Heterotic Orbifold Models and Discrete Family Symmetries

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Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Referent:	Prof. Dr. Hans Peter Nilles
Korreferent :	Prof. Dr. Rainald Flume

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Chapter 1

Introduction

For years physicists have been searching for a unified description of nature, *one theory* which explains all four fundamental forces. Up to now all attempts have finally failed. But superstring theories are deemed good candidates to achieve this aim because they establish a consistent framework for grand unified theories (GUT) as well as for gravity.
They are also interesting for aesthetical reasons because they automatically include many areas of modern physics such as supersymmetry, GUTs or extra dimensions. Nevertheless they are not free of problems: For instance, superstring theories require a ten dimensional target space for a mathematically consistent formulation. Therefore the need arises to compactify six dimensions and make them small so that they cannot be observed. To retain $N = 1$ SUSY in four dimensions the compactification manifold has to be a Calabi-Yau manifold. A special limit of certain classes of these leads to the so-called orbifolds [1, 2] which are flat everywhere, except for the fixed points where the curvature is concentrated. For this reason calculations are much easier on orbifolds. The spectrum can be explicitly derived and the interaction terms of the states can be calculated. Therefore orbifolds are a good starting point to consider phenomenological questions from a string theoretical viewpoint.
There are still some open questions about the standard model or some grand unified theory, one of which is the origin of additional discrete symmetries. They are somehow artificially introduced in order to explain the derived mass patterns, a realistic CKM phase or proton decay rate. Orbifold models naturally lead to such symmetries due to selection rules for the couplings. The investigation of these discrete symmetries is one of the two main topics of this thesis.

To be more precise, the focus of this study is to clarify how discrete non-Abelian family symmetries arise from orbifold models. Therefore a construction already presented in the literature [3] is applied to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold and a certain symmetry group $\Delta(54)$ for models of a special type is derived. Finally this construction is generalised in order to apply to other orbifold set-ups as well. A certain class of discrete non-Abelian family symmetries, which emerge due to the fixed point structure of orbifolds, is derived in Sec. 4.3 and Sec. 4.3.1. To discuss the properties of the $\Delta(54)$ group we construct explicit examples of

three-generation models, based on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, for which this symmetry is realised. From the discussion in Sec. 4.4 it should become clear that this is not just a reformulation of old results but has its own power.
The second main part of this work concerns discrete torsion, a strange degree of freedom for orbifold models, already observed by Vafa in 1986 [4]. The result of our consideration of discrete torsion is that it is not really an additional degree of freedom, but can be absorbed in transformations of the gauge embedding shifts.
Both topics appear in $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold compactifications. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold combines the two features of multiplicity three for states in the twisted sectors and of being a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold. It exhibits an enormous variety of different models with many allowed coupling terms. Therefore on the one hand the fixed point structure should facilitate to construct three-generation models, on the other hand the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold provides us with far more possibilities for explicit model building.

The thesis is organised as follows. In chapter 2 and Sec. 4.1 and Sec. 4.2 of chapter 4 we review the relevant facts concerning orbifold compactification of the heterotic string theory, which we will need for our further considerations. We place special emphasis on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold. Particular three-generation $\mathbb{Z}_3 \times \mathbb{Z}_3$ models, to which we refer in later sections, are constructed in chapter 3. There we also correct some statements in the literature. The main part of chapter 3 is the discussion of discrete torsion in Sec. 3.3 and its equivalent formulation in terms of embedding shifts. In chapter 4, after stating the well known selection rules, a new view on the treatment of interaction terms on orbifolds is achieved via the identification of certain discrete symmetry groups and the construction of the mentioned class of non-Abelian family symmetries which arises. In App. A.1 the projection conditions for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold are derived and in App. A.2 some particular features of the twisted sectors explicitly calculated. The remaining appendices on the one hand contain results of group theory, App. B.1, which are used to construct the discrete symmetry groups in App. B.2, and on the other hand contain the tables with the matter content of the models to which we refer in the previous chapters (App. C).
To summarise, the own ideas are concentrated in Sec. 3.3, Sec. 3.4, Sec. 4.3 and Sec. 4.4.

Chapter 2

Towards realistic models from string theory

In this first chapter we briefly review the heterotic string construction [5, 6] and explain the need and the difficulties of compactifying six space dimensions. Thus we are directly lead to orbifold compactifications [1, 2], which are a certain limit of compactifications on Calabi-Yau manifolds. The advantage of orbifold compactifications is that they are much easier to handle and calculations on an orbifold are less involved.
The present chapter is just a review of results from the literature, which are relevant for our further considerations. Very detailed surveys of heterotic orbifolds can be found in [7, 8, 9, 10].

2.1 Compactification of the heterotic string

For the heterotic string there is the need to formulate the theory in ten spacetime dimensions. Compactification then means to make the six extra space dimensions small, in order that they cannot be observed in experiments at energy scales, which are possible nowadays. Nevertheless the phenomenology of the effective four dimensional theory, resulting from compactifying a higher dimensional one, can be verified or disproven.

2.1.1 The heterotic string

The heterotic string consists of a left moving bosonic string and a right moving type-II B superstring. The critical dimension of the bosonic string is $d = 26$ while the superstring lives in a 10 dimensional target space. This set-up is clarified in Fig. (2.1) below. Thus the heterotic string construction provides us with the following degrees of freedom:

$$\begin{array}{lll} X_L^\mu\,, & X_L^i\,, & X_L^I\,, \\ X_R^\mu\,, & X_R^i\,, & \\ \Psi_R^\mu\,, & \Psi_R^i\,. & \end{array}$$

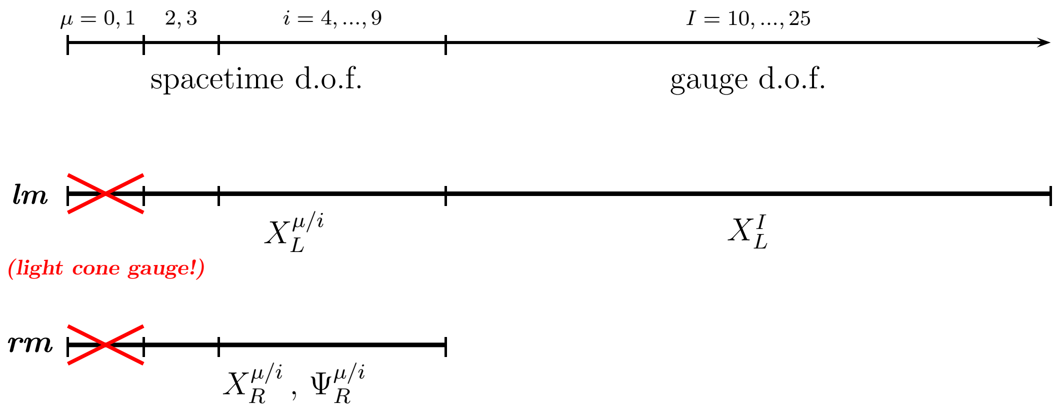


Figure 2.1: Left and right moving degrees of freedom of the heterotic string.

They are mappings from the string worldsheet, parameterised by τ, σ, to the ten dimensional target space and should therefore be properly denoted $X(\tau, \sigma)$, $\Psi(\tau, \sigma)$. The lower indices R/L for the bosonic (X) and the fermionic (Ψ) degrees of freedom indicate right and left moving respectively. The upper indices μ/i correspond to the first ten degrees of freedom while I describes the left moving bosonic degrees of freedom which have no right moving counterpart. In light cone gauge the index sets for the dynamical degrees of freedom are $\mu = 2, 3$; $i = 4, ..., 9$; $I = 10, ...25$. The splitting into the indices μ/i is due to our aim to further compactify six of these eight degrees of freedom. In the heterotic string construction the 16 left moving degrees of freedom X_L^I are compactified on a 16 dimensional torus T^{16}. This toroidal compactification constrains the corresponding momenta p^I to lie on a 16 dimensional integer, even and selfdual lattice. There are only two such lattices, the spin(32)/$\mathbb{Z}_2$ and the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice.[1]
The appearance of some additional massless fields (gauge fields) in the spectrum after this compactification is a stringy phenomenon due to possible windings around the torus. By the so called Frenkel-Kač-Segal construction the generators of a gauge group can be explicitly constructed from these massless fields. This is the way how in the heterotic string construction a gauge group arises automatically. In this work we will only consider the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, with $\mathrm{E}_8 \times \mathrm{E}_8$ being the gauge group arising in this manner.

After solving the string equations of motion and quantizing the appropriate field expansions the massless spectrum is derived by tensoring together the massless right and massless left moving states

$$|\text{massless right moving state}\rangle \otimes |\text{massless left moving state}\rangle \; .$$

From the discussion of the superstring we know that for the right moving superstring

[1]For details on the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice see [9, 11].

states there are two different types of massless ground states $b^{\mu/i}_{-\frac{1}{2}}|0\rangle_R$ (R) and $d^{\mu/i}_0|0\rangle_R$ (NS), leading to two different sectors of states, namely the Ramond sector (R) and the Neveu-Schwarz sector (NS). These two types of states constitute $\mathbf{8_v}$ and $\mathbf{8_s}$ representations (eight dimensional vector and spinor representation) of SO(8), the little group for massless states in $d = 10$. For our further considerations we denote the ground states by their SO(8) weights $q_{v/s}$ and achieve

$$\begin{aligned} b^{\mu/i}_{-\frac{1}{2}}|0\rangle_R &= |q_v\rangle_R = |(\underline{\pm 1, 0, 0, 0})\rangle_R = (\text{NS}) \ , \\ d^{\mu/i}_0|0\rangle_R &= |q_s\rangle_R = |(\underline{\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}})\rangle_R = (\text{R}) \ . \end{aligned}$$

Here the underlining means all possible permutations of the underlined values. Therefore the eight states of $|(\underline{\pm 1, 0, 0, 0})\rangle_R$ constitute the $\mathbf{8_v}$ vector representation of the Neveu-Schwarz sector and those of $|(\underline{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}})\rangle_R$, with an even number of plus signs, the spinor representation $\mathbf{8_s}$ of the Ramond sector. In this notation it is obvious how the states of the heterotic string arise as superpartners of each other due to the superstring included in the construction. For the left moving bosonic string there are the massless states $\tilde{\alpha}^{\mu/i}_{-1}|0\rangle_L$, $\tilde{\alpha}^I_{-1}|0\rangle_L$ and $|p\rangle_L$. Therefore the full *massless spectrum of the 10 dimensional heterotic string* is given by

$$\begin{array}{cc} \text{SUGRA-multiplet} & |q_{v/s}\rangle_R \otimes \tilde{\alpha}^{\mu/j}_{-1}|0\rangle_L \ , \\ \text{16 uncharged gauge bosons/gauginos} & |q_{v/s}\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L \ , \\ 240 + 240 \text{ charged gauge bosons/gauginos} & |q_{v/s}\rangle_R \otimes |p\rangle_L \ . \end{array}$$

The states of the uncharged gauge bosons correspond to the 16 generators of the Cartan subalgebra of $E_8 \times E_8$, those of the charged gauge bosons to the $240 + 240$ roots p (the momentum p of the state must be an $E_8 \times E_8$ root).

2.1.2 Notes on compactification

By compactifying 16 of the 26 left moving bosonic degrees of freedom of the heterotic string on a torus T^{16} we arrived at an equal dimensionality of right and left moving spacetime degrees of freedom. But the heterotic string still lives in a ten dimensional Minkowskian spacetime $\mathbb{M}^{10}$. To finally end up with our observable four dimensional world $\mathbb{M}^4$ there is the need to compactify six further dimensions on some compact internal manifold X:

$$\mathbb{M}^{10} \longrightarrow \mathbb{M}^4 \times X \ .$$

The tangent space group SO(9,1) of the ten dimensional space now breaks as follows

$$\text{SO(9,1)} \longrightarrow \text{SO(3,1)} \times \text{SO(6)} \ ,$$

where SO(6) $\backsimeq$ SU(4). Consideration of the $d = 10, N = 1$ SUSY parameter ε with respect to this breaking will now motivate the introduction of an additional discrete symmetry and thus lead to the orbifold.

The left-handed Majorana-Weyl spinor ε, a spinor representation **16**, decomposes as

$$\mathbf{16} \longrightarrow (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}) \ ,$$

where the $(\mathbf{2}, \mathbf{1})$ denotes a left-handed Majorana-Weyl spinor of SU(2) $\times$ SU(2) $\simeq$ SO(3,1). Thus it yields four Majorana-Weyl spinors of each chirality, corresponding to the **4** under the internal SU(4). Hence for some general compactification scheme (e.g. a simple toroidal compactification) the resulting supersymmetry in $d = 4$ could be $N = 4$ SUSY. For phenomenological reasons we would prefer $d = 4, N = 1$ SUSY. To obtain this $N = 1$ SUSY there must be exactly one covariantly constant spinor. Therefore the holonomy group of the compactification manifold must leave exactly one component of the **4** invariant. The subgroup of SU(4) which does this job is SU(3) and therefore finally we must demand the further breaking

$$\mathrm{SO(9,1)} \longrightarrow \mathrm{SO(3,1)} \times \mathrm{SO(6)} \longrightarrow \mathrm{SO(3,1)} \times \mathrm{SU(3)} \times \mathrm{U(1)}$$

under which the **16** finally decomposes as

$$\mathbf{16} \longrightarrow (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}) \longrightarrow (\mathbf{2}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}) \ .$$

This is the reason for taking a discrete subgroup $\Gamma \in$ SU(3) of the holonomy group and demanding invariance of the theory under it. Then only the $(\mathbf{2}, \mathbf{1})$ representation survives this projection and the corresponding generators are those of a $d = 4, N = 1$ SUSY. We will proceed in this manner in defining the orbifold in the next section. The above arguments are also nicely explained in [8, 12].

2.2 Geometric orbifolds

In this section we consider the space on which we will compactify six of the ten heterotic string coordinates without referring to the heterotic string construction.

Definition 1: An *orbifold* O is defined to be the quotient space of a manifold M and a discrete group Γ, called *point group*, which acts on M

$$O \equiv M/\Gamma \ .$$

The simplest example of an one dimensional orbifold is the sphere S^1, divided by the order two cyclic group, $S^1/\mathbb{Z}_2$. It is depicted in Fig. (2.2). We allow arbitrary radii R and therefore describe the sphere by the set of parameters $[-\pi R, \pi R]$. Dividing out the group $\mathbb{Z}_2$ means to identify points in this set of parameters which differ by a sign. By looking at Fig. (2.2) we notice, that there are 2 fixed points under the action of the discrete group. They are transformed into itself by acting with the group elements on them. These are the points $\{0, \pi R\}$.

In this work we will take the manifold M as a six dimensional torus T^6, which can be described as $T^6 = \mathbb{R}^6/\Lambda$, where $\Lambda = \{n_\alpha e_\alpha | e_\alpha \in \mathbb{R}^6, n_\alpha \in \mathbb{Z}\}$ is some lattice defining the torus. The vectors $\{e_\alpha\}$ with $\alpha = 1, ..., 6$ are the basis vectors of the lattice. Then the point group transformations must act as isometries of the torus lattice. By defining the space group we are led to an equivalent thinking of the orbifold.

Definition 2: The *space group* S is defined to be the semi-direct product[2] of the point group Γ and the translation group Λ

$$S \equiv \Gamma \ltimes \Lambda \ .$$

By use of the space group the orbifold O can be written as

$$O = \mathbb{R}^6/S \ ,$$

which in fact is nothing else than the previous definition. Due to the properties of the semi-direct product an element $g \in S$ can be written as $g = (\theta, n_\alpha e_\alpha)$, where $\theta \in \Gamma$ and $n_\alpha e_\alpha \in \Lambda$. The product of two elements $g_1, g_2 \in S$ then can be calculated as

$$g_1\, g_2 = (\theta_1, n_\alpha^{(1)} e_\alpha)\,(\theta_2, n_\alpha^{(2)} e_\alpha) = (\theta_1\theta_2, n_\alpha^{(2)}\theta_1 e_\alpha + n_\alpha^{(1)} e_\alpha) \ . \tag{2.1}$$

From the definition above it should be clear that the six dimensional orbifold is described by the point group Γ and the six basis vectors of the torus lattice e_α. We then denote a

[2] To have the space group formulae in accordance with the literature, we define the semi-direct product for the space group as in Eq. (2.1). This definition is slightly different from the one in App. B.1 which is used to investigate discrete symmetries in chapter 4. Both definitions are equivalent.

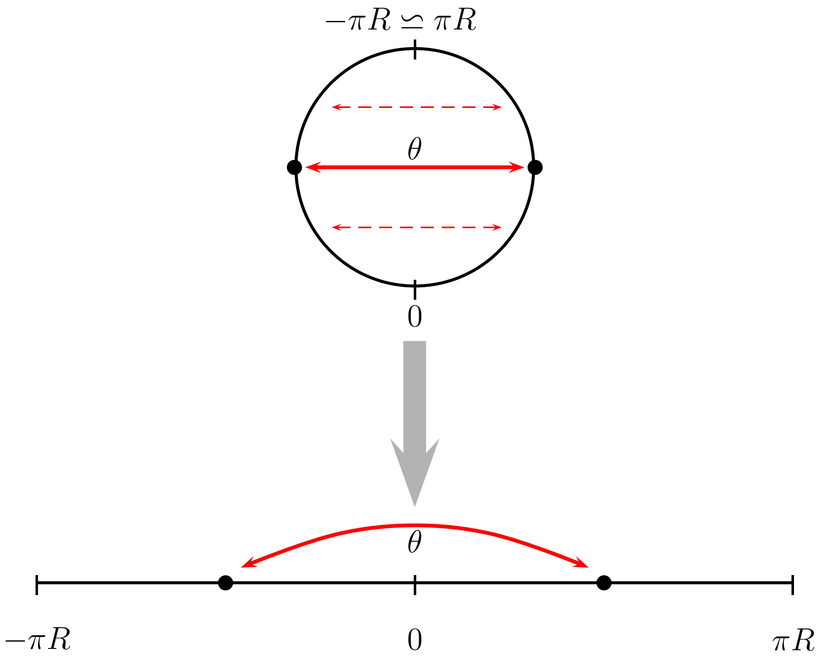


Figure 2.2: Deriving the simplest orbifold $S^1/\mathbb{Z}_2$ from the sphere S^1 by identifying points under the action of θ.

general vector of the orbifold space as X^i, with $i = 1, ..., 6$. Later the heterotic string degrees of freedom $X^i \equiv X^i_R + X^i_L$, with $i = 4, ..., 9$, will be taken as these orbifold coordinates.

There are some general restrictions on the choice of possible point groups. In Sec. 2.1.2 we learned that the point group, acting on the six bosonic heterotic string coordinates, must be taken as a subgroup of the holonomy group of the compactification manifold. In our case this means $\Gamma \subset \mathrm{SO}(6) \simeq \mathrm{SU}(4)$. If we consider only the case of Abelian point groups (and we will do so in this work), an element $\theta \in \mathrm{SO}(6)$ may be written as $\theta = \exp[2\pi i(v^1 J_{12} + v^2 J_{34} + v^3 J_{56})]$ with J_{12}, J_{34}, J_{56} the elements of the Cartan subalgebra of SO(6). To end up with a $d = 4$, $N = 1$ supersymmetric theory we further had to demand

$$\Gamma \subset \mathrm{SU}(3) \ ,$$

leading to the constraint on the transformation parameters v^a

$$v^1 \pm v^2 \pm v^3 = 0 \tag{2.2}$$

for some choice of signs. Later we will choose the v^a such that $v^1 + v^2 + v^3 = 0$, in agreement with the literature.

To define the action of this SU(3) subgroup on the orbifold coordinates we build three complex coordinates Z^a, with $a \in \{1,2,3\}$, out of the six real coordinates X^i:

$$\begin{aligned} Z^a &= \frac{1}{\sqrt{2}}(X^{2a+2} + iX^{2a+3}) \qquad a \in \{1,2,3\} \\ Z^{\bar{a}} &= \frac{1}{\sqrt{2}}(X^{2a+2} - iX^{2a+3}) \ . \end{aligned}$$

For the vectors e_1, e_2 defining the two dimensional torus lattice, this means that we build $e_1 + ie_2$ and consider a one dimensional complex space instead of the two dimensional real one. On the complex coordinates the above rotation just acts as a multiplication by a phase

$$\theta Z^a \mapsto e^{2\pi i v^a} Z^a \ .$$

The parameters v^a correspond to the rotation angles in the three tori,[3] the vector $v \equiv (v^1, v^2, v^3)$ is called *twist vector* and the transformation to which it belongs *spacetime twist*.
For the identification of points on the orbifold under the point group to be well defined the point group must be an automorphism of the underlying torus lattice. The requirements that the point group acts in this crystallographic manner and of $N = 1$ SUSY lead to restrictions on the possible point groups. Thus the only allowed point groups are $\mathbb{Z}_N$ with $N = 3, 4, 6, 7, 8, 12$ and $\mathbb{Z}_N \times \mathbb{Z}_M$ with M a multiple of $N = 2, 3, 4, 6$, defined as follows:

$\mathbb{Z}_\mathbf{N}$: The cyclic group of order N is defined to be

$$\mathbb{Z}_N = \{\theta^k \,|\, k = 0, ..., N-1\}$$

with $\theta^N = \mathbb{1}$. Therefore the element θ can be seen as a rotation by $2\pi/N$.

$\mathbb{Z}_\mathbf{N} \times \mathbb{Z}_\mathbf{M}$: The direct product of an order N and an order M cyclic group is described by two independently acting twists θ and ω

$$\mathbb{Z}_N \times \mathbb{Z}_M = \{\theta^k \circ \omega^l \,|\, k = 0, ..., N-1; l = 0, ..., M-1\} \ .$$

In some cases there is more than one possible twist vector. All possibilities are tabulated e.g. in [2, 7].
Due to the existence of fixed points under its action the space group is said to act non freely. The underlying torus is flat everywhere, only at the fixed points the curvature does not vanish. Thus there is a curvature singularity and an orbifold is not a manifold.[4]

[3] Among the heterotic string states there are the ones $b^i_{-\frac{1}{2}}|0\rangle_R \otimes \tilde{\alpha}^j_{-1}|0\rangle_L$ with $i, j = 1, ..., 6$ corresponding to the six compactified dimensions. They are called moduli. Assigning vevs to these moduli corresponds to fixing free parameters of the compactified space, like radii and angles.

[4] By parallel transporting vectors around different closed curves on the orbifold (ordinary closed and closed under point group action) this curvature singularity can be seen from the non-triviality of the holonomy group [13].

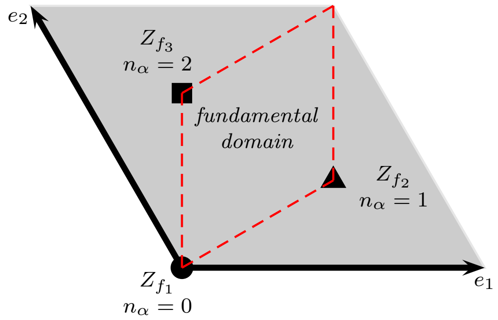


Figure 2.3: The three fixed points of the orbifold $T^2/\mathbb{Z}_3$.

It is instructive to study the example of a two dimensional orbifold $T^2/\mathbb{Z}_3$. A torus lattice on which the $\mathbb{Z}_3$ point group acts crystallographically is the SU(3) root lattice, because the two simple roots of SU(3) include an angle of 120°. We normalise the two basis vectors of the SU(3) root lattice such that $|e_\alpha|^2 = 2$. It is not difficult to show (see [9]) that there are the following 3 fixed points under the action of the space group in the complex plane of the torus T^2:

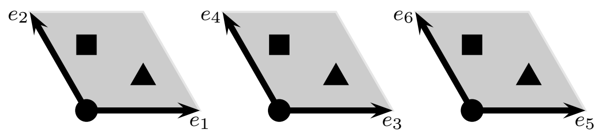
$$\begin{aligned} Z_{f_1} &= 0 \\ Z_{f_2} &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{6}} \\ Z_{f_3} &= i\sqrt{\frac{2}{3}} \; . \end{aligned}$$

The situation is depicted in Fig. (2.3). There is a one-to-one correspondence between a fixed point and the space group element under whose action it is fixed. This element is called the *constructing element* and denoted $\hat{g}$. In other words for each fixed point Z_f there is a $\hat{g} = (\theta^k \omega^l, n_\alpha e_\alpha) \in S$, such that

$$Z_f = \hat{g} Z_f = \theta^k \omega^l Z_f + n_\alpha e_\alpha \; .$$

The appropriate space group elements for the $\mathbb{Z}_3$ fixed points are

$$\begin{aligned} Z_{f_1} &\Longleftrightarrow (\mathbb{1}, 0) \\ Z_{f_2} &\Longleftrightarrow (\theta, e_1) \\ Z_{f_3} &\Longleftrightarrow (\theta, e_1 + e_2) \; . \end{aligned}$$



27 fixed points

$$\theta = \mathrm{diag}\left(e^{\frac{2\pi i}{3}} \quad , \quad e^{\frac{2\pi i}{3}} \quad , \quad e^{\frac{-4\pi i}{3}} \right)$$

Figure 2.4: The three tori of the six dimensional $\mathbb{Z}_3$ orbifold with its 27 fixed points.

For later reasons we label the three fixed points by a number $n_\alpha = 0, 1, 2$ which gives the number of torus shifts in the translational part of the constructing element (see App. A.1).

The dashed rhomboid in Fig. (2.3) is called *fundamental domain.* Under the action of the space group each point of the orbifold is equivalent to a point in the fundamental domain. Some edges of the fundamental domain are identified under the space group action in such a way that the orbifold is finally folded to a ravioli-like object.
After this preparatory work it is now straightforward to generalise the two dimensional $\mathbb{Z}_3$ orbifold to six dimensions, the case we are interested in. To define a six dimensional $\mathbb{Z}_3$ orbifold we take the six torus T^6 as the product of three two dimensional tori T^2 and let the twist θ act on each of them as a 120°-rotation. Acting on complex coordinates the twist reads $\theta = \mathrm{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{-4\pi i}{3}})$ and can therefore be represented by the twist vector

$$v = \frac{1}{3}(1, 1, -2) \ .$$

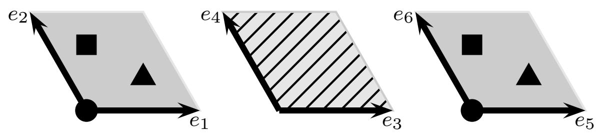
The choice of the -2 instead of 1 in the last entry is due to the $N = 1$ SUSY condition Eq. (2.2). There are now 27 fixed points of the six dimensional $\mathbb{Z}_3$ orbifold, corresponding to the $3 \times 3 \times 3$ possible combinations of the fixed points in the three complex planes in Fig. (2.4).

2.2.1 The $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold

Now we are in position to define the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, the object to be considered in this work. We choose two independently acting $\mathbb{Z}_3$ groups as point group, such that each $\mathbb{Z}_3 \times \mathbb{Z}_3$ twist must be described by two independent twists θ and ω

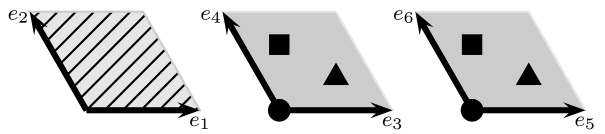
$$\begin{aligned} \mathbb{Z}_3 \times \mathbb{Z}_3 &\equiv \{\theta^k \omega^l | \, \theta \in \mathbb{Z}_3 \, , \omega \in \mathbb{Z}_3 \, ; k, l = 0, 1, 2\} \\ &= \{\mathbb{1}, \theta, \omega, \theta^2, \omega^2, \theta\omega^2, \theta^2\omega, \theta\omega, \theta^2\omega^2\} \ . \end{aligned}$$

Thus in total the point group consists of 9 elements.



9 fixed tori

$$\theta \;=\; \mathrm{diag}\Big(e^{\frac{2\pi i}{3}} \quad , \quad 1 \quad , \quad e^{\frac{-2\pi i}{3}} \Big)$$



9 fixed tori

$$\omega \;=\; \mathrm{diag}\Big(1 \quad , \quad e^{\frac{2\pi i}{3}} \quad , \quad e^{\frac{-2\pi i}{3}} \Big)$$

Figure 2.5: The three tori of the six dimensional $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with its 9 fixed tori for each of the two independently acting twists θ, ω.

The action of the two independent twists $\theta = e^{2\pi i v_1}$ and $\omega = e^{2\pi i v_2}$ is now specified to be

$$\begin{aligned} v_1 &= \frac{1}{3}(1, 0, -1) \ , \\ v_2 &= \frac{1}{3}(0, 1, -1) \ , \end{aligned}$$

such that the first twist does not rotate the second torus and the second twist leaves the first torus unrotated, as can be seen from Fig. (2.5). For this reason there are no proper fixed points under the action of these two twists but in each case 9 two dimensional fixed tori. The twist $\theta\omega$ corresponds to the twist vector $v_1 + v_2 = \frac{1}{3}(1, 1, -2)$ which is just the twist vector of the $\mathbb{Z}_3$ orbifold we considered before. Under $\theta\omega$ and $\theta^2\omega^2$ we therefore really have 27 fixed points. This is in contrast to the simpler case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold where there are no proper fixed points under the action of any space group element but only fixed tori [9].

2.3 Orbifold meets heterotic string

To construct a four dimensional theory from the ten dimensional heterotic string theory, we take the six bosonic heterotic string degrees of freedom $X^i = X^i_R + X^i_L$ to be coordinates of an orbifold space O. The target space of the heterotic string is therefore divided into two factors, as indicated before,

$$\mathbb{M}^{10} \longrightarrow \mathbb{M}^4 \times O \ .$$

From the real heterotic string degrees of freedom we build complex ones, as before in the case of the geometric orbifold

$$\begin{aligned} X^i = X^i_R + X^i_L \ &\Longrightarrow \ Z^a \, , \, Z^{\bar{a}} \qquad i = 2, ...9 \ \ a = 0, ..., 3 \\ \Psi^i_R \ &\Longrightarrow \ \psi^a_R, \psi^{\bar{a}}_R \ , \end{aligned}$$

where $Z^a \equiv \frac{1}{\sqrt{2}} \left(X^{2a+2} + iX^{2a+3} \right)$ and $\psi^a_R \equiv \frac{1}{\sqrt{2}} \left(\Psi^{2a+2}_R + i\Psi^{2a+3}_R \right)$. The three degrees of freedom Z^a with $a = 1, 2, 3$ are the coordinates of the orbifold space while Z^0 corresponds to the Minkowski space $\mathbb{M}^4$ and is therefore not affected by the action of the space group. The identification of points in spacetime leads to *projection conditions* for the states of the heterotic string. In a consistent theory on an orbifold only space group invariant states are allowed. States which are not invariant are projected out and this projection induces a gauge group breaking and new matter representations. To explain this mechanism we have to investigate the action of the space group on the heterotic string degrees of freedom and the projection conditions. We will find out that there is a new kind of closed string on the orbifold, the so called *twisted string*. In the following section some general features of orbifolded heterotic string theories are derived and stated but we will often switch to the particular case of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.

2.3.1 Space group action on the states

By considering the geometric orbifold we derived the action of the spacetime twist θ on the complex coordinates with the help of the SO(6) Cartan generators. Now the bosonic string degrees of freedom Z^a play the role of these coordinates and therefore the twist must act on them in the same manner. The space group of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold thus acts on the Z^a as

$$(\theta^k \omega^l, n_\alpha e_\alpha) Z^a = e^{2\pi i (k v_1 + l v_2)} Z^a + n_\alpha e^a_\alpha \ .$$

To assure worldsheet SUSY and hence the invariance of the worldsheet supercurrent under the twist, its action on the right moving fermionic degrees of freedom must be identical

$$(\theta^k \omega^l, n_\alpha e_\alpha) \psi^a_R = e^{2\pi i (k v_1 + l v_2)} \psi^a_R \ .$$

It is of course an equivalent description to take three Z^a with $a = 1, 2, 3$ and $v = (v^1, v^2, v^3)$ or four Z^a with $a = 0, 1, 2, 3$ and the twist $v = (0, v^1, v^2, v^3)$. Therefore in some situations we will write the twist as a four and in some as a three dimensional vector.

There is the possibility to let the point group act asymmetrically on the right and left moving degrees of freedom [14, 15]. We leave this possibility aside and consider only *symmetric orbifolds*.
After specifying the space group action on the six bosonic and six fermionic compactified degrees of freedom, the question arises if there is also an associated transformation on the sixteen gauge degrees of freedom X_L^I, where $I = 10, ..., 25$. To ensure invariance of the closed string one-loop amplitude under modular transformations,[5] the spacetime twist must be embedded into the gauge degrees of freedom. Otherwise the modular invariance conditions, which will be explained later in Sec. 3.3.1, cannot be fulfilled. The gauge degrees of freedom of the heterotic string correspond to the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice and Cartan generators. Thus in general the space group is embedded into the automorphisms of $\mathrm{E}_8 \times \mathrm{E}_8$

$$S \longrightarrow G \subset \mathrm{Aut}(\mathrm{E}_8 \times \mathrm{E}_8) \ .$$

In the case of inner automorphisms (and we will only consider this case) this embedding can be realised as a *shift embedding*.[6] That means that to each space group element there is a corresponding element

$$(\theta^k \omega^l, n_\alpha e_\alpha) \longmapsto (\mathbb{1}; kV_1 + lV_2 + n_\alpha A_\alpha) \ ,$$

which acts on the gauge degrees of freedom just as a shift $X_L^I \longmapsto X_L^I + kV_1^I + lV_2^I + n_\alpha A_\alpha^I$. Here the V_i are called *shift vectors* and the A_α *Wilson lines*. Wilson lines are background fields on the torus. They induce an additional transformation phase on the gauge degrees of freedom, if these are transformed around non-contractible loops on the torus [16]. The first entry on the right hand side corresponds to rotations of the root lattice and in this work always equals the identity. A proper derivation of this shift embedding, making use of a theorem by Kač, can be found in [17].
The derived transformation properties of the heterotic string degrees of freedom give rise to a transformation of the states in the spectrum. In the right moving sector the Abelian point group must be contained in the Cartan subalgebra of $\mathrm{SO}(6) \subset \mathrm{SO}(8)$. Thus, with the Cartan generators H_a, the twist can be written as $\theta = \exp(2\pi i(kv_1^a + lv_2^a)H_a)$. Its action on the right moving weight states is then given by

$$|q\rangle_R \longmapsto e^{2\pi i(kv_1+lv_2)\cdot q}|q\rangle_R \ .$$

In a similar way one derives the transformation of the left moving states $|p\rangle_L$. The left moving oscillators[7] $\tilde{\beta}_n^a$ correspond to the compactified dimensions and must therefore transform analogously to the Z^a. Finally the transformation behaviour, necessary for our further

[5]For details on modular invariance see Sec. 3.3.

[6]An inner automorphism on a Lie algebra g is an automorphism which acts on the generators as

$$T^a \mapsto R T^a R^{-1} \ ,$$

with $T^a \in g$ and $R \in G$, the corresponding Lie group. An automorphism which is not of this form is called an *outer automorphism*. E.g. complex conjugation is an outer automorphism.

[7]For the definition of the complexified oscillators see App. A.2.

considerations, can be summarised as

$$\begin{array}{rcl} |q\rangle_R & \longmapsto & e^{2\pi i(kv_1+lv_2)\cdot q}|q\rangle_R \\ |p\rangle_L & \longmapsto & e^{-2\pi i(kV_1+lV_2+n_\alpha A_\alpha)\cdot p}|p\rangle_L \\ \tilde{\beta}^a_n & \longmapsto & e^{2\pi i(kv_1^a+lv_2^a)}\tilde{\beta}^a_n \ . \end{array}$$

A general state of the form $|q\rangle_R \otimes \tilde{\beta}^a_n|p\rangle_L$ therefore transforms as

$$|q\rangle_R \otimes \tilde{\beta}^a_n|p\rangle_L \mapsto e^{2\pi i[(kv_1+lv_2)\cdot q-(kV_1+lV_2+n_\alpha A_\alpha)\cdot p\,\pm\,(kv_1^a+lv_2^a)]}|q\rangle_R \otimes \tilde{\beta}^a_n|p\rangle_L \ , \qquad (2.3)$$

where the $\pm$ takes account of oscillators, possibly being complex conjugated. In the case that there are more than one oscillator in the left moving part of the state, their phases add up. The minus sign in the phase for the left moving states is due to the fact that the metric of the even self-dual lattice of the combined left-right momenta has Lorentzian signature and therefore distinguishes scalar products of q and p by a sign [14, 7].

2.3.2 Untwisted and twisted sectors

An entirely new feature of a string theory on an orbifold is the existence of twisted strings. For the degrees of freedom of the heterotic string compactified on a torus there are the boundary conditions

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + n_\alpha e_\alpha \ .$$

The corresponding strings are closed in the usual sense and closed up to torus shifts respectively. If the underlying space is an orbifold space there will be the possibility to close the strings up to space group transformations. The most general boundary condition therefore reads (with the constructing element $\hat{g} = (\theta^k\omega^l, n_\alpha e_\alpha) \in S$)

$$Z(\tau, \sigma + 2\pi) = \hat{g}Z(\tau, \sigma) = \theta^k\omega^l Z(\tau, \sigma) + n_\alpha e_\alpha \ .$$

Thus there are again the two kinds of strings we already had before. But there are as well strings which are only closed under the action of the twist $\theta^k\omega^l$, the so called *twisted strings*. They are entirely new objects in the theory. The situation is depicted in Fig. (2.6).

Strings which are closed up to the action of a nontrivial $\hat{g} \in S$ are said to lie in the *twisted sector* $T_{\hat{g}}$. In the case of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold without Wilson lines there are 9 twisted sectors due to the point group consisting of nine elements. By twisted sectors we will always mean these twisted sectors due to the point group elements. Each of these sectors is further divided into subsectors due to the translational part of the space group. The boundary condition corresponding to the identity element is the trivial one and the appropriate sector is denoted *untwisted sector* or *bulk*. In the following we will write the eight twisted sectors as $T_{(k,l)}$ $(k, l = 0, 1, 2)$ and the untwisted sector as $T_{(0,0)} \equiv U$.
We also consider orbifolds with Wilson lines (nonvanishing A_α for some α) and these background fields will discriminate between the various fixed points of the eight twisted sectors due to the additional phase in Eq. (2.3). Therefore they split those sectors up into

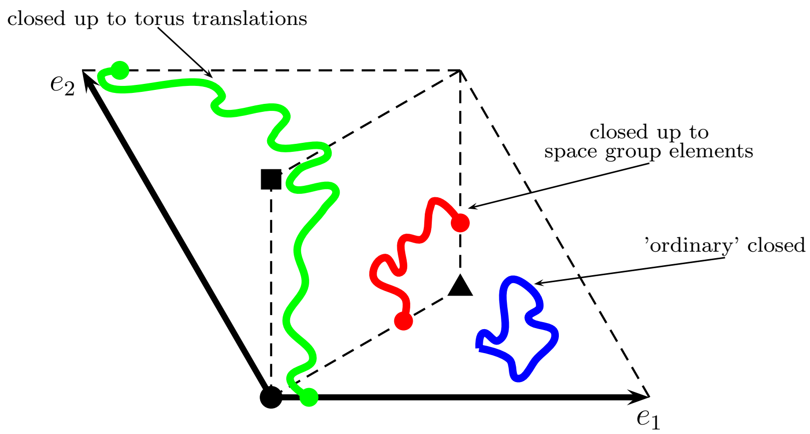


Figure 2.6: There are three kinds of possible closed strings on an orbifold, the 'ordinary' closed strings, strings which are closed on the torus and such which are only closed up to point group elements.

subsectors. In the next sections we will see how at fixed points, corresponding to different space group elements, different gauge group representations will arise due to the fact that the Wilson lines do not vanish.
States in the untwisted sector U are not localised at all. Six of the eight twisted sectors will provide us with states which are localised in two of the three tori but bulk states in the remaining one. These states are localised at two dimensional fixed tori. It is a very interesting feature in the $\mathbb{Z}_3 \times \mathbb{Z}_3$-construction that there are two twisted sectors $T_{(1,1)}$ and $T_{(2,2)}$ which contain 27 proper fixed points. Thus the states of $T_{(1,1)}$ and $T_{(2,2)}$ are completely localised. Comparing with the fixed point structure of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we realise that this is a new property in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case as explained in Sec. (2.2.1). In $\mathbb{Z}_2 \times \mathbb{Z}_2$ there are no completely localised states [9], in $\mathbb{Z}_3 \times \mathbb{Z}_3$ there are.

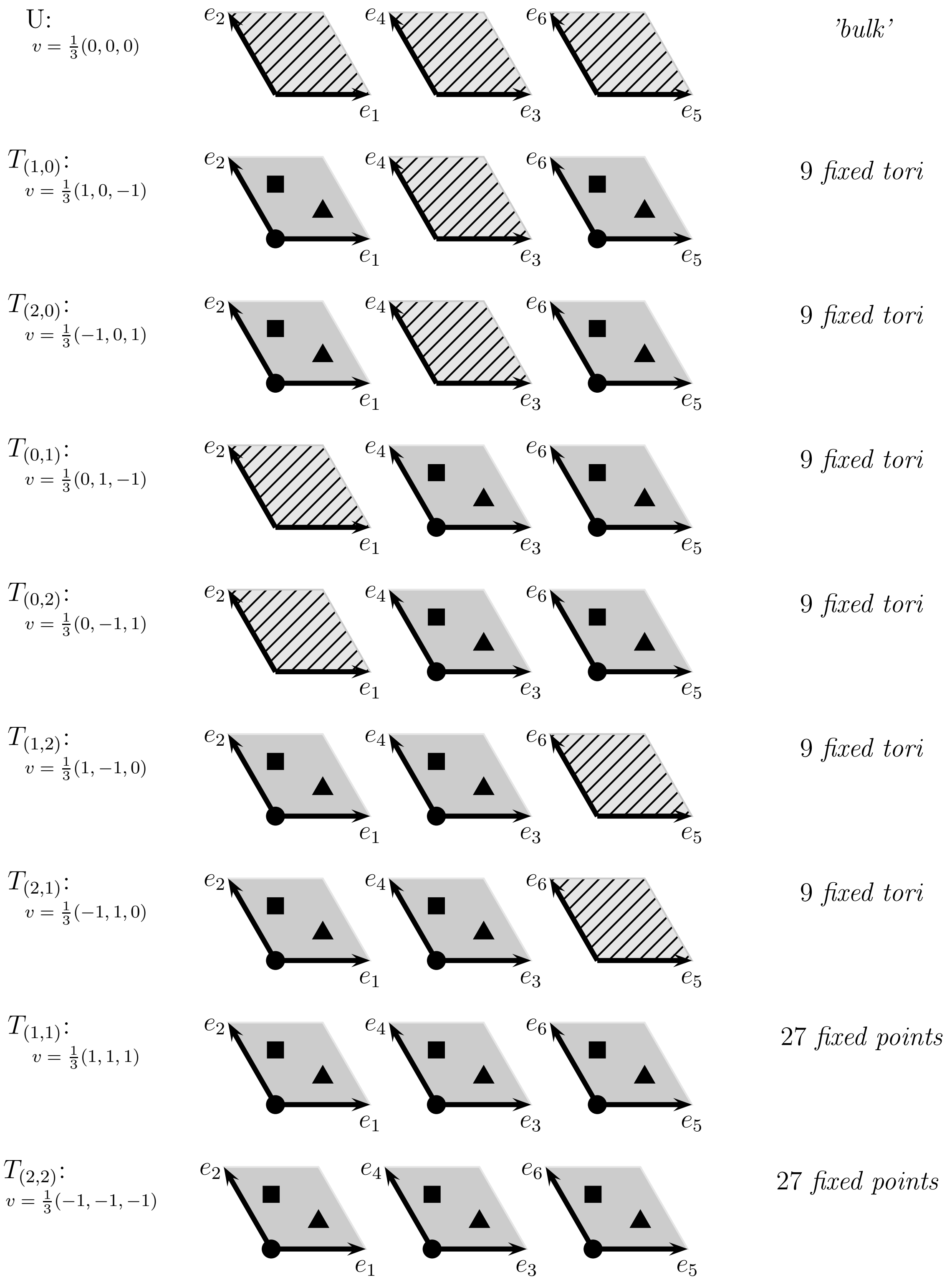


Figure 2.7: The fixed point structure of the nine different sectors of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.

2.3.3 Projecting out states

After the discussion of the geometric and the string theoretical background of orbifold compactifications, we will now calculate the spectrum of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold. In this section we will describe how to calculate the spectrum on orbifolds more or less generally before doing the calculation then explicitly for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ standard embedding case. Performing this calculation we will precisely derive the projection conditions for the states on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.
As mentioned before, the main point is that only states which are invariant under the action of the space group are compatible with the orbifold construction and are therefore allowed on the orbifold. All other states are forbidden and projected out. Thus the identification of points of the underlying spacetime leads to projection conditions of the states in the spectrum. These projection conditions are constraints on the transformation phases of the states under elements of the space group. If we denote the transformation phase of the right moving state by γ, that of the left moving one by $\tilde{\gamma}$, a generic state will transform as

$$|q\rangle_R \otimes \tilde{\alpha}|p\rangle_L \longmapsto (\gamma|q\rangle_R) \otimes (\tilde{\gamma}\,\tilde{\alpha}|p\rangle_L) = \gamma\,\tilde{\gamma}\,(|q\rangle_R \otimes \tilde{\alpha}|p\rangle_L)\ .$$

Obviously only states whose transformation phases satisfy

$$\gamma\tilde{\gamma} = 1$$

will survive the orbifold projection and enter the orbifold spectrum. In the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case $\gamma^3 = 1 = \tilde{\gamma}^3$ due to the order of the point group. In the first subsection we will now consider only the untwisted sector and explain how the $d = 4$, $N = 1$ SUGRA multiplet, the gauge bosons of the unbroken group and charged matter representations arise from the heterotic string states. Then we will turn to the twisted sectors where we have to solve a changed mass equation to find the massless modes.

Untwisted sector

The spectrum of the heterotic string consists of the $d = 10$, $N = 1$ SUGRA multiplet, the 16 uncharged gauge bosons and the 240+240 charged gauged bosons of $\mathrm{E}_8\times\mathrm{E}_8$. Considering the theory on the orbifold now means to forbid all states which are not invariant under the space group. The different transformation behaviour of these states leads to interesting consequences in the spectrum of the four dimensional orbifolded theory.

SUGRA:

$$\begin{array}{ccc} \underline{|(\pm1,0,0,0)\rangle_R \otimes \tilde{\beta}^{\mu/j}_{-1}|0\rangle_L} & \longmapsto & |(\pm1,0,0,0)\rangle_R \otimes \tilde{\beta}^{a}_{-1}|0\rangle_L \\ \text{d=10, N=1 SUGRA} & \longmapsto & \text{d=4, N=1 SUGRA (and singlets)}\ . \end{array}$$

Here the index μ/j on the left hand side takes the values $2, ..., 9$ (we splitted the spacetime index of the ten dimensional heterotic string into μ/j, corresponding to the non-compactified and compactified degrees of freedom respectively). On the right hand side $\mu = 0, \overline{0}$. The possible right moving states on the right hand side are only $|(\pm 1, 0, 0, 0)\rangle_R$ because they must correspond to right moving oscillators in the Minkowski spacetime degrees of freedom ($\mu = 0, \overline{0}$). Therefore the right moving state is invariant under the space group (which acts trivially in the non-compactified dimensions) and the only matching combination with an invariant left moving state is of the above form. The corresponding states arising from the Ramond sector of the right moving superstring have right moving states $| \pm (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rangle_R$. The other combinations (corresponding to $|(0, \underline{\pm 1, 0, 0})\rangle_R \otimes \tilde{\beta}^a_{-1}|0\rangle_L$) give rise to the gauge singlets, as discussed before. They are the already mentioned moduli, which determine the radii and the complex structure of the underlying space.

16 uncharged gauge bosons:

$$
\begin{array}{rcl}
|\underline{(\pm 1, 0, 0, 0)}\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L & \longmapsto & |(\pm 1, 0, 0, 0)\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L \\
|\underline{(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})}\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L & \longmapsto & | \pm (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L \; . \\
\text{16 uncharged gauge bosons/gauginos in d=10} & \longmapsto & \text{16 uncharged gauge bosons/gauginos in } d = 4
\end{array}
$$

The left moving states of the 16 uncharged gauge bosons transform with a trivial phase. Therefore the only appropriate right moving states are the two given above. They again correspond to the oscillators in the uncompactified dimensions and the combined states are 16 uncharged gauge bosons of the gauge group in $d = 4$. This is the reason why in orbifold compactification of this type the rank of the gauge group can never be reduced.[8]

240+240 charged gauge bosons:

$$|q\rangle_R \otimes |p\rangle_L \quad \longmapsto \quad \text{charged gauge bosons and charged matter of the unbroken group}$$

By considering the transformation properties of the left moving states $|p\rangle_L$ we notice that there are left moving states which transform with a trivial phase $\tilde{\gamma} = 1$ and thus correspond to roots p which stay invariant under space group transformations. Therefore the corresponding right moving states also must transform with a trivial phase (thus they correspond to the weights $q = (\pm 1, 0, 0, 0), q = \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$). These states are charged gauge bosons and gauginos of the unbroken gauge group in $d = 4$. From these space group invariant roots we can construct the gauge group in $d = 4$ [9]. Demanding invariance under the space group therefore breaks the gauge group $E_8 \times E_8$ in $d = 10$ to some subgroup, the *unbroken gauge group* in $d = 4$.
But there are also left moving states which transform with a nontrivial phase $\tilde{\gamma} \neq 1$ and it is only the tensor product of the left and the right moving state which

[8] Nevertheless it is possible to reduce the rank of the gauge group in orbifold compactifications. Therefore the spacetime twist has to be realised as a rotation in the gauge degrees of freedom and combined with a Wilson line. If the Wilson line transforms nontrivially under the rotation, the rank can be reduced [18, 17].

stays invariantly, such that $\gamma\tilde{\gamma} = 1$. These states fall into representations of the unbroken gauge group. They are called *charged matter* representations. The above explanations can be resumed in the following breaking of the 248+248 dimensional adjoint representation of $E_8 \times E_8$ under the inner automorphism corresponding to the embedded action of the space group on the gauge degrees of freedom [19]:

$$\begin{array}{ccccc} \mathbf{248} & = & \mathrm{Ad}(\mathrm{G}_{unbroken}) & + & (\text{charged matter reps. under } \mathrm{G}_{unbroken}) \ . \\ (248\ \text{ p's}) & = & (\text{p's with } \tilde{\gamma} = 1) & & (\text{p's with } \tilde{\gamma} \neq 1, \text{but } \gamma\tilde{\gamma} = 1) \end{array}$$

Each of the two independent E_8 groups breaks in this manner. The unbroken gauge group in $d = 4$ arising from the first E_8 is called *observable sector* gauge group that from the second E_8 *hidden sector* gauge group.

Twisted sectors

The changed boundary conditions for the closed strings of the twisted sector $T_{(k,l)}$

$$Z(\tau, \sigma + 2\pi) = \hat{g}Z(\tau, \sigma) = \theta^k \omega^l Z(\tau, \sigma) + n_\alpha e_\alpha$$

lead to new features of the mode expansions of the bosonic and fermionic degrees of freedom and of the corresponding states. For example the mode expansion for the twisted bosonic degrees of freedom reads

$$Z^a(\tau, \sigma) = z_f^a + \frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \left(\frac{1}{n + v^a} \beta^a_{n+v^a} e^{-2\pi i(n+v^a)(\tau-\sigma)} + \frac{1}{n - v^a} \tilde{\beta}^a_{n-v^a} e^{-2\pi i(n-v^a)(\tau+\sigma)} \right) \ .$$

In the following the results emerging from these changed boundary conditions[9] and mode expansions are just stated. For a derivation of them see App. A.2.

- The centre of mass coordinate z_f^a is attached to a fixed point coordinate. Thus the twisted strings are localised at the fixed points for $\hat{g}$.
- The right and left moving momenta are shifted by the respective twist or shift vector corresponding to the constructing element

$$\begin{array}{ccccl} q & \longmapsto & q_{sh} & = & q + kv_1 + lv_2 \ , \\ p & \longmapsto & p_{sh} & = & p + kV_1 + lV_2 + n_\alpha A_\alpha \ . \end{array}$$

  A twisted states is written $|q_{sh}\rangle \otimes |p_{sh}\rangle$.
- The left moving oscillators $\tilde{\beta}$ are fractionally moded (explicit derivation in App. A.2). In the sector $T_{(k,l)}$ there are only the oscillators

$$\tilde{\beta}^a_{n-(kv_1+lv_2)} \ , \qquad \tilde{\beta}^{\bar{a}}_{n+(kv_1+lv_2)} \qquad n \in \mathbb{Z} \ .$$

[9]The twisted boundary conditions for the left moving degrees of freedom read $X_L^I(\tau, \sigma+2\pi) = X_L^I(\tau, \sigma) + kV_1^I + lV_2^I + n_\alpha A_\alpha^I$ due to the shift embedding.

This fractional moding causes non-integer eigenvalues of the *number operator* $\tilde{N}$, defined in App. A.2, which acts on the oscillators as

$$\begin{aligned} \tilde{N}\,\tilde{\beta}^a_{n-v^a} &= -(n-v^a)\tilde{\beta}^a_{n-v^a} \quad n-v^a<0\ , \\ \tilde{N}\,\tilde{\beta}^{\bar{a}}_{n+v^a} &= -(n+v^a)\tilde{\beta}^{\bar{a}}_{n+v^a} \quad n+v^a<0\ , \end{aligned}$$

where $v^a = kv_1^a + lv_2^a$. This will be important for the determination of the massless states, because the number operator appears in the mass equation.

- Due to the modified mode expansions, the generators of the Virasoro algebra L_0 and $\tilde{L_0}$ change. The consequences are modified mass formulae for the right and left moving states in the twisted sectors

$$\begin{aligned} \frac{m_R^2}{4} &= \frac{1}{2}q_{sh}^2 - \frac{1}{2} + \delta c\ , \\ \frac{m_L^2}{4} &= \frac{1}{2}p_{sh}^2 + \tilde{N} - 1 + \delta c\ . \end{aligned}$$

  The constant $\delta c = \frac{1}{2}\sum_{a=1}^{3}||kv_1^a + lv_2^a||(1-||kv_1^a + lv_2^a||)$ is called correction to the zero-point energy. The notation $||v||$ means that the value of v is integrally shifted into the interval $[0,1]$. It arises due to the changed calculation of the normal ordering constants and is the same for the right and the left movers. The new masslessness conditions $\frac{1}{4}m_R^2 = 0$ and $\frac{1}{4}m_L^2 = 0$ must now be solved for each sector and, in the presence of Wilson lines, even for each fixed point. They are constraint equations for the shifted SO(8) weights q_{sh} and $\mathrm{E}_8 \times \mathrm{E}_8$ roots p_{sh}. Therefore determining the momenta for the massless states, which fall into representations of the unbroken gauge group, leads to the matter representations which are located at the appropriate fixed point.

Having determined the massless states at a particular fixed point we finally must project onto space group invariant states to retain only the states allowed on the orbifold. This procedure turns out to be a tricky one. In this paragraph it will be explained for general orbifold constructions, in the next section explicitly for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case. In contrast to the untwisted sector the situation now is somewhat more involved. For a certain state in the twisted sector $T_{\hat{g}}$ of the Hilbert space of states, corresponding to the space group element $\hat{g}$, at first instance we do not have to guarantee invariance under all elements of the space group. There must be a distinction between elements $h \in S$ which commute with the constructing element $\hat{g}$ and such which do not, as will be illuminated now.
The complex coordinate $Z(\tau,\sigma)$ is assumed to have a mode expansion corresponding to the boundary condition $Z(\tau,\sigma+2\pi) = \hat{g}Z(\tau,\sigma)$. Then the question arises to which boundary condition $hZ(\tau,\sigma)$, for another $h \in S$ belongs. From a brief calculation

$$hZ(\tau,\sigma+2\pi) = h\hat{g}Z(\tau,\sigma) = (h\hat{g}h^{-1})\,hZ(\tau,\sigma)$$

it becomes clear that $hZ(\tau,\sigma)$ closes under $h\hat{g}h^{-1} \in S$, which is just another element in the conjugation class of $\hat{g}$. Thus the states corresponding to $Z(\tau,\sigma)$ lie in the Hilbert space sector $T_{\hat{g}}$ and those belonging to $hZ(\tau,\sigma)$ in $T_{h\hat{g}h^{-1}}$. There are now two cases to distinguish:

- $[\hat{\mathbf{g}},\mathbf{h}] = \mathbf{0}$: in this case $hZ(\tau,\sigma)$ also closes under $\hat{g}$. The corresponding states belong to $T_{\hat{g}}$. Those elements $h \in S$ which commute with $\hat{g}$ are said to lie in the *centralizer* of the constructing element:
$$h \in Z_{\hat{g}} \equiv \{\tilde{h} \in S | [\hat{g},\tilde{h}] = 0\} \ . \tag{2.4}$$
- $[\hat{\mathbf{g}},\mathbf{h}] \neq \mathbf{0}$: Here $hZ(\tau,\sigma)$ closes under $h\hat{g}h^{-1}$ and the twisted sector $T_{h\hat{g}h^{-1}}$, in which the states lie, belongs to $h\hat{g}h^{-1}$. Therefore h maps states from one twisted sector to another
$$h : \ T_{\hat{g}} \longrightarrow T_{h\hat{g}h^{-1}} \ .$$

How do the above considerations influence the construction of invariant states? Each state in $T_{\hat{g}}$ must of course be invariant under $Z_{\hat{g}}$ and all states arising from solving the mass equations must be projected onto centralizer invariant states.
But due to our above discussion there is a new possibility to ensure the invariance under $h \neq Z_{\hat{g}}$! If there are states in $T_{\hat{g}}$, $T_{h\hat{g}h^{-1}}$, $T_{h^2\hat{g}(h^{-1})^2}, \ldots$ which are not invariant under h, there could be nevertheless a linear combination of them which is. Therefore for each noncommuting element $h \in S$ we have to build linear combinations of states in $T_{\hat{g}}$, $T_{h\hat{g}h^{-1}}, \ldots, T_{h^{N-1}\hat{g}(h^{-1})^{N-1}}$ which are invariant under h (for $h^N = \mathