VF) = -2\left(\mathrm{tr}_{(\mathbf{1},\mathbf{6})_{(1)}} + \mathrm{tr}_{(\mathbf{1},\bar{\mathbf{6}})_{(-1)}}\right)(H_VF) &(6.107)\\ &= -24F. &(6.108)\end{aligned}$$

Furthermore we note that $\mathrm{Tr}\,(H_VF) = 30\mathrm{tr}\,(H_VF) = 360F$. Again we do not have to calculate the contribution of the complex conjugate representations $\overline{R}$ separately. They yield the same anomaly contribution, thus, we take twice the contribution of the representation R. First we obtain

$$\frac{1}{96}\mathrm{Tr}\left(\left(\frac{1}{8}H_V^2 - \frac{3}{40}\right)F^2\right) = -\frac{1}{128}F_{\mathbf{20}}^2 + \frac{3}{64}F_{\mathbf{6}}^2 + \frac{21}{32}F^2\,, \tag{6.109}$$

$$\frac{1}{6}\mathrm{Tr}\left(H_VF^3\right) = 8F_{\mathbf{6}}^3 + 6F_{\mathbf{20}}^2F + 36F_{\mathbf{6}}^2F + 120F^3\,, \tag{6.110}$$

$$\frac{1}{15}\mathrm{Tr}F^2 = 2\mathrm{tr}F^2 = 2F_{\mathbf{20}}^2 + 4F_{\mathbf{6}}^2 + 24F^2\,. \tag{6.111}$$

This yields for the anomaly

$$\begin{aligned}\hat{I}_8^{\mathrm{res}} = &\left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \mathrm{tr}R^2\right) \\ &\times\left(-\frac{1}{128}\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + \frac{9}{32}(iF)^2 - \frac{1}{256}\mathrm{tr}R^2\right) \\ &+\frac{1}{8}(iF)\left[48(iF)^3 + 8\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^3 + 24\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2(iF) - 3(iF)\mathrm{tr}R^2\right].\end{aligned} \tag{6.112}$$

Resolution Model 2C I - SU(15)×U(1)^2

The breaking of the 10-dimensional gauge group SO(32) results in the representations $\mathbf{15}_{(-1,6)}$, $\mathbf{105}_{(1,2)}$, $\mathbf{15}_{(2,-4)}$ and their complex conjugate ones. There are the adjoint representation $\mathbf{224}_{(0,0)}$ and the singlets of the U(1)-factors. The multiplicity of each representation is counted automatically in the anomaly formula by the Dirac index. We write $F_{\mathbf{15}}$, F and F' to denote the field strengths of the various gauge factors. Again we have chosen the U(1)-basis such that the first U(1) is identical to the direction of the background bundle defined by V.

<u>**1**. First summand:</u>

$$\frac{1}{96}\mathrm{tr}_{\mathbf{105}_{(1,2)}}F^2 = \frac{13}{96}F_{\mathbf{15}}^2 + \frac{35}{32}F^2 + \frac{35}{8}FF' + \frac{35}{8}F'^2 \tag{6.113}$$

$$\frac{1}{96}\mathrm{tr}_{\mathbf{15}_{(-1,6)}}F^2 = \frac{1}{96}F_{\mathbf{15}}^2 + \frac{5}{32}F^2 - \frac{15}{8}FF' + \frac{45}{8}F'^2 \tag{6.114}$$

$$\frac{1}{96}\mathrm{tr}_{\mathbf{15}_{(2,-4)}}F^2 = \frac{1}{96}F_{\mathbf{15}}^2 + \frac{5}{8}F^2 - \frac{5}{2}FF' + \frac{5}{2}F'^2 \tag{6.115}$$

$$\frac{1}{96}\mathrm{tr}_{\mathbf{224}_{(0,0)}}F^2 = \frac{5}{16}F_{\mathbf{15}}^2 \tag{6.116}$$

Again we use App. B to relate the antisymmetric tensor **105** to the fundamental **15**. Applying H_V on the representations above we obtain the first U(1)-charge and calculate the multiplicity factor $\frac{1}{8}H_V^2 - \frac{3}{40}$. We obtain

Representations	$\frac{1}{8}H_V^2 - \frac{3}{40}$
$\mathbf{15}_{(-1,6)}$, $\mathbf{105}_{(1,2)}$	1/20
$\mathbf{15}_{(2,-4)}$	17/40
$\mathbf{224}_{(0,0)}$	−3/40

The branching of the fundamental is given by $\mathbf{32} \to \mathbf{15}_{(1/2,1)} \oplus \overline{\mathbf{15}}_{(-1/2,-1)} \oplus \mathbf{1}_{(-3/2,5)} \oplus \mathbf{1}_{(3/2,-5)}$ to obtain

$$X_{0,4} = \mathrm{tr}_{\mathbf{32}}F^2 - \mathrm{tr}R^2 = 2F_{\mathbf{15}}^2 + 12F^2 + 80F'^2 - \mathrm{tr}R^2. \tag{6.117}$$

Again we note $\mathrm{tr}_{(R_1,R_2)_{(q_1,q_2)}}H_V F = |R_1||R_2|q_1\,(q_1F + q_2F')$.

2. Second Summand:
We apply $\mathrm{tr}F^3 = 0$ for the real representation **224**:

$$\frac{1}{6}\mathrm{tr}_{\mathbf{105}_{(1,2)}}F^3 = \frac{11}{6}F_{\mathbf{15}}^3 + \frac{13}{2}F_{\mathbf{15}}^2F + 13F_{\mathbf{15}}^2F' + \frac{35}{2}F^3 + 105F^2F' + 210FF'^2 + 140F'^3 \quad (6.118)$$

$$\frac{1}{6}\mathrm{tr}_{\mathbf{15}_{(-1,6)}}F^3 = \frac{1}{6}F_{\mathbf{15}}^3 - \frac{1}{2}F_{\mathbf{15}}^2F + 3F_{\mathbf{15}}^2F' - \frac{5}{2}F^3 + 45F^2F' - 270FF'^2 + 540F'^3 \quad (6.119)$$

$$\frac{1}{6}\mathrm{tr}_{\mathbf{15}_{(2,-4)}}F^3 = \frac{1}{6}F_{\mathbf{15}}^3 + F_{\mathbf{15}}^2F - 2F_{\mathbf{15}}^2F' + 20F^3 - 120F^2F' + 240FF'^2 - 160F'^3\,. \quad (6.120)$$

Using the branching of **32** as given above again we deduce

$$\begin{aligned} X_2^{\mathrm{non}} &= -2\mathrm{tr}_{\mathbf{32}}\left(H_VF\right) & (6.121)\\ &= -2\left(\mathrm{tr}_{\mathbf{15}_{(1/2,1)}} + \mathrm{tr}_{\overline{\mathbf{15}}_{(-1/2,-1)}} + \mathrm{tr}_{\mathbf{1}_{(-3/2,5)}} + \mathrm{tr}_{\mathbf{1}_{(3/2,-5)}}\right)\left(H_VF\right) & (6.122)\\ &= -24F\,. & (6.123)\end{aligned}$$

Again we obtain $\mathrm{Tr}\left(H_VF\right) = 30\mathrm{tr}\left(H_VF\right) = 360F$. The contribution of the complex conjugate representations $\overline{R}$ is the same as those of the representations R. As a first step we obtain

$$\frac{1}{96}\mathrm{Tr}\left(\left(\frac{1}{8}H_V^2 - \frac{3}{40}\right)F^2\right) = \frac{21}{32}F^2 - \frac{15}{8}FF' + \frac{25}{8}F'^2\,, \quad (6.124)$$

$$\frac{1}{6}\mathrm{Tr}\left(H_VF^3\right) = 4F_{\mathbf{15}}^3 + 18F_{\mathbf{15}}^2F + 120F^3 + 1920FF'^2 - 1440F'^3 + 12F_{\mathbf{15}}^2F' - 360F^2F'\,, \quad (6.125)$$

$$\frac{1}{15}\mathrm{Tr}F^2 = 2\mathrm{tr}F^2 = 4F_{\mathbf{15}}^2 + 24F^2 + 160F'^2\,. \quad (6.126)$$

This yields the anomaly

$$\begin{aligned}\hat{I}_8^{\mathrm{res}} = {}& \left(2\mathrm{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 + 12(iF)^2 + 80(iF')^2 - \mathrm{tr}R^2\right) \quad (6.127)\\ &\times\left(\frac{9}{32}(iF)^2 - \frac{15}{8}(iF)(iF') + \frac{25}{8}(iF')^2 - \frac{1}{256}\mathrm{tr}R^2\right)\\ &+\frac{1}{8}(iF)\left[48(iF)^3 - 1440(iF')^3 + 4\mathrm{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^3 + 6\mathrm{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2(iF)\right.\\ &\left.+12\mathrm{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2(iF') - 360(iF)^2(iF') + 1440(iF)(iF')^2 - 3(iF)\mathrm{tr}R^2\right]\,.\end{aligned}$$

Chapter 7

The Explicit Model Matching

Here we will present the blowing up of the orbifold models. Then we will see how the anomalous field redefinitions can be used to match the resolution spectra and anomalies.

Before going into the details let us just note that a first step in the matching procedure is the identification of the U(1)-bundle V on the resolution with an orbifold shift V_{orb}. However, V_{orb} does in general not obey the integrated Bianchi identity. But we are allowed to change the orbifold shift to V'_{orb} by adding 2λ for a weight vector λ such that it fulfills the Bianchi identity. Thus we can infer $V_{\text{orb}} = V \mod 2\lambda$ up to irrelevant sign flips of V_{orb}. Hence, we can think of the U(1)-bundle on the resolution as a gauge flux on the orbifold that disappears into the orbifold singularity in the blow down limit [43]. However, this level of matching is not very instructive and we will see many more details of the matching explicitly.

7.1 Matching of 6D Models

First we will discuss the details of the matching only for model **2B** as all steps except the relations of the axions and B-fields apply similarly to the other models. They will be discussed afterwards in section 7.3, where also all calculations can be found. After the discussion of model **2B** we will address also the 4D models focusing on the results.

7.1.1 Blowing Up Model 2B

The Orbifold Model

Consider the orbifold model **2B** of Tab. 5.1 with $V_{\text{orb}} = (0^{10}, 1^6)$, gauge group SU(16) $\times$ U(1) and spectrum

$$\frac{1}{16}(\mathbf{20}, \mathbf{12}) + \frac{1}{2}(\mathbf{1}, \mathbf{32}). \tag{7.1}$$

There is only one twisted field, consequently only one possible way to assign a vev and to blow up the orbifold. The heterotic anomaly polynomial is given in the factorised form (6.67) that can be cancelled by the B-field h_2 on the orbifold.

The Resolution Model

We will see, that the resolution model obtained by blowing up this orbifold model is **2B I**. It is described by $V = (0^{10}, 1^6)$, the gauge group $\mathrm{SO}(20) \times \mathrm{SU}(6) \times \mathrm{U}(1)_V$, U(1)-generator $Q_V = (0^{10}, 1^6)$ and the spectrum

$$\frac{1}{8}(\mathbf{20}, \mathbf{6})_{(1)} + \frac{7}{8}(\mathbf{1}, \mathbf{15})_{(2)}. \tag{7.2}$$

We denote the U(1) and its generator Q with an index V to indicate, that this is precisely the U(1) of the background bundle, that is also present in the 6D gauge group, as mentioned earlier. The anomaly is given by $\hat{I}_8^{\mathrm{res}} = \left(\hat{I}_8^{\mathrm{uni}} + \hat{I}_8^{\mathrm{non}}\right)$ with

$$\begin{aligned} \hat{I}_8^{uni} &= \left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \mathrm{tr}R^2\right) \\ &\times \left(-\frac{1}{128}\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{6}}(iF_6)^2 + \frac{9}{32}(iF)^2 - \frac{1}{256}\mathrm{tr}R^2\right) \end{aligned} \tag{7.3}$$

and

$$\hat{I}_8^{non} = -\frac{1}{8}(iF)\left[-48(iF)^3 - 8\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^3 - 24(iF)\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 3(iF)\mathrm{tr}R^2\right], \tag{7.4}$$

compare to Tab. 7.2 and section 6.3. Note that the latter one is proportional to iF, the field strength of the anomalous $\mathrm{U}(1)_V$ (there is a $(iF)^4$-term). After the branching of the spectrum (7.1) under the breaking of the gauge group due to the vev the difference in the anomalies $\hat{I}_8^{\mathrm{het}}$ and $\hat{I}_8^{\mathrm{res}}$ is only due to the anomalous field redefinition of the twisted fields. Hence the structure of $\hat{I}_8^{\mathrm{non}} = (iF)[\ldots]$ is wanted. The calculations needed to obtain the anomaly $\hat{I}_8^{\mathrm{res}}$ are presented in section 6.3.

Blowing Up the Orbifold

The field theoretic blowing up of the orbifold model **2B** can now be performed. First of all, we obtain the gauge group of the resolution model **2B I** by the breaking

$$\mathrm{SO}(12) \to \mathrm{SU}(6) \times \mathrm{U}(1). \tag{7.5}$$

The U(1) will be broken explicitly, i.e. the rank is reduced. This symmetry breaking can be obtained by identifying the twisted field **32** as the Higgs. Assigning a vev to an adequate component of this twisted field will produce precisely the desired symmetry breaking. Following section 6.2 we will call this component the blow-up mode. It is one component of the twisted field **32** after the branching into full hypermultiplets

$$\frac{1}{2}\mathbf{32} \to \mathbf{15}_{(-1)} + \mathbf{1}_{(3)} \tag{7.6}$$

due to (7.5). We take $\mathbf{1}_{(3)}$ as the blow-up mode and reproduce the breaking (7.5) with a vev v at the string scale

$$(\mathbf{1},\mathbf{1})_{(3)} = e^T v. \tag{7.7}$$

Furthermore, we introduce a twisted superfield T to parametrise the fluctuations around v. It contains the twisted axion a^T. Of course, this vev breaks the U(1) in (7.5) explicitly. Then we have to analyse the branching of the other fields in the spectrum under (7.5), including the adjoint representation given by

$$\begin{array}{rcl} \mathbf{Adj}\,(\mathrm{SO}(20)) = \mathbf{66} & \rightarrow & \mathbf{35}_{(0)} + \mathbf{1}_{(0)} + \mathbf{15}_{(2)} + \overline{\mathbf{15}}_{(-2)} \\ \mathbf{12} & \rightarrow & \mathbf{6}_{(1)} + \overline{\mathbf{6}}_{(-1)} \end{array} \tag{7.8}$$

The details of all above branchings (7.6), (7.8) are given below at the end of this chapter. Thus, taking into account all branchings we already obtain the bulk matter of resolution model **2B I**, i.e. the matter with non-integral multiplicity. Furthermore, the heterotic anomaly $\hat{I}_8^{\mathrm{het}}$ is branched into the representations of the new gauge group

$$\begin{aligned} \hat{I}_8^{\mathrm{het}} &= \left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \mathrm{tr}R^2\right) \\ &\quad\times\left(-\frac{1}{128}\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + \frac{9}{32}(iF)^2 - \frac{1}{256}\mathrm{tr}R^2\right). \end{aligned} \tag{7.9}$$

Note that we use the same symbol $\hat{I}_8^{\mathrm{het}}$ to denote the original orbifold anomaly as well as the branched orbifold anomaly. However, it does not yet match the anomaly $\hat{I}_8^{\mathrm{res}}$ of model **2B I** given in (6.112). The missing ingredient is to match also the local matter (of integral multiplicity) by making the anomalous field redefinition

$$(\mathbf{1},\mathbf{15})'_{(2)} := e^T\,(\mathbf{1},\mathbf{15})_{(-1)}\,. \tag{7.10}$$

This contributes an additional part to the heterotic anomaly $\hat{I}_8^{\mathrm{het}}$ of the structure

$$\hat{I}_8^{\mathrm{red}} = 3(iF)\left[2(iF)^3 + \frac{1}{3}\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^3 + (iF)\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 - \frac{1}{8}(iF)\mathrm{tr}R^2\right], \tag{7.11}$$

what will also be derived at the end of this chapter.

Finally, we obtain the right resolution model **2B I** from the blown up orbifold model **2B**

$$\frac{1}{8}(\mathbf{20},\mathbf{6})_{(1)} + \frac{7}{8}(\mathbf{1},\mathbf{15})'_{(2)} \tag{7.12}$$

where we already omitted vectorlike matter. This matter decouples from the massless spectrum due to the Higgs effect. The gauginos originating from the breaking of the gauge group (7.8) have chirality opposite to those of the matter[1] obtained from the branching of the orbifold untwisted and twisted matter (7.6), (7.8). Furthermore, we have two broken

[1]This is due to the decomposition of the $\mathcal{N}=(1,1)$, $D=6$ vector multiplet into the vector- and hypermultiplet of $\mathcal{N}=1$, $D=6$. The details can be found in App. B of [51].

Orbifold	Resolution
$\mathrm{SO}(28) \times \mathrm{SU}(2) \times \mathrm{SU}(2)'$ $\frac{1}{16}(\mathbf{28}, \mathbf{2}, \mathbf{2})$ $+\frac{1}{2}\underline{(\mathbf{28}, \mathbf{1}, \mathbf{2})} + 2\,(\mathbf{1}, \mathbf{2}, \mathbf{1})$	$\mathrm{SO}(26) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2$ $\frac{1}{8}\left[(\mathbf{26}, \mathbf{2})_{(1,1)} + (\mathbf{1}, \mathbf{2})_{(1,-2)} + (\mathbf{1}, \mathbf{2})_{(3,0)}\right]$ $+2(\mathbf{1}, \mathbf{2})_{(3,0)} + \frac{7}{8}(\mathbf{1}, \mathbf{1})_{(2,2)} + \frac{7}{8}(\mathbf{26}, \mathbf{1})_{(2,-1)}$
$\mathrm{SO}(20) \times \mathrm{SO}(12)$ $\frac{1}{16}\,(\mathbf{20}, \mathbf{12}) + \frac{1}{2}\underline{(\mathbf{1}, \mathbf{32})}$	$\mathrm{SO}(20) \times \mathrm{U}(6)$ $\frac{1}{8}\left[(\mathbf{20}, \mathbf{6})_{(1)}\right] + \frac{7}{8}(\mathbf{1}, \mathbf{15})_{(2)}$
$\mathrm{SU}(16) \times \mathrm{U}(1)$ $\frac{1}{8}\mathbf{120}_{(2)} + \underline{\mathbf{16}_{(-3)}}$	$\mathrm{SU}(15) \times \mathrm{U}(1)^2$ $\frac{1}{8}\left[\mathbf{15}_{(-1,6)} + \mathbf{105}_{(1,2)}\right] + \frac{7}{8}\mathbf{15}_{(2,-4)}$

Table 7.1: Matching of all heterotic SO(32)-models on $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{M}^2$. The underlined states denote the twisted fields, that contain the blow-up mode. They are the Higgs fields of the symmetry breaking that leads to the gauge group of the resolution model.

gauginos in the representation $(\mathbf{1}, \mathbf{15})_{(2)}$ in the bulk, thus, they occur with a multiplicity of $2 \times \frac{1}{16} = \frac{1}{8}$. Therefore, $\frac{1}{8}$ of the Higgs in the representation $(\mathbf{1}, \mathbf{15})'_{(2)}$ is eaten to provide the degrees of freedom for massive gauge fields. Finally, this is the promised explanation for the multiplicity 7/8 in the spectrum of the resolution models in Tab. 5.3. It can be understood as a localised state with multiplicity 1 and two bulk states of opposite chirality, hence, with a multiplicity $-\frac{1}{8}$ that leave over a state with multiplicity $1 - \frac{1}{8} = \frac{7}{8}$.

Note that we always make the choice to denote all representations of the hypermultiplets in (7.12) with R not the complex conjugate $\overline{R}$, e.g. we write $(\mathbf{20}, \overline{\mathbf{6}})_{(-1)}$ as $(\mathbf{20}, \mathbf{6})_{(1)}$. The results apply similarly to all other models as summarised in Tab. 7.1.

The Anomaly Polynomial and the Axions

Adding the term (7.11) originating from the anomalous field redefinition to $\hat{I}_8^{\text{het}}$ given in Tab. 7.2 we obtain the expected identity

$$\hat{I}_8^{\text{het}} + \hat{I}_8^{\text{red}} =: \hat{I}_8^{\text{blow}} \overset{!}{=} \hat{I}_8^{\text{res}} = \hat{I}_8^{\text{uni}} + \hat{I}_8^{\text{non}}. \tag{7.13}$$

As $\hat{I}_8^{\text{het}}$ is identical to $\hat{I}_8^{\text{uni}}$ and consequently according to (7.13) also $\hat{I}_8^{\text{non}} = \hat{I}_8^{\text{red}}$, we read off

$$X_4^{\text{uni}} = X_4^{\text{het}}, \;\; X_6^{\text{non}} = -\frac{1}{8} X_6^{\text{red}}. \tag{7.14}$$

This applies immediately to the axion- and B-field couplings. Hence we deduce

$$b_2 = h_2, \;\; a^{\text{md}} = -8a^T. \tag{7.15}$$

This is a very simple form, that is unfortunately not obtained for the other models.

	Anomaly Polynomials of Orbifold and Resolution
2A	$\hat{I}_8^{\text{het}} = \frac{1}{2^8}\left[\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 2\text{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 12(iF)^2 + 6(iF')^2 - \text{tr}R^2\right] \times \left[-2\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 12\text{tr}_{\mathbf{2}}\text{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 72(iF)^2 + 192(iF)(iF') + 84(iF')^2 - \text{tr}R^2\right]$ $\hat{I}_8^{\text{uni}} = \frac{1}{2^8}\left[\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 2\text{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 12(iF)^2 + 6(iF')^2 - \text{tr}R^2\right] \times \left[-2\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 12\text{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 264(iF)^2 + 84(iF')^2 - 192(iF)(iF') - \text{tr}R^2\right]$ $\hat{I}_8^{\text{non}} = \frac{1}{8}(iF)\Big\{144(iF)^3 + 24\left[(iF) + (iF')\right](iF_2)^2 - 144\left[(iF) - (iF')\right](iF)(iF') - 3(iF)\text{tr}R^2\Big\}$
2B	$\hat{I}_8^{\text{het}} = \frac{1}{2^8}\left[\text{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \text{tr}R^2\right] \times \left[-2\text{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 12\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 72(iF)^2 - \text{tr}R^2\right]$ $\hat{I}_8^{\text{uni}} = \frac{1}{2^8}\left[\text{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \text{tr}R^2\right] \times \left[-2\text{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 12\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 72(iF)^2 - \text{tr}R^2\right]$ $\hat{I}_8^{\text{non}} = \frac{1}{8}(iF)\Big\{48(iF)^3 + 8\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^3 + 24(iF)\text{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 - 3(iF)\text{tr}R^2\Big\}$
2C	$\hat{I}_8^{\text{het}} = \frac{1}{2^8}\left[2\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 + 12(iF)^2 + 80(iF')^2 - \text{tr}R^2\right]\left[72(iF)^2 + 480(iF)(iF') + 800(iF')^2 - \text{tr}R^2\right]$ $\hat{I}_8^{\text{uni}} = \frac{1}{2^8}\left[2\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 + 12(iF)^2 + 80(iF')^2 - \text{tr}R^2\right]\left[72(iF)^2 - 480(iF)(iF') + 800(iF')^2 - \text{tr}R^2\right]$ $\hat{I}_8^{\text{non}} = \frac{1}{8}(iF)\Big\{48(iF)^3 - 1440(iF')^3 + 4\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^3 + 6\left[(iF) + 2(iF')\right]\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 - 360\left[(iF) - 4(iF')\right](iF)(iF') - 3iF)\text{tr}R^2\Big\}$

Table 7.2: The anomaly polynomials of the six-dimensional SO(32) orbifold and resolution models. We summarise the orbifold anomaly polynomial $\hat{I}_8^{\text{het}}$ after the branching due to the blowing up and the resolution anomaly polynomial split into the universal $\hat{I}_8^{\text{uni}}$ and non-universal $\hat{I}_8^{\text{non}}$ parts.

Generalised D-flatness

The issue of flatness is encoded in the so-called D-term potential, cf. App. B of [51]

$$D^{a,\rho} = \Phi^*_{i,M}\sigma^\rho_{MN}T^{a,ij}\Phi_{j,N}\,. \tag{7.16}$$

Here M, N are labels of the $\text{SU}(2)_R$-symmetry of the $d = 6, \mathcal{N} = 1$ theory from the $\mathcal{N} = 2$ point of view in 4D, σ^ρ denote the Pauli matrices, T^a the generator of the gauge symmetry and Φ the complex doublet containing two half-hypers. Then the potential is simply $V = D^{a,\rho}D^{a,\rho}$. However, it is shown in [32] in case of a single blow-up mode charged under an U(1), that the potential V cannot vanish. Furthermore, it is argued that $V = 0$ can be obtained in case of more than one blow up mode. Thus, this local picture seems to make sense only for the application to build a global resolution model. Then one needs at least two blown-up fixed points and blow-up modes. For a more detailed discussion, we refer to [40].

7.1.2 The Axion Relations in General

According to Tab. 7.2 and section 7.3 the anomalies of the blown up orbifold including the anomalous field redefinitions and the resolution models match. However, the forms X_4^{uni}, X_4^{het} and X_6^{red}, X_6^{non} are not proportional, respectively. There are several terms, that prohibit that. Hence, a mixing between the axions and B-fields happens, what is not surprising due to the experience with the 4D models, compare section 7.2. But the matching is much more complicated as we will see below.

Let us begin the discussion by listing the gauge transformations of the axions and B-fields

$$\begin{aligned} \text{Orbifold} &: \quad \delta_\Lambda h_2 \nearrow X_{0,4}, \quad \delta_\Lambda a^T \nearrow 3(iF); \\ \text{Resolution} &: \quad \delta_\Lambda b_2 \nearrow X_{0,4}, \quad \delta_\Lambda a^{\text{md}} \nearrow X_2^{\text{non}}, \end{aligned} \tag{7.17}$$

where the arrow $\nearrow$ indicates, that the descent equations have to be used. δ_Λ denotes a combined gauge and Lorentz transformation. The gauge transformations of the B-fields are model-independent and fixed both on the orbifold and resolution side. Hence, we have to change our description of the physical degrees of freedom to overcome this rigidity of (7.17) to be able to formulate relations between the fields on both sides. This can be achieved by passing to the dual forms, namely the two-forms $\tilde{h}_2$, $\tilde{b}_2$ and the four-forms $\tilde{a}_4 := \tilde{a}_4^T$, $\tilde{c}_4 := \tilde{a}_4^{\text{md}}$. The duality transformation interchanges couplings and gauge transformations (on the level of the anomaly polynomial). Thus, the gauge transformations read as

$$\text{Orbifold} : \quad \delta_\Lambda \tilde{h}_2 \nearrow X_4^{\text{het}}, \quad \delta_\Lambda \tilde{a}_4 \nearrow X_6^{\text{red}}; \tag{7.18}$$
$$\text{Resolution} : \quad \delta_\Lambda \tilde{b}_2 \nearrow X_4^{\text{uni}}, \quad \delta_\Lambda \tilde{c}_4 \nearrow X_6^{\text{non}}, \tag{7.19}$$

whereas the couplings are now given by the forms in (7.17). Precisely these gauge transformations will guide us how to relate the orbifold and resolution fields appropriately. We have to express $\tilde{h}_2$ and $\tilde{a}_4$ by their counterparts $\tilde{b}_2$, $\tilde{c}_4$ and some appropriate combinations of the U(1) gauge fields, such that both sides of these expression transform identical under gauge transformations. Then their gauge transformation produce the same term on the level of the anomaly polynomial and the anomaly is cancelled.

We now apply this method to model **2C I** to deduce the axion and B-field relations. The detailed calculation leading to the results presented here can be found in section 7.3. The gauge transformations for $\tilde{b}_2$ and $\tilde{h}_2$ are given in this case, such that the difference is just

$$\delta_\Lambda \left(\tilde{b}_2 - \tilde{h}_2 \right) \nearrow X_4^{\text{uni}} - X_4^{\text{het}} = -\frac{15}{4}(iF)(iF'), \tag{7.20}$$

compare Tab. 7.2. Hence we can apply the descent equations to obtain the relation between the B-fields

$$\tilde{b}_2 = \tilde{h}_2 - \frac{15}{4}(iA)(iA'). \tag{7.21}$$

To relate also $\tilde{c}_4$ and $\tilde{a}_4$ we make the ansatz

$$\tilde{c}_4 = -\frac{1}{8}\tilde{a}_4 + \gamma(iF')\tilde{h}_2 + (iA')Y_3 \tag{7.22}$$

for $\gamma \in \mathbb{R}$ a free parameter and Y_3 a gauge invariant three-form. Using (7.18) we can consider the terms induced by a gauge transformation on the level of the anomaly polynomial. This yields a factorised expression, which justifies our ansatz (7.22)

$$\begin{aligned} X_6^{\text{non}} + \frac{1}{8} X_6^{\text{red}} &= (iF') \left[-\frac{5}{16} \text{tr}_{\mathbf{15}} \left(iF_{\mathbf{15}} \right)^2 - \frac{15}{8} (iF)^2 - \frac{25}{2} (iF')^2 + \frac{5}{32} \text{tr} R^2 \right] && (7.23) \\ &\overset{!}{=} \gamma (iF') \tilde{h}_2 + (iF') Y_4, && (7.24) \end{aligned}$$

where the explicit form of X_6^{non} and X_6^{red} and $Y_4 := dY_3$ were used. However, naive use of the descent equations for $(iF')^3$ yields $(iA') \left[(iA')(iF') \right] = 0$, because iA' is an Abelian gauge field. Thus, this term cannot be obtained from the anomalous variation of $(iA')Y_3$ as Y_3. Consequently, $(iF')^3$ has to be generated by the anomalous transformation of $\tilde{h}_2$ that is determined by X_4^{het}. We can use this observation to determine γ. Fortunately, X_4^{het} contains a term $\frac{25}{8}(iF')^2$, such that we can write

$$\begin{aligned} X_6^{\text{non}} + \tfrac{1}{8} X_6^{\text{red}} \supset -\tfrac{25}{2} (iF')^3 \overset{!}{=} \gamma \tfrac{25}{8} (iF')^3 \subset \gamma (iF') X_4^{\text{het}} && (7.25) \\ \Longrightarrow \gamma = -4. && (7.26) \end{aligned}$$

Now, we are able to calculate Y_4 by plugging in all results. Finally, we obtain

$$\begin{aligned} (iF') Y_4 &= X_6^{\text{non}} + \frac{1}{8} X_6^{\text{red}} + 4 (iF') X_4^{\text{het}} && (7.27) \\ &= (iF') \left[-\frac{5}{16} \text{tr}_{\mathbf{15}} \left(iF_{\mathbf{15}}^2 \right) - \frac{3}{4} (iF)^2 + \frac{15}{2} (iF)(iF') + \frac{9}{64} \text{tr} R^2 \right]. && (7.28) \end{aligned}$$

This results can now be used for the descent equations to obtain the expression for Y_3 in terms of the Chern-Simons forms for the various characteristic classes appearing in Y_4.

The methods presented above can also be used to relate the axions and B-fields of model **2A** and its resolution. However, the inconvenient numerical effort of this calculations will prevent us from addressing this here. Let us just note that analogous relations to the ones given above can be deduced and that $\tilde{c}_4 = -\frac{1}{8}\tilde{a}_4 + \ldots$ occurs. Of course, this new description reproduces the results of section 7.1.1 for model **2B I** as we can write $\tilde{h}_2 = \tilde{b}_2$ and $\tilde{c}_4 = -\frac{1}{8}\tilde{a}_4$. Both sides of the equations automatically have the same gauge transformations. After dualisation of this relations we recover our old results (7.15), because the factor $-\frac{1}{8}$ converts to -8.

Hence, it seems also for the 6D models to be a model-independent statement, that $a^{\text{md}} = -8a^T + \ldots$, although the precise relations are much more complicated.

7.2 Matching in 4D

The steps of the matching recipe of chapter 6 apply analogously to the four-dimensional models. First one can relate the orbifold shift V_{orb} up to a lattice vector λ with an allowed V defining the U(1)-bundle on the resolution, where again λ has to be added to obtain a V

obeying the Bianchi identity. Then we can identify a twisted non-Abelian singlet acquiring a non-vanishing vev v and a new field T parametrising the fluctuations around the vev, that contains the axion a^T. This induces the symmetry breaking necessary to match the resolution gauge group by breaking a U(1) explicitly. Again vectorlike matter decouples from the massless spectrum on the orbifold due to the Higgs effect and the corresponding trilinear couplings in the superpotential on the blown up orbifold. Furthermore, the spectra completely match after the anomalous field redefinitions, that generate a new anomaly contribution $\hat{I}_6^{\text{red}}$ to match also the resolution anomaly. Furthermore, the axion relations can be worked out directly according to section 6.2.2. The details of this matching including the anomalies and axion relations can be found in Tab. A.1 and Tab. A.2. The novelty here is the possibility of at most two anomalous U(1)-directions on the resolution, that correspond to the heterotic axion a^{het} already present on the orbifold and the new axion a^T. This is contrary to the situation on the orbifold, where at most only one local anomalous U(1) is possible [23, 41].

Furthermore, the issue of F-flatness is analysed in [40] in detail. There it can be shown using the orbifold selection rules [29, 12] and gauge invariance that F-flatness is guaranteed. The auxiliary fields $F_i = \frac{\partial \mathcal{W}}{\partial \varphi_i^*}$ vanish in the vacuum as only the single blow-up mode has a non-vanishing vev. The couplings containing it are at most linear in the blow-up mode and at least trilinear in total, such that the F-terms vanish. In more specific cases, the couplings are even more restricted or vanishing in identically. Also D-terms are considered in [40]. The analysis shows, that at most only one non-vanishing D-term is generated in the blow up procedure. It corresponds precisely to the U(1) under which Ψ_q is charged. However, there is a possibility to explain this D-term on the level of matching local models only. There is the possibility of one anomalous U(1) on the resolution additional to the one, that can be present on the orbifold. Thus, the additional anomalous U(1) on the resolution produces [9, 3] one additional Fayet-Iliopoulos term (FI-term), that is not present on the orbifold. Therefore, we can interpret the D-term produced due to the blow-up mode Ψ_q as the necessary ingredient to match the FI-term of this anomalous U(1) on the resolution. However, we emphasise again that we are working in the picture of local blown-up orbifold and resolution models. The matching on the level of non-vanishing D-terms is just another tool to relate the orbifold to corresponding resolution models.

Finally, we conclude with the summary, that all resolution models with U(1)-brackground can be matched by an appropriate orbifold model. It is even possible to blow-up orbifold model **3C** in two inequivalent ways yielding different resolution models. This happens not in the case of six-dimensional models where we observe a one-to-one matching.

7.3 Blowing Up Details in 6D

In the following we present the details for the blowing up of all six-dimensional orbifold models and match the resolution models. First we will present the necessary symmetry breaking, all field branchings and redefinitions. Then we will also give the detailed calculations for the branching of the orbifold anomalies $\hat{I}_8^{\text{het}}$ and show the results for $\hat{I}_8^{\text{res}}$. We will make use of the details given in section 5.2. Furthermore, we will use the anomalies worked out in section 6.3 for the respective models and present all necessary formulæ for the matching of the models on the level of the anomalies.

7.3.1 Blowing Up Model 2A

The Orbifold Model and its Blow Up

Start with the orbifold model defined by $V_{\mathrm{orb}} = (0^{13}, 1^2, 0)$ that yields the gauge group $\mathrm{SO}(28) \times \mathrm{SU}(2) \times \mathrm{SU}(2)'$ and the matter

$$\frac{1}{16}(\mathbf{28}, \mathbf{2}, \mathbf{2}) + \frac{1}{2}(\mathbf{28}, \mathbf{1}, \mathbf{2}) + 2(\mathbf{1}, \mathbf{2}, \mathbf{1}). \tag{7.29}$$

We will see that it can be matched with resolution model **2A I**. The starting point of the matching procedure is again the symmetry breaking of the orbifold to the resolution gauge group by

$$\mathrm{SO}(28) \times \mathrm{SU}(2)' \to \mathrm{SO}(26) \times \mathrm{U}(1)' \times \mathrm{U}(1)'', \tag{7.30}$$

what is achieved by identifying the blow-up mode and assigning a vev to it. This induces the branchings

$$\begin{array}{ccccccc} (\mathbf{378}, \mathbf{1}, \mathbf{1}) & \to & (\mathbf{325}, \mathbf{1})_{(0,0)} & +(\mathbf{26}, \mathbf{1})_{(2,-1)} & +(\mathbf{26}, \mathbf{1})_{(-2,1)} & +(\mathbf{1}, \mathbf{1})_{(0,0)} \\ (\underline{\pm 1, \pm 1, 0^{11}}, 0^2, \underline{0}) & & (\underline{\pm 1, \pm 1, 0^{11}}, 0^3) & (\underline{\pm 1, 0^{12}}, 0^2, 1) & (\underline{\pm 1, 0^{12}}, 0^2, -1) & (0^{16}) \end{array} \tag{7.31}$$

$$\begin{array}{ccccc} (\mathbf{1}, \mathbf{1}, \mathbf{3}) & \to & (\mathbf{1}, \mathbf{1})_{(2,2)} & +(\mathbf{1}, \mathbf{1})_{(0,0)} & +(\mathbf{1}, \mathbf{1})_{(-2,-2)} \\ (0^{13}, \pm(1,1), 0) & & (0^{13}, 1^2, 0) & (0^{16}) & (0^{13}, (-1)^2, 0) \end{array} \tag{7.32}$$

where we explicitly present all root vectors. Here we show the charges according to (Q_V, Q) after performing a redefinition of the U(1)-generators, compare to (7.33). The U(1)-generators that are natural after the breaking of the SO(28) and SU(2)′, respectively, are $Q' = (0^{15}, -2)$ and $Q'' = (0^{13}, 1, 1, 0)$. Both are adequate sums of roots of SO(28) or SU(2)′. They are orthogonal to each other and the roots of the non-Abelian gauge factors. However, one U(1) should equal the generator Q_V of the $\mathrm{U}(1)_V$ on the resolution and the other one, Q, should reproduce the U(1)-charge of the bulk matter. Thus, we rotate and rescale them[2] in the following way

$$\begin{pmatrix} Q \\ Q_V \end{pmatrix} = C \cdot \begin{pmatrix} Q' \\ Q'' \end{pmatrix} = \begin{pmatrix} (0^{13}, 1^2, -1) \\ (0^{13}, 1^2, 2) \end{pmatrix}, \qquad C := \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{pmatrix}. \tag{7.33}$$

[2]The bilinear form encoding the scalar product of the vectors Q', Q'' is $A := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. The matrix $C := \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{pmatrix}$ transfers it into $C^T A C = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$. Thus, the transformed vectors Q, Q_V are again orthogonal and rescaled to length $\sqrt{3}$ and $\sqrt{6}$, respectively.

This yields the branching of the matter representations as

$$\begin{array}{ccccccc} (\mathbf{28},\mathbf{2},\mathbf{2}) & \rightarrow & (\mathbf{26},\mathbf{2})_{(1,1)} & +(\mathbf{1},\mathbf{2})_{(3,0)} & +(\mathbf{1},\mathbf{2})_{(-1,2)} & +\text{c. c.} \\ (\underline{1,0^{12}},\utilde{\pm 1,0},\underline{0}) & & (\underline{1,0^{12}},\utilde{1,}0,0) & (0^{13},\underline{1,0},1) & (0^{13},\underline{1,0},-1) & -(\dots) \end{array} \tag{7.34}$$

$$\begin{array}{ccccccc} (\mathbf{28},\mathbf{1},\mathbf{2}) & \rightarrow & (\mathbf{26},\mathbf{1})_{(-1,-1)} & +(\mathbf{1},\mathbf{1})_{(3,0)} & +(\mathbf{1},\mathbf{1})_{(-1,2)} & +\text{c. c.} \\ (\underline{1,0^{12}},\pm(\tfrac{1}{2},\tfrac{1}{2}),\underline{0}) & & (\underline{1,0^{12}},(\tfrac{1}{2})^2,0) & (0^{13},1/2^2,1) & (0^{13},\tfrac{1}{2}^2,-1) & -(\dots) \end{array} \tag{7.35}$$

$$\begin{array}{ccc} (\mathbf{1},\mathbf{2},\mathbf{1}) & \rightarrow & (\mathbf{1},\mathbf{2})_{(0,0)} \\ (0^{13},\pm(\tfrac{1}{2},\tfrac{1}{2}),0) & & (0^{13},\pm(\tfrac{1}{2},\tfrac{1}{2}),0). \end{array} \tag{7.36}$$

We denote here the complex conjugate representations $\overline{R}$ by "c. c." because the corresponding weight vector is just the negative $-(\dots)$ of the one for R. All quantum numbers are inverted. The branched components of the half-hyper $(\mathbf{28},\mathbf{1},\mathbf{2})$ combine to full hypermultiplets of representation R by pairing up with the complex conjugate ones $\overline{R}$. Again we are free to denote the full hypers either by R or $\overline{R}$. Furthermore, we rewrite all hypermultiplets in representations $\overline{R}$ as those in R. We will show our choice of representations R below in the full spectrum. The respective multiplicity doubles and we obtain a multiplicity of $1/8$ for bulk matter. At this stage only this matter matches the resolution model. However, the twisted, localised matter does not.

Now we recover the twisted field $(\mathbf{28},\mathbf{1},\mathbf{2})$ as the Higgs of the symmetry breaking necessary to match the gauge group of the resolution model. First we assign a vev at the string scale by

$$(\mathbf{1},\mathbf{1})_{(3,0)} = e^T v \tag{7.37}$$

to obtain the desired symmetry breaking. Then we can use the new superfield T to perform the anomalous field redefinitions

$$\begin{aligned} (\mathbf{26},\mathbf{1})'_{(2,-1)} &:= e^T (\mathbf{26},\mathbf{1})_{(-1,-1)}\,, && (7.38)\\ (\mathbf{1},\mathbf{2})'_{(3,0)} &:= e^T (\mathbf{1},\mathbf{2})_{(0,0)}\,, && (7.39)\\ (\mathbf{1},\mathbf{1})'_{(2,2)} &:= e^T (\mathbf{1},\mathbf{1})_{(-1,2)}\,. && (7.40) \end{aligned}$$

Now we see that some components of the (localised) Higgs, namely $\frac{1}{8}(\mathbf{26},\mathbf{1})'_{(2,-1)}$ and $\frac{1}{8}(\mathbf{1},\mathbf{1})'_{(2,2)}$ are eaten by the broken (bulk) gauginos of the breaking (7.31) and (7.32). They are of opposite chirality and acquire a mass such that they decouple from the massless spectrum. Consequently, we obtain the final spectrum of resolution model **2A I**:

$$\frac{1}{8}\left[(\mathbf{26},\mathbf{2})_{(1,1)} + (\mathbf{1},\mathbf{2})_{(-1,2)} + (\mathbf{1},\mathbf{2})_{(3,0)}\right] + 2(\mathbf{1},\mathbf{2})'_{(3,0)} + \frac{7}{8}(\mathbf{1},\mathbf{1})'_{(2,2)} + \frac{7}{8}(\mathbf{26},\mathbf{1})'_{(2,-1)}. \tag{7.41}$$

The Branched Anomaly and the Matching

To match also the anomalies we have to calculate the orbifold anomaly in the branched representations due to the symmetry breaking of the orbifold gauge group. We have to use the branchings of all occurring representations and the anomaly formula for $\hat{I}_8^{\rm het}$ deduced in section 6.3.1

$$\begin{aligned}\hat{I}_8^{\rm het} &= \left(\mathrm{tr}_{\mathbf{28}}(iF_{\mathbf{28}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF'_{\mathbf{2}})^2 - \mathrm{tr}R^2\right) \\ &\times \left(\frac{3}{64}\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + \frac{11}{64}\mathrm{tr}_{\mathbf{2}}(iF'_{\mathbf{2}})^2 - \frac{1}{128}\mathrm{tr}_{\mathbf{28}}(iF_{\mathbf{28}})^2 - \frac{1}{256}\mathrm{tr}R^2\right).\end{aligned} \tag{7.42}$$

Using the details of the previous section we deduce the needed branchings

$$(\mathbf{28},\mathbf{1},\mathbf{1}) \;\rightarrow\; (\mathbf{26},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1})_{(2,0)} + (\mathbf{1},\mathbf{1})_{(-2,0)} \tag{7.43}$$

$$(\mathbf{1},\mathbf{1},\mathbf{2}) \;\rightarrow\; (\mathbf{1},\mathbf{1})_{(0,1)} + (\mathbf{1},\mathbf{1})_{(0,-1)} \tag{7.44}$$

of the fundamentals of SO(28) and SU(2)'. Denoting the field strengths by $F_{\mathbf{26}}$, $F_{\mathbf{2}}$, $\tilde{F}'$ and F'' we obtain

$$\mathrm{tr}_{\mathbf{28}}F_{\mathbf{28}}^2 = \mathrm{tr}_{\mathbf{26}_{(0,0)}}F^2 + \mathrm{tr}_{\mathbf{1}_{(2,0)}}F^2 + \mathrm{tr}_{\mathbf{1}_{(-2,0)}}F^2 = F_{\mathbf{26}}^2 + 8\tilde{F}'^2 \tag{7.45}$$

$$\mathrm{tr}_{\mathbf{2}}F'^2_{\mathbf{2}} = 2F''^2, \tag{7.46}$$

where we denote charges ordered according to (Q', Q''). Furthermore, we have redefined the U(1)-generators with a matrix C as given in (7.33). This has to be applied here, too. The redefinition of the U(1) basis does not change the physics, thus, the kinetic term T has to be invariant. It can be expressed as

$$T \sim \tilde{F}'^2Q'^2 + F''^2Q''^2 = \left(\tilde{F}'Q' + F''Q''\right)^2 = \left(\begin{pmatrix}\tilde{F}'\\ F''\end{pmatrix}^T \cdot \begin{pmatrix}Q'\\ Q''\end{pmatrix}\right)^2, \tag{7.47}$$

where $\left(\tilde{F}'Q' + F''Q''\right)$ denotes a 16-dimensional vector, hence, the square has to be understood as the scalar product on the 16-dimensional vectorspace of Q' and Q''. Then we can write this in the (formal) matrix notation in (7.47). Here F', F denote the field strengths in the basis Q, Q_V. Transforming the U(1)-generators as $(Q, Q_V)^T = C(Q', Q'')^T$ invariance of (7.47) requires the formal vector to transform as $C^T(F', F)^T \stackrel{!}{=} (\tilde{F}', F'')^T$. Thus, we obtain the relation

$$\begin{pmatrix}\tilde{F}'\\ F''\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -1\\ 1 & 1\end{pmatrix} \cdot \begin{pmatrix}F'\\ F\end{pmatrix} = \begin{pmatrix}\frac{1}{2}F' - F\\ F + F'\end{pmatrix}. \tag{7.48}$$

This yields $\tilde{F}'^2 = \frac{1}{4}F'^2 - F'F + F^2$ and $F''^2 = F^2 + 2FF' + F'^2$. Then we insert all results into the anomaly $\hat{I}_8^{\rm het}$ and obtain the branched anomaly in the U(1)-basis of $\mathrm{U}(1)_V \times U(1)$

$$\begin{aligned}\hat{I}_8^{\rm het} &= \left(\mathrm{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + 2\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + 12(iF)^2 + 6(iF')^2 - \mathrm{tr}R^2\right) \\ &\times \Bigg(-\frac{1}{128}\mathrm{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 + \frac{9}{32}(iF)^2 + \frac{3}{4}(iF)(iF') \\ &+\frac{21}{64}(iF')^2 - \frac{1}{256}\mathrm{tr}R^2\Bigg).\end{aligned} \tag{7.49}$$

This is clearly different from the anomaly $\hat{I}_8^{\text{uni}}$ in section 6.3.2. Now we are ready to read off the missing anomaly $\hat{I}_8^{\text{red}}$ that is generated due to the anomalous field redefinitions of the twisted, local matter. Comparison with $\hat{I}_8^{\text{res}}$ yields

$$\begin{aligned}\hat{I}_8^{\text{red}} &= (2\pi i)^3\left(I_8^{\text{res}} - I_8^{\text{het}}\right) &(7.50)\\ &= 3(iF)\Big[\frac{1}{4}(iF)\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 - \frac{1}{2}(iF')\text{tr}_{\mathbf{26}}(iF_{\mathbf{26}})^2 + \frac{3}{2}(iF)\text{tr}_{\mathbf{2}}(iF_{\mathbf{2}})^2 &(7.51)\\ &\quad +9(iF)^3 - 12(iF)^2(iF') + \frac{15}{2}(iF)(iF')^2 - 3(iF')^3 - \frac{1}{8}\text{tr}R^2\left(3(iF) - 4(iF')\right)\Big]\end{aligned}$$

The term in the brackets is X_6^{red}. This allows for the matching of the anomalies in the form

$$\hat{I}_8^{\text{het}} + \hat{I}_8^{\text{red}} =: \hat{I}_8^{\text{blow}} = \hat{I}_8^{\text{res}}. \tag{7.52}$$

7.3.2 Blowing Up Model 2B

Here we just give the details that are necessary to verify the argumentation of the previous section. We will present the root vectors of the gauge group, the branching of all representations including their weights and the anomaly calculations.

The Blown Up Orbifold

The model is defined by $V_{\text{orb}} = (0^{10}, 1^6)$ with gauge group and spectrum as discussed above. Let us repeat the needed symmetry breaking

$$\text{SO}(12) \;\rightarrow\; \text{SU}(6)\times\text{U}(1). \tag{7.53}$$

This implies the branchings

$$\begin{array}{ccccccc} (\mathbf{1},\mathbf{66}) & \rightarrow & (\mathbf{1},\mathbf{35})_{(0)} & +(\mathbf{1},\mathbf{15})_{(2)} & +(\mathbf{1},\overline{\mathbf{15}})_{(-2)} & +(\mathbf{1},\mathbf{1})_{(0,0)} \\ (0^{10},\underline{\pm1,\pm1,0^4}) & & (0^{10},\pm(\underline{1,0^4,-1})) & (0^{10},\underline{1,1,0^4}) & (0^{10},\underline{0^4,-1,-1}) & (0^{16}) \end{array} \tag{7.54}$$

where the root vectors are given below. The SO(20) is unbroken. Its roots are given by $(\underline{\pm1,\pm1,0^8},0^6)$, what corresponds to the 180 roots. The CSA contributes another 10 operators to recover the full adjoint **190**. The natural generator of the single U(1)-factor is given by $Q = (0^{10}, 1^6)$, i.e. the operator is $H_Q = H^I Q_I$. It is orthogonal to the roots of the non-Abelian gauge factors. We note that this is already equal to the gauge embedding vector V of the resolution model **2B I**.

Furthermore, we obtain the branching of the matter representations

$$\begin{array}{cccc} \mathbf{12} & \rightarrow & \mathbf{6}_{(1)} & +\overline{\mathbf{6}}_{(-1)} \\ (\underline{\pm1,0^5}) & & (\underline{1,0^5}) & (\underline{0^5,-1}) \end{array} \tag{7.55}$$

$$\begin{array}{cccccc} \mathbf{32} & \rightarrow & \mathbf{15}_{(-1)} & +\overline{\mathbf{15}}_{(1)} & +\mathbf{1}_{(3)} & +\mathbf{1}_{(-3)} \\ ((\pm\frac{1}{2})^6) & & (\underline{\tfrac{1}{2}^2,-\tfrac{1}{2}^4}) & (\underline{\tfrac{1}{2}^4,-\tfrac{1}{2}^2}) & (\tfrac{1}{2}^6) & (-\tfrac{1}{2}^6) \end{array} \tag{7.56}$$

where we only present the last six components of the weight vectors and the SO(12) and SU(6)×U(1) representations. This is sufficient because of the structure of the roots: the first 10 entries describe SO(20), the last six SU(6). Furthermore, the latter define the U(1)-charge completely. Thus, the SO(20) representations survive as given initially on the orbifold. For the further blowing up procedure we refer to the main text in section 7.1.

The Branched Anomaly and the Matching

Here we calculate the anomaly $\hat{I}_8^{\mathrm{het}}$ in the branched representations due to the symmetry breaking of the orbifold gauge group. Let us start with the orbifold anomaly of section 6.3.1

$$\begin{aligned} \hat{I}_8^{\mathrm{het}} &= \left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + \mathrm{tr}_{\mathbf{12}}(iF_{\mathbf{12}})^2 - \mathrm{tr}R^2\right) \\ &\quad \times \left(\frac{3}{128}\mathrm{tr}_{\mathbf{12}}(iF_{\mathbf{12}})^2 - \frac{1}{128}\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 - \frac{1}{256}\mathrm{tr}R^2\right). \end{aligned} \tag{7.57}$$

The necessary branchings are

$$(\mathbf{20},\mathbf{1}) \rightarrow (\mathbf{20},\mathbf{1})_{(0)} \tag{7.58}$$

$$(\mathbf{1},\mathbf{12}) \rightarrow (\mathbf{1},\mathbf{6})_{(1)} + (\mathbf{1},\overline{\mathbf{6}})_{(-1)} \tag{7.59}$$

Denoting the field strengths on the resolution by $F_{\mathbf{20}}$, $F_{\mathbf{6}}$ and F we obtain

$$\mathrm{tr}_{\mathbf{12}}F_{\mathbf{12}}^2 = \mathrm{tr}_{\mathbf{6}_{(1)}}F^2 + \mathrm{tr}_{\overline{\mathbf{6}}_{(-1)}}F^2 = 2F_{\mathbf{6}}^2 + 12F^2. \tag{7.60}$$

Of course, $\mathrm{tr}_{\mathbf{20}}F_{\mathbf{20}}^2$ is invariant under the branching. Then we insert all results into the anomaly $\hat{I}_8^{\mathrm{het}}$ and obtain the branched anomaly in the U(1)-basis of $\mathrm{U}(1)_V \times \mathrm{U}(1)$

$$\begin{aligned} \hat{I}_8^{\mathrm{het}} &= \left(\mathrm{tr}_{\mathbf{20}}(iF_{\mathbf{20}})^2 + 2\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + 12(iF)^2 - \mathrm{tr}R^2\right) \\ &\quad \times \left(-\frac{1}{128}\mathrm{tr}_{\mathbf{28}}(iF_{\mathbf{20}})^2 + \frac{3}{64}\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 + \frac{9}{32}(iF)^2 - \frac{1}{256}\mathrm{tr}R^2\right). \end{aligned} \tag{7.61}$$

We use this to calculate the anomaly $\hat{I}_8^{\mathrm{red}}$ corresponding to the anomalous field redefinitions of the twisted matter. We finally deduce

$$\hat{I}_8^{\mathrm{red}} = (2\pi i)^3\left(I_8^{\mathrm{res}} - I_8^{\mathrm{het}}\right) \tag{7.62}$$

$$= 3(iF)\left[2(iF)^3 + \frac{1}{3}\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^3 + (iF)\mathrm{tr}_{\mathbf{6}}(iF_{\mathbf{6}})^2 - \frac{1}{8}(iF)\mathrm{tr}R^2\right] \tag{7.63}$$

The term in the brackets is X_6^{red}.

7.3.3 Blowing Up Model 2C

The Orbifold Model and its Blow Up

The orbifold model is given by $V_{\text{orb}} = \left(\frac{1}{2}^{15}, -\frac{3}{2}\right)$, the gauge group SU(16) × U(1) and the matter

$$\frac{1}{8}\mathbf{120}_{(2)} + \mathbf{16}_{(-3)}. \tag{7.64}$$

It will be matched with resolution model **2C I**. The matching requires the symmetry breaking

$$\mathrm{SU}(16) \to \mathrm{SU}(15) \times \mathrm{U}(1)'. \tag{7.65}$$

Let us analyse the branching of all representations

$$\begin{array}{ccccccc} \mathbf{255}_{(0)} & \to & \mathbf{224}_{(0,0)} & +\mathbf{15}_{(2,-4)} & +\overline{\mathbf{15}}_{(-2,4)} & +\mathbf{1}_{(0,0)} \\ \pm(\underline{1,-1,0^{14}}) & & (\pm(\underline{1,-1,0^{13}}),0) & (\underline{1,0^{14}},-1) & (\underline{0^{14},-1},1) & (0^{16}) \end{array} \tag{7.66}$$

where also the root vectors are given. The U(1)-generators are again redefined to (Q_V, Q), compare (7.67). The U(1)-generator after the breaking (7.65) of the SU(16) is $Q'' = (1^{15}, -15)$. It is orthogonal to the initial U(1) on the orbifold given by $Q' = (1^{16})$ and the roots of the non-Abelian gauge factor SU(15). Our aim is to obtain one generator Q_V equal to the $\mathrm{U}(1)_V$ on the resolution. The other one, Q, is chosen to reproduce the U(1)-charges of the bulk matter on the resolution. We redefine them in the following way

$$\begin{pmatrix} Q_V \\ Q \end{pmatrix} = C \cdot \begin{pmatrix} Q' \\ Q'' \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}^{15}, -\frac{3}{2}) \\ (1^{15}, 5) \end{pmatrix}, \qquad C := \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{5}{4} & -\frac{1}{4} \end{pmatrix}. \tag{7.67}$$

Then the branchings of the matter representations read as

$$\begin{array}{ccccc} \mathbf{120}_{(2)} & \to & \mathbf{105}_{(1,2)} & +\mathbf{15}_{(-1,6)} \\ (\underline{1,1,0^{14}}) & & (\underline{1,1,0^{13}},0) & (\underline{1,0^{13}},1) \end{array} \tag{7.68}$$

$$\begin{array}{ccccc} \mathbf{16}_{(-3)} & \to & \mathbf{15}_{(-1,-4)} & +\mathbf{1}_{(-3,0)} \\ (\underline{\frac{3}{4},-\frac{1}{4}^{15}}) & & (\underline{\frac{3}{4},-\frac{1}{4}^{14}},-\frac{1}{4}) & (-\frac{1}{4}^{15},\frac{3}{4}) \end{array} \tag{7.69}$$

The charges of the orbifold representations are of course charges under the initial U(1) with generator (1^{16}).

Now we identify the Higgs of the symmetry breaking as the twisted field $\mathbf{16}_{(-3)}$. First we assign a vev v to the blow-up mode

$$(\mathbf{1},\mathbf{1})_{(3,0)} = e^T v\,, \tag{7.70}$$

that is contained in the Higgs. Then we obtain the desired symmetry breaking. Now we use the new superfield T to perform the anomalous field redefinition

$$\mathbf{15}'_{(2,-4)} := e^{-T}\mathbf{15}_{(-1,-4)} \tag{7.71}$$

Again the components $\frac{1}{8}\mathbf{15}'_{(2,-4)}$ of the localised Higgs are eaten by the broken gauginos of the breaking (7.66) that become massive. Only the chiral matter remains massless. Finally we obtain the spectrum of resolution model **2C I**:

$$\frac{1}{8}\left[\mathbf{105}_{(1,2)} + \mathbf{15}_{(-1,6)}\right] + \frac{7}{8}\mathbf{15}'_{(2,-4)}. \tag{7.72}$$

The Branched Anomaly and the Matching

The orbifold anomaly is given by

$$\hat{I}_8^{\text{het}} = \left(2\text{tr}_{\mathbf{16}}(iF_{\mathbf{16}})^2 + 32(iF)^2 - \text{tr}R^2\right)\left(2(iF)^2 - \frac{1}{256}\text{tr}R^2\right) \tag{7.73}$$

For the matching of the orbifold and resolution anomaly we have to use the branching

$$\mathbf{16}_{(1)} \quad \rightarrow \quad \mathbf{15}_{(1,1)} + \mathbf{1}_{(1,-15)}\,, \tag{7.74}$$

where we use the charges with respect to (Q', Q''). However, the charges under Q' are irrelevant for the following and suppressed. We denote the corresponding field strengths by $F_{\mathbf{15}}$, $\tilde{F}'$ and F'' and obtain

$$\text{tr}_{\mathbf{16}}F_{\mathbf{16}}^2 = \text{tr}_{\mathbf{15}_{(1)}}F^2 + \text{tr}_{\mathbf{1}_{(-15)}}F^2 + \text{tr}_{\mathbf{1}_{(-2,0)}}F^2 = F_{\mathbf{15}}^2 + 240F''^2\,. \tag{7.75}$$

The redefinition of the U(1)-generators with the matrix C as given in (7.67) has to be taken into account. Let F, F' denote the field strengths in the basis Q_V, Q. Then we obtain

$$\begin{pmatrix} \tilde{F}' \\ F'' \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{4} \\ \frac{1}{8} & -\frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} F \\ F' \end{pmatrix} = \begin{pmatrix} \frac{3}{8}F + \frac{5}{4}F' \\ \frac{1}{8}F - \frac{1}{4}F' \end{pmatrix}. \tag{7.76}$$

This yields $\tilde{F}'^2 = \frac{9}{64}F^2 + \frac{15}{16}F'F + \frac{25}{16}F^2$ and $F''^2 = \frac{1}{64}F^2 - \frac{1}{16}FF' + \frac{1}{16}F'^2$. Then the final result for the branched anomaly $\hat{I}_8^{\text{het}}$ in the basis (Q_V, Q) reads

$$\begin{aligned} \hat{I}_8^{\text{het}} &= \left(2\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 + 12(iF)^2 + 80(iF')^2 - \text{tr}R^2\right) \\ &\quad \times \left(\frac{9}{32}(iF)^2 + \frac{15}{8}(iF)(iF') + \frac{25}{8}(iF')^2 - \frac{1}{256}\text{tr}R^2\right). \end{aligned} \tag{7.77}$$

Note that the $(iF)(iF')$-term has the opposite sign in the anomaly $\hat{I}_8^{\text{uni}}$ of section 6.3.2. Finally we read off the anomaly contribution $\hat{I}_8^{\text{red}}$ to match the resolution anomaly

$$\hat{I}_8^{\text{red}} = (2\pi i)^3\left(I_8^{\text{res}} - I_8^{\text{het}}\right) \tag{7.78}$$

$$\begin{aligned} &= 3(iF)\left[\frac{1}{6}\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^3 + \frac{1}{4}(iF)\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2 - 2(iF')\text{tr}_{\mathbf{15}}(iF_{\mathbf{15}})^2\right. \\ &\quad \left. +2(iF)^3 - 30(iF)^2(iF') + 60(iF)(iF')^2 - 160(iF')^3 - \frac{1}{8}\text{tr}R^2\left((iF) - \frac{5}{4}(iF')\right)\right]. \end{aligned} \tag{7.79}$$

The term in the brackets is X_6^{red}.

Chapter 8

Inclusion of Wilson Lines

We are now in the position to apply the models on the local non-compact resolution to construct also global resolution models. We will present a construction scheme of these global resolution models for the four-dimensional case by gluing the local models. Furthermore, we will allow for different local U(1)-bundles. This will enable us to match the blown up models of heterotic orbifold models on the compact $T^6/\mathbb{Z}_3$ in the presence of Wilson lines[1]. We will explicitly demonstrate that a blown up $E_8 \times E_8$ orbifold model with one Wilson line can be matched completely with a resolution built according to our construction rules. Finally, we give an outlook for further applications and research.

8.1 Construction of Global Resolution Models

Here we describe the construction of a consistent resolution model taking into account an orbifold Wilson line. We will follow the discussion of [40]. First we have to understand how this Wilson line, that is formulated in the language of orbifold models, can emerge on the compact resolution. The general idea is to transfer concepts known from orbifolds to the situation of smooth resolutions.

The presence of Wilson lines A_{orb} implies different local orbifold shifts V^i_{orb} at the various fixed points, that are labelled by i. On $T^6/\mathbb{Z}_3$ the situation is completely described by four quantities describing a specific model. They are the orbifold shift vector V_{orb} and the three Wilson lines A^i_{orb}, $i = 1, 2, 3$. Then the local shift at a given fixed point can be expressed by $V^i_{\text{orbi}} = V_{\text{orb}} + n_i A^i_{\text{orb}}$ with $n_i \in \mathbb{Z}$. However, possible choices V_{orb}, A^i_{orb} are restricted by the conditions of modular invariance and $\mathbb{Z}_3$ periodicity, cf. section 3.2.2.

But how can we translate this picture into a reasonable one on the resolution? First, we are able to match a local orbifold with shift vector V_{orb} with a corresponding local resolution model with a U(1)-bundle defined by $V = V_{\text{orb}} \mod 3\lambda$, up to irrelevant sign flips. We will write this as $V \equiv V_{\text{orb}}$ in the following. Then we can simply apply this to the construction of a global resolution model: We glue allowed local resolutions with different

[1]For an overview of $T^6/\mathbb{Z}_3$ orbifold models with Wilson lines see for example [21]

U(1)-backgrounds $V^i \equiv V^i_{\text{orbi}}$ in an appropriate way. We just some of the $\mathcal{M}^3$ resolution models classified in section 5.1.3 and patch them together using adequate rules.

8.1.1 Gluing Rules and Projection Conditions

Now we present the rules that are necessary to obtain reasonable compact resolutions and heterotic models on them. The first consistency condition is again the integrated Bianchi identity. Locally this is given by $(V_i)^2 = 12$ for the resolution $\mathcal{M}^3$ at each resolved fixed point. However, new compact 4-cycles are present on the compact resolution that occur by combining the non-compact 4-cycles of the resolution $\mathcal{M}^3$. New conditions emerge but it can be shown [39, 43] for the case of $T^6/\mathbb{Z}_3$ that they are automatically satisfied. Hence, we have to take only locally allowed resolutions to build a global one.

Co-cycle Condition

Another consistency condition emerges due to the consistency requirement for the bundle over the compact resolution space. The gauge bundle is defined on the global manifold, thus, we have to consider changes of local coordinates on the sections of two charts U_i, U_j. The local resolution models have to be interpreted as such charts U_i around a resolved fixed point of the orbifold. Consequently, we have to define the transition functions g_{ij} on $U_i \cap U_j$ that define the global structure of the bundle constructed out of the local pieces at each U_i. They simply map fibre coordinates in one patch to those in another patch. Here they are given by

$$g_{ij} = e^{2\pi i A^I_{ij} H_I/3}, \tag{8.1}$$

where A_{ij} describes a function on $U_i \cap U_j$ defining the topology of the bundle. It simply provides the contribution to glue a patch with a bundle of generator $V_i^I H_I$ and another one with $V_j^I H_I$. Furthermore, we can identify it in the blow down limit with the orbifold Wilson line A_{orb} that lies along the direction connecting the fixed points in U_i and U_j. In this limit A_{ij} is constant. Therefore, we will refer to the functions A_{ij} also as Wilson lines to express the connection to orbifold Wilson lines.

In the case of three arbitrary intersecting patches we have the so-called co-cycle condition

$$g_{ij} \circ g_{jk} \circ g_{ki} \stackrel{!}{=} 0, \tag{8.2}$$

and additionally $(g_{ij})^{-1} = g_{ji}$ and $g_{ii} = 1$. These conditions are necessary for a consistent global bundle. The co-cycle yields the condition $A_{ij} + A_{jk} + A_{ki} = 0 \mod 3\lambda$ for λ in the weight lattice.

The Homotopy Condition

The resolution $\mathcal{M}^3$ is a line bundle over $\mathbb{CP}^2$ that is a compact space. Hence, $\mathcal{M}^3$ is non-compact due to its non-compact fibre $\mathbb{C}$. We have to glue the resolutions $\mathcal{M}^3$ such that the non-compact fibre directions of different resolved fixed points combine into new

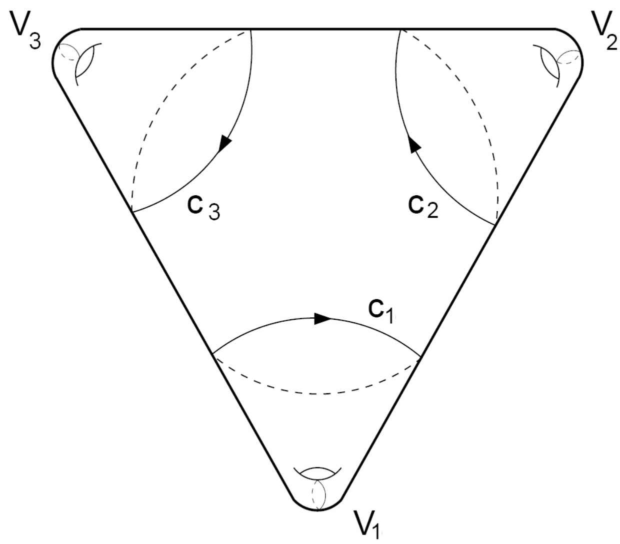

Figure 8.1: The two-dimensional illustration of the resolved $T^6/\mathbb{Z}_3$ as $T^2/\mathbb{Z}_3$ in one plane. This illustration is taken from [40] with minor changes.

compact cycles. It is illustrative to analyse the global resolution in a two-dimensional projection. Thus, we fix the coordinates in two of the three complex planes of the resolved $T^6/\mathbb{Z}_3$ such that the fibre coordinate of the bundle is along the remaining plane. We choose coordinates of a resolved fixed point labelled by (f_2, f_3) in the two planes. The geometry of this can depicted as in Fig. 8.1 and is the resolved $T^2/\mathbb{Z}_3$. On this space we can identify three 1-cycles c_i, $i = 1, 2, 3$ that encircle the resolved fixed point (i, f_2, f_3). However, they are not independent and related by $(c_1 \circ c_2) = c_3^{-1}$ in the homotopy group. Hence, the path $c_1 \circ c_2 \circ c_3$ is contractible. These paths c_i lie at the "boundary" of the patches U_i. Then the presence of a gauge flux in the patch U_i can be expressed [43] as

$$\int_{c_i} i\mathcal{A}_{(i,f_2,f_3)} = \frac{2\pi i}{3} V_i^I H_I \quad \mathrm{mod}\ 3\lambda^I H_I. \tag{8.3}$$

Here we denoted the gauge field by the chosen fixed point coordinates (f_2, f_3) in the two planes and the considered patch on the resolved $T^2/\mathbb{Z}_3$. Then $c_1 \circ c_2 \circ c_3$ immediately translates into

$$\sum_i 3 \int_{c_i} i\mathcal{A}_{(i,f_2,f_3)} = 2\pi i \left(V_1^I + V_2^I + V_3^I\right) H_I = 3\lambda^I H_I \tag{8.4}$$

with λ a vector of the $\mathrm{E}_8\times\mathrm{E}_8$ lattice. The vector λ has to be taken into account because the identification (8.3) is only valid up to root lattice vectors. Thus we obtain

$$V_1 + V_2 + V_3 \equiv 0 \tag{8.5}$$

Of course we can write down analogous conditions for the other complex planes. We can rewrite this conditions by introducing analogues of Wilson lines by $V_i - V_j = A_{ij}$. These are the only conditions restricting the choice of the V_i as they are a priori completely independent.

All these steps define a construction scheme for a global resolution. However, we are not able to give a full classification of all resolutions fulfilling these conditions.

Spectrum and Projection Conditions

Now we want to obtain also a reasonable model on these resolutions. A first step towards this aim is to simply add the spectra of all local resolution models. In general these are different in the various local patches. However, this is not sufficient as we need for example a definition of a 4D gauge group. We will see that there are appropriate projection conditions for the global gauge group and the massless states. Let us just motivate the necessary projections and refer to [40] for the technical details. We will see that they are quite illustrative.

- The 4D gauge group is determined by a maximal projection condition. This is obvious as the gauge bosons are bulk fields. They can move freely on the whole resolution space, i.e. their profile is completely de-localised. Hence, they are affected by all local gauge fluxes V_i as they can enter each local patch. Gauge bosons of the 4D gauge group have always to commute with the gauge background that is here defined by the glued U(1)-bundles. Thus, we have to demand

$$V_i \cdot p = 0 \tag{8.6}$$

  for all local backgrounds V_i. Again we denote the roots of the gauge bosons by p. It is convenient to rewrite this for the case of one Wilson line as $V_1 =: V$, $V_2 =: V + A$ and $V_3 = V + 2A + 3\lambda'$. We will see that it is allowed to choose the orbifold Wilson A_{orb} as this A. Using this new variables we obtain the equivalent condition

$$V \cdot p = 0, \quad A \cdot p = 0, \quad \lambda' \cdot p = 0\,. \tag{8.7}$$

  . The rank of the gauge group is not reduces as there is no projection condition on the gauge bosons corresponding to the Cartan generators H_I. Again this projections are more restrictive than the corresponding orbifold conditions as they have only to be obeyed up integers. Thus, the 4D gauge group on the resolution is smaller than the orbifold one. Furthermore, trivial Wilson lines from the orbifold point of view, i.e. Wilson lines that are multiples of $E_8 \times E_8$ lattice vectors are non-trivial for the resolution model. They produce different 4D gauge groups on the resolution.

- The matter on the resolution is also subject to projection conditions. However, we have to distinguish between localised and bulk matter. We mention again that localised states occur with an integral multiplicity and bulk matter with fractional.

In the case of $T^6/\mathbb{Z}_3$ multiples of 1/9 occur. The localised matter has a vanishing profile outside a local patch. Hence, it feels only the influence of the gauge flux in its own local patch. Consequently, there are no further projection conditions and it has only to be branched according to the breaking of the local into the 4D gauge group.

However, the bulk matter can move over the whole resolution space. Therefore, it feels the projection due to the transition functions between the local gauge backgrounds. This can be stated most easily again using $V_1 = V$, $V_2 = V + A$

$$A \cdot p = 0 \mod 3\,, \tag{8.8}$$

where we denote the weights of the matter representations by p. The mod-condition occurs because $\frac{2\pi i}{3} A \cdot p$ is just the phase the matter picks up and the occurrence of 3λ in all conditions for the bundle on the resolution.

This concludes the summary of projection conditions for gauge group and matter. Now we will apply them to match the blow up of an $\mathrm{E}_8 \times \mathrm{E}_8$ orbifold model with one Wilson line. The general recipe is to blow up the local orbifold models at the various fixed point of the Wilson line model by identifying blow-up modes and apply the whole machinery developed for the local models as before. Then the choices of blow-up modes guide us which local resolution models have to be glued. Finally we add the spectra of the resolution models, calculate the 4D gauge group and matter. At the end we will obtain a matching with the blown up orbifold model.

8.1.2 A one Wilson Line Example

Let us start with the orbifold model defined by $V_{\text{orb}} = (2, 2, 0^6)(2, 0^7)$ and the Wilson line $A_{\text{orb}} = (0, -4, 2, 0^5)(-2, 0^7)$ that is chosen to be in the first plane along the basis vectors e_1 as well as e_2. The Wilson line splits the 27 fixed point into three groups[2] of inequivalent fixed points, labelled collectively by the space group elements g_1, g_2 and g_3, respectively, under which they are fixed in the first plane. This means that there are different local orbifold shifts V_{orb}, $V_{\text{orb}} + A_{\text{orb}}$ and $V_{\text{orb}} + 2A_{\text{orb}}$ yielding different local models for each of the three groups of fixed points but the same model at the fixed points of one group. The details are given in Tab. 8.1. There we show the untwisted sector U with the 4D gauge group of the orbifold model, the three local models at the fixed points labelled by the space group element given in the form $g_i = (\theta, n_1 e_1 + n_2 e_2)$, the local gauge group, local matter and their branchings due to the blowing up. Furthermore, we give the blow-up mode, the field redefinitions of the twisted fields and the local resolution model that is chosen at the fixed point. Again there a vectorlike states that acquire a mass and decouple from the massless spectrum. Let us now blow up this model. We identify the blow-up modes at each of the orbifold fixed point by

$$g_1: \quad (\mathbf{1};\mathbf{1})_{(-4,0,0)}\,, \quad g_2: \quad (\mathbf{27};\mathbf{1})_{(0,0,0)} \tag{8.9}$$

$$g_3: \quad (\mathbf{1};\mathbf{1})_{(0,-4,0)}\,. \tag{8.10}$$

[2]Each group consists of 9 equivalent fixed points. Then there are $3 \times 3 = 9$ possible choices for a fixed point in the two other planes.

fixed point (loc.) gauge group	matter mult.	local matt.	4D matt.	decomposition to blow-up group	field redefinition
U Sector	3		$(\mathbf{27},\mathbf{1})_{(2,2,0)}$	$(\mathbf{16},\mathbf{1})_{(2,2,0,-1)}$	
$\mathrm{E}_6\times\mathrm{SO}(14)\times\mathrm{U}(1)^3$				$(\mathbf{10},\mathbf{1})_{(2,2,0,2)}$	
				$(\mathbf{1},\mathbf{1})_{(2,2,0,-4)}$	
$g_1=(\theta,0)$	1	$(\mathbf{1},\mathbf{14})_{(2,0)}$	$(\mathbf{1},\mathbf{14})_{(2,0,0)}$	$(\mathbf{1},\mathbf{14})_{(2,0,0,0)}$	$=e^{-\frac{1}{2}T_1}(\mathbf{1},\mathbf{14})'^m_{(0,0,0,0)}$
$\mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)^2$	1	$(\mathbf{1},\mathbf{1})_{(-4,0)}$	$(\mathbf{1},\mathbf{1})_{(-4,0,0)}$	$(\mathbf{1},\mathbf{1})_{(-4,0,0,0)}$	$=v_1e^{T_1}$
	3	$(\mathbf{1},\mathbf{1})_{(0,4)}$	$(\mathbf{1},\mathbf{1})_{(0,2,-2)}$	$(\mathbf{1},\mathbf{1})_{(0,2,-2,0)}$	$=e^{T_1}(\mathbf{1},\mathbf{1})'_{(4,2,-2,0)}$
local blow-up at g_1	**3C I**				
$g_2=(\theta,e_1)$	1	$(\mathbf{27},\mathbf{1},\mathbf{1})$	$(\mathbf{27},\mathbf{1})_{(0,0,0)}$	$(\mathbf{16},\mathbf{1})_{(0,0,0,-1)}$	$=e^{T_2}(\mathbf{16},\mathbf{1})'_{(0,0,0,3)}$
$\mathrm{E}_6\times\mathrm{SU}(3)\times\mathrm{E}_8$				$(\mathbf{10},\mathbf{1})_{(0,0,0,2)}$	$=e^{-\frac{1}{2}T_2}(\mathbf{10},\mathbf{1})'^m_{(0,0,0,0)}$
				$(\mathbf{1},\mathbf{1})_{(0,0,0,-4)}$	$=v_2e^{T_2}$
	3	$(\mathbf{1},\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{1})_{(-2,-2,0)}$	$(\mathbf{1},\mathbf{1})_{(-2,-2,0,0)}$	$=e^{T_2}(\mathbf{1},\mathbf{1})'_{(-2,-2,0,4)}$
			$(\mathbf{1},\mathbf{1})_{(0,2,2)}$	$(\mathbf{1},\mathbf{1})_{(0,2,2,0)}$	$=e^{T_2}(\mathbf{1},\mathbf{1})'_{(0,2,2,4)}$
			$(\mathbf{1},\mathbf{1})_{(2,0,-2)}$	$(\mathbf{1},\mathbf{1})_{(2,0,-2,0)}$	$=e^{T_2}(\mathbf{1},\mathbf{1})'_{(2,0,-2,4)}$
local blow-up at g_2	**3A I**				
$g_3=(\theta,e_1+e_2)$	1	$(\mathbf{1},\mathbf{14})_{(0,2)}$	$(\mathbf{1},\mathbf{14})_{(0,2,0)}$	$(\mathbf{1},\mathbf{14})_{(0,2,0,0)}$	$=e^{-\frac{1}{2}T_3}(\mathbf{1},\mathbf{14})'^m_{(0,0,0,0)}$
$\mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)^2$	1	$(\mathbf{1},\mathbf{1})_{(0,-4)}$	$(\mathbf{1},\mathbf{1})_{(0,-4,0)}$	$(\mathbf{1},\mathbf{1})_{(0,-4,0,0)}$	$=v_3e^{T_3}$
	3	$(\mathbf{1},\mathbf{1})_{(4,0)}$	$(\mathbf{1},\mathbf{1})_{(2,0,2)}$	$(\mathbf{1},\mathbf{1})_{(2,0,2,0)}$	$=e^{T_3}(\mathbf{1},\mathbf{1})'_{(2,4,2,0)}$
local blow-up at g_3	**3C I**				

Table 8.1: The four U(1)-generators of the 4D gauge group are $Q_1=(2,2,0^6)(2,0^7)$, $Q_2=(2,0,-2,0^5)(-2,0^7)$, $Q_3=(0,-2,-2,0^5)(2,0^7)$ and $Q_4=(2,-2,2,0^5)(0^8)$. The $\mathrm{U}(1)^2$- generators of the local models at g_1 and g_3 are Q_1, Q_2-Q_3 and Q_1+Q_3, Q_2, respectively. The superscript m indicates matter, that acquires a mass.

The vevs of these fields break the orbifold gauge groups at the fixed points as follows

$$g_1:\ \mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)^2 \rightarrow \mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)\ , \tag{8.11}$$

$$g_2:\ \mathrm{E}_6\times\mathrm{SU}(3)\times\mathrm{E}_8 \rightarrow \mathrm{SO}(10)\times\mathrm{SU}(3)\times\mathrm{E}_8\times\mathrm{U}(1)\ , \tag{8.12}$$

$$g_3:\ \mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)^2 \rightarrow \mathrm{E}_7\times\mathrm{SO}(14)\times\mathrm{U}(1)\ . \tag{8.13}$$

We obtain the 4D gauge group

$$\mathrm{SO}(10)\times\mathrm{SO}(14)\times\mathrm{U}(1)^4, \tag{8.14}$$

as the intersection of the local ones. Note that three U(1)-factors are broken explicitly due to the vevs.

This choices of the blow-up modes fix the local resolution models that we glue together. We choose $V_1=V_{\mathrm{orb}}$, $V_2=V_{\mathrm{orb}}+A_{\mathrm{orb}}$ and $V_3=V_{\mathrm{orb}}+2A_{\mathrm{orb}}+3\lambda'=(2,0-2,0^5)(-2,0^7)$, where we added $3\lambda'$ to obtain $V_3^2 = 12$. These vectors V_i correspond to the resolution models that are used in the three local patches. We use model **3C I** at the nine fixed points denoted by g_1 and g_3, respectively, and model **3A I** at the nine of g_2. We simply apply the results of section 7 for the resolution models and then use the projection conditions of section 8.1.1 to obtain the 4D gauge group and the matter at the blown up fixed points g_i

$$\mathrm{SO}(10) \times \mathrm{SO}(14) \times \mathrm{U}(1)^4 \tag{8.15}$$

$$g_1 \;:\; \left[(\mathbf{16};\mathbf{1})_{(2,2,0,\text{-}1)} + (\mathbf{10};\mathbf{1})_{(2,2,0,2)} + (\mathbf{1};\mathbf{1})_{(2,2,0,\text{-}4)}\right] + 27\,(\mathbf{1};\mathbf{1})_{(4,2,\text{-}2,0)}\;, \tag{8.16}$$

$$g_2 \;:\; \left[(\mathbf{16};\mathbf{1})_{(2,2,0,\text{-}1)} + (\mathbf{10};\mathbf{1})_{(2,2,0,2)} + (\mathbf{1};\mathbf{1})_{(2,2,0,\text{-}4)}\right] \tag{8.17}$$

$$+9(\mathbf{16};\mathbf{1})_{(0,0,0,3)} + 27\,(\mathbf{1};\mathbf{1})_{(\text{-}2,\text{-}2,0,4)} + 27\,(\mathbf{1};\mathbf{1})_{(0,2,2,4)} + 27\,(\mathbf{1};\mathbf{1})_{(2,0,\text{-}2,4)}\;, \tag{8.18}$$

$$g_3 \;:\; \left[(\mathbf{16};\mathbf{1})_{(2,2,0,\text{-}1)} + (\mathbf{10};\mathbf{1})_{(2,2,0,2)} + (\mathbf{1};\mathbf{1})_{(2,2,0,\text{-}4)}\right] + 27\,(\mathbf{1};\mathbf{1})_{(2,4,2,0)}\;. \tag{8.19}$$

Here we multiplied the spectrum already by nine to take into account the nine different fixed points with these matter. The untwisted matter is given by the representations in the brackets. It is the same for all $3 \times 9 = 27$ local patches. Note that a lot of untwisted matter is projected out by the projection rules of the gluing. Finally, this spectrum matches the blown up orbifold after applying the necessary field redefinitions of the twisted fields, compare to the last column of Tab. 8.1.

Finally, we have to analyse F- and D-flatness of the blown-up global orbifold model. The details can again be found in [40]. Let us just state that the blown up global orbifold is F-flat. There are no allowed non-vanishing couplings that contain high enough powers of the blow-up modes and are only linear in the other fields such that a non-vanishing F-term could occur. Again, gauge invariance plays an important role in this analysis. Furthermore, the issue of D-flatness can be solved in the context of global orbifolds. We give here only a qualitative discussion according to [40]. However, we have to admit that an interpretation from the point of view of the global resolution is still missing, where all D-terms are understood as FI-terms due to the anomalous U(1)-factors. A complete analysis is still missing. Let us consider the global orbifold model. First we note, that there is an anomalous U(1) denoted by Q_{an}, that corresponds to $Q_{\mathrm{an}} = Q_1 + Q_2$ according to our choice of the U(1)-basis in Tab. 8.1. The sum of all charges under this anomalous is positive, i.e. $\mathrm{Tr}\,(Q_{\mathrm{an}}) > 0$. The anomalous U(1) produces a FI-term [9, 3], that we have to cancel by assigning vevs to some fields of the spectrum. We will argue that our choice of blow-up modes and vevs is compatible with D-flatness. However, it is necessary to assign a vev also to the untwisted state $(\mathbf{10},\mathbf{1})_{(2,2,0,2)}$ of Tab. 8.1. We branch it as

$$\mathbf{10} \to \mathbf{8}_{(0)} + \mathbf{1}_{(2)} + \mathbf{1}_{(-2)}\;, \tag{8.20}$$

and assign the same vev v_a to the two singlets. This induces the breaking SO(10) $\to$ SO(8) $\times$ U(1), where the U(1) is broken explicitly. This choice of vevs guarantees, that the D-term of the new U(1) is cancelled as both components have opposite charge but equal vev. Furthermore, it is be possible to fix the ratios of v_a and v_1, v_2 and v_3 of the blow-up modes such that the D-terms including the FI-term of the anomalous U(1) cancel. The blown up local orbifold is D-flat. This is one possible choice to obtain a D-flat global orbifold model using the considered blow-up modes.

8.2 Outlook

Here we list a few of the open questions and give an outlook for further research. In general, the interpretation of D-flatness in the global case is still an open point and further analysis is necessary, especially from the point of view of the global resolution. However, the complete list of matched local six-dimensional and four-dimensional models is a promising result and may also guide the way for a complete understanding of the issue of D-flatness. Furthermore, it is still unclear, whether all possible compact resolution models according to the construction rules given here can really be matched by blown-up orbifold models, that are F- and D-flat at the end. Vice versa it has to be analysed, whether all blown-up orbifold models can be interpreted as a resolution model with a (local) U(1)-bundle. It might be necessary to analyse more complicated and even non-Abelian bundles. Also a general statement about the number of anomalous U(1)'s and 4D axions on the compact resolution and the connection to the number of orbifold Wilson lines is still missing. This is equivalent to a complete analysis of the structure of the global anomaly polynomials of the orbifold and resolution. The phenomenological implications of these questions have to be considered, too.

Concerning the local resolution models, the analysis of more general geometries using the methods of toric geometry is a possible generalisation to more interesting orbifolds like $T^6/\mathbb{Z}_{6-II}$. A more detailed inspection of the concrete relationship between the vev of the blow-up mode and the size of the exceptional divisor of the resolution is another open point in the context of the explicit formulæ of [43]. Furthermore, it should be analysed how far the matching of blown-up orbifold and resolution models can be extended and whether they are really identical, namely the physics beyond the considered matching steps has to be investigated.

Appendix A

Tables of the 4D Matching

	orbifold model	resolution model	field redefinitions
3A ↓ **3A I**	$E_6 \times SU(3) \times E_8$	$SO(10) \times U(3) \times E_8$	$(\mathbf{27},\mathbf{1};\mathbf{1}) \to \begin{cases} (\mathbf{1},\mathbf{1};\mathbf{1})_{-4} = e^T v \\ (\mathbf{16},\mathbf{1};\mathbf{1})_{-1} = e^T (\mathbf{16},\mathbf{1};\mathbf{1})_3 \\ (\mathbf{10},\mathbf{1};\mathbf{1})_2 = e^{-T/2} (\mathbf{10},\mathbf{1};\mathbf{1})^m_0 \end{cases}$
	$\frac{1}{9}(\mathbf{27},\mathbf{3};\mathbf{1}) + (\mathbf{27},\mathbf{1};\mathbf{1}) + 3(\mathbf{1},\overline{\mathbf{3}};\mathbf{1})$	$\frac{1}{9}[(\mathbf{16},\mathbf{3};\mathbf{1})_{-1} + (\mathbf{10},\mathbf{3};\mathbf{1})_2 + (\mathbf{1},\mathbf{3};\mathbf{1})_{-4}] + (\mathbf{16},\mathbf{1};\mathbf{1})_3 + 3(\mathbf{1},\overline{\mathbf{3}};\mathbf{1})_4$	$(\mathbf{1},\overline{\mathbf{3}};\mathbf{1}) \to (\mathbf{1},\overline{\mathbf{3}};\mathbf{1})_0 = e^T (\mathbf{1},\overline{\mathbf{3}};\mathbf{1})_4$
3B ↓ **3B I**	$[E_6 \times SU(3)]^2$	$[E_6 \times U(2)]^2$	$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3}) \to \begin{cases} (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{-4,0} = e^T v \\ (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{-1,3} = e^T (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{3,3} \\ (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{-1,-3} = e^T (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{3,-3} \\ (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})_{2,0} = e^{-T/2} (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})^m_{0,0} \end{cases}$
	$\frac{1}{9}[(\mathbf{27},\overline{\mathbf{3}};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{27},\overline{\mathbf{3}})] + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	$\frac{1}{9}[(\mathbf{27},\mathbf{2};\mathbf{1},\mathbf{1})_{-1,-1} + (\mathbf{27},\mathbf{1};\mathbf{1},\mathbf{1})_{2,2} + (\mathbf{1},\mathbf{1};\mathbf{27},\mathbf{2})_{-1,1} + (\mathbf{1},\mathbf{1};\mathbf{27},\mathbf{1})_{2,-2}] + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{3,3} + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{3,-3}$	
3C ↓ **3C I**	$E_7 \times SO(14) \times U(1)^2$	$E_7 \times SO(14) \times U(1)^2$	$(\mathbf{1};\mathbf{1})_{-4,0} = e^T v$ $(\mathbf{1};\mathbf{1})_{0,4} = e^T (\mathbf{1};\mathbf{1})_{4,4}$ $(\mathbf{1};\mathbf{14})_{2,0} = e^{-T/2} (\mathbf{1};\mathbf{14})^m_{0,0}$
	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}] + (\mathbf{1};\mathbf{14})_{2,0} + (\mathbf{1};\mathbf{1})_{-4,0} + 3(\mathbf{1};\mathbf{1})_{0,4}$	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}] + 3(\mathbf{1};\mathbf{1})_{4,4}$	
3C ↓ **3C II**	$E_7 \times SO(14) \times U(1)^2$	$E_7 \times SO(12) \times U(1)^3$	$(\mathbf{1};\mathbf{14})_{2,0} \to \begin{cases} (\mathbf{1};\mathbf{1})_{-4,0,0} = e^T v \\ (\mathbf{1};\mathbf{12})_{-1,0,-2} = e^T (\mathbf{1};\mathbf{12})_{3,0,-2} \\ (\mathbf{1};\mathbf{1})_{2,0,-4} = e^{-T/2} (\mathbf{1};\mathbf{1})^m_{0,0,-4} \end{cases}$
	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{14})_{2,-4}] + (\mathbf{1};\mathbf{14})_{2,0} + (\mathbf{1};\mathbf{1})_{-4,0} + 3(\mathbf{1};\mathbf{1})_{0,4}$	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{-1,2,-2} + (\mathbf{1};\mathbf{32})_{-1,2,-2} + (\mathbf{1};\overline{\mathbf{32}})_{2,2,0} + (\mathbf{1};\mathbf{1})_{-4,-4,0} + (\mathbf{1};\mathbf{12})_{-1,-4,-2} + (\mathbf{1};\mathbf{1})_{2,-4,\pm 4}] + (\mathbf{1};\mathbf{12})_{3,0,-2} + 3(\mathbf{1};\mathbf{1})_{4,4,0}$	$(\mathbf{1};\mathbf{1})_{-4,0} \to (\mathbf{1};\mathbf{1})_{2,0,4} = e^{-T/2} (\mathbf{1};\mathbf{1})^m_{0,0,4}$ $(\mathbf{1};\mathbf{1})_{0,4} \to (\mathbf{1};\mathbf{1})_{0,4,0} = e^T (\mathbf{1};\mathbf{1})_{4,4,0}$
3D ↓ **3D I**	$SU(9) \times SO(14) \times U(1)$	$SU(8) \times SO(14) \times U(1)^2$	$(\overline{\mathbf{9}};\mathbf{1})_{-4/3} \to \begin{cases} (\mathbf{1};\mathbf{1})_{-4,0} = e^T v \\ (\overline{\mathbf{8}};\mathbf{1})_{-1,3} = e^T (\overline{\mathbf{8}};\mathbf{1})_{3,3} \end{cases}$
	$\frac{1}{9}[(\mathbf{84};\mathbf{1})_0 + (\mathbf{1};\mathbf{64})_{-1} + (\mathbf{1};\mathbf{14})_2] + (\overline{\mathbf{9}};\mathbf{1})_{-4/3}$	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{-1,-1} + (\mathbf{28};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}] + (\overline{\mathbf{8}};\mathbf{1})_{3,3}$	

Table A.1: This is the matching of the 4D orbifold and resolution models. In the first column we present the matched models, then we give the orbifold and resolution spectra and gauge group in the second and third columns, respectively. The field redefinitions to to match the two spectra are given in the last column. For blow-up models **3C II** and **3D I** a change of the U(1) basis has been done during the branching, that is indicated by →. This is done to ensure that the blow-up mode is charged under the first blowup U(1) only. The superscript m refers to non-chiral states acquiring a mass in the blow-up. This table is taken from [40] with minor changes to match our notations.

	Anomaly Polynomials of Orbifold and Resolution	α
3A $\downarrow$ **3A I**	$\hat{I}_6^{\rm het}=0$ $\hat{I}_6^{\rm uni}=\frac{3}{2}(iF)\left[24(iF)^2+{\rm tr}_{\bf 10}(iF_{\bf 10})^2+2{\rm tr}_{\bf 3}(iF_{\bf 3})^2+{\rm tr}_{\bf 8}(iF_{\bf 8})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm non}=\frac{1}{4}(iF)\left[528(iF)^2+6{\rm tr}_{\bf 10}(iF_{\bf 10})^2+12{\rm tr}_{\bf 3}(iF_{\bf 3})^2-6{\rm tr}_{\bf 8}(iF_{\bf 8})^2-R^2\right]$	$-\frac{3}{8}$
3B $\downarrow$ **3B I**	$\hat{I}_6^{\rm het}=0$ $\hat{I}_6^{\rm uni}=0$ $\hat{I}_6^{\rm non}=\frac{1}{4}(iF)\left[96(iF)^2+288(iF')^2-{\rm tr}R^2\right]$	1
3C $\downarrow$ **3C I**	$\hat{I}_6^{\rm het}=\frac{2}{3}(iF+iF')\left[24(iF)^2+48(iF')^2+\frac{1}{6}{\rm tr}_{\bf 7}(iF_{\bf 7})^2+{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm uni}=\frac{1}{6}(iF+4iF')\left[24(iF)^2+48(iF')^2+\frac{1}{6}{\rm tr}_{\bf 7}(iF_{\bf 7})^2+{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm non}=\frac{1}{4}(iF)\left[144(iF)^2+480(iF')^2+384(iF)(iF')+\frac{1}{3}{\rm tr}_{\bf 7}(iF_{\bf 7})^2-2{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$	$\frac{1}{8}$
3C $\downarrow$ **3C II**	$\hat{I}_6^{\rm het}=-\frac{1}{3}(iF-\frac{1}{2}iF'+2iF'')\times$ $\times\left[24(iF)^2+3(iF')^2+32(iF'')^2+\frac{1}{6}{\rm tr}_{\bf 7}(iF_{\bf 7})^2+{\rm tr}_{\bf 12}(iF_{\bf 12})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm uni}=\frac{1}{12}(17iF+2iF'-8iF'')\times$ $\times\left[24(iF)^2+3(iF')^2+32(iF'')^2+\frac{1}{6}{\rm tr}_{\bf 7}(iF_{\bf 7})^2+{\rm tr}_{\bf 12}(iF_{\bf 12})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm non}=\frac{1}{4}(iF)\left[288(iF)^2+12(iF')^2+128(iF'')^2+96(iF)(iF')\right.$ $\left.-384(iF)(iF'')-\frac{2}{3}{\rm tr}_{\bf 7}(iF_{\bf 7})^2+4{\rm tr}_{\bf 12}(iF_{\bf 12})^2-{\rm tr}R^2\right]$	$-\frac{1}{4}$
3D $\downarrow$ **3 DI**	$\hat{I}_6^{\rm het}=-\frac{1}{3}(iF-2iF')\left[24(iF)^2+48(iF')^2+2{\rm tr}_{\bf 8}(iF_{\bf 8})^2+{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm uni}=\frac{1}{6}(iF+4iF')\left[24(iF)^2+48(iF')^2+2{\rm tr}_{\bf 8}(iF_{\bf 8})^2+{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$ $\hat{I}_6^{\rm non}=\frac{1}{4}(iF)\left[192(iF)^2+480(iF')^2+384(iF)(iF')+4{\rm tr}_{\bf 8}(iF_{\bf 8})^2-2{\rm tr}_{\bf 14}(iF_{\bf 14})^2-{\rm tr}R^2\right]$	$-\frac{1}{8}$

Table A.2: These are the anomaly polynomials of the 4D resolution models $\hat{I}_6^{\rm blow}$ split into the universal part $\hat{I}_6^{\rm uni}$ and the non-universal part $\hat{I}_6^{\rm non}$ and the orbifold models $\hat{I}_6^{\rm het}$ after the branching due to the blowing-up. We also show the model-dependent axion redefinition parameter α of (6.42).

Appendix B

Trace Relations

Here we cite just the results of [16] needed for our calculations. We also adopt the notation used there. However, let us note that it is possible to verify them by explicit calculations with Chern characters. Unfortunately, the details of these calculations would exceed the scope of this thesis.

$$
\begin{aligned}
\mathrm{Tr}F^2_{\mathrm{SU}(N)} &= 2N\,\mathrm{tr}F^2_{\mathrm{SU}(N)} && \text{(B.1)}\\
\mathrm{Tr}F^2_{\mathrm{SO}(N)} &= (N-2)\,\mathrm{tr}F^2_{\mathrm{SO}(N)} && \text{(B.2)}\\
\mathrm{Tr}F^4_{\mathrm{SU}(N)} &= 2N\,\mathrm{tr}F^4_{\mathrm{SU}(N)} + 6\,\left(\mathrm{tr}F^2_{\mathrm{SU}(N)}\right)^2 && \text{(B.3)}\\
\mathrm{Tr}F^4_{\mathrm{SO}(N)} &= (N-8)\,\mathrm{tr}F^4_{\mathrm{SO}(N)} + 3\,\left(\mathrm{tr}F^2_{\mathrm{SO}(N)}\right)^2 && \text{(B.4)}\\
\mathrm{tr}_{a^{ij}}F^3_{\mathrm{SU}(N)} &= (N-4)\,\mathrm{tr}F^3_{\mathrm{SU}(N)}\,, \;\; (N\geq 3) && \text{(B.5)}\\
\mathrm{tr}_{\mathbf{2}^{(N-1)}}F^2_{\mathrm{SO}(2N)} &= 2^{(N-4)}\,\mathrm{tr}F^2_{\mathrm{SO}(2N)} && \text{(B.6)}\\
\mathrm{tr}_{\mathbf{2}^{(N-1)}}F^4_{\mathrm{SO}(2N)} &= -2^{(N-5)}\,\mathrm{tr}F^4_{\mathrm{SO}(2N)} + 3\cdot 2^{(N-7)}\,\left(\mathrm{tr}F^2_{\mathrm{SO}(2N)}\right)^2 && \text{(B.7)}
\end{aligned}
$$

Here a^{ij} denotes the two-times antisymmetric tensor of SU(N) and $\mathbf{2}^{(N-1)}$ the spinor representation of SO(2N). One can often use simple group theory arguments like $\mathbf{N}\otimes\overline{\mathbf{N}}-\mathbf{1}$ for the adjoint of SU(N) to obtain the corresponding trace relations.

The following traces have been calculated using the properties of Chern characters. Of course, they reproduce some results given above.

$$
\begin{aligned}
\mathrm{Tr}_{a^{ij}}F^4_{\mathrm{SU(N)}} &= (N-8)\,\mathrm{tr}F^4_{\mathrm{SU(N)}} + 3\,\left(\mathrm{tr}F^2_{\mathrm{SU(N)}}\right)^2 && \text{(B.8)}\\
\mathrm{Tr}_{a^{ij}}F^2_{\mathrm{SU(N)}} &= (N-2)\,\mathrm{tr}F^2_{\mathrm{SU(N)}}\,. && \text{(B.9)}
\end{aligned}
$$

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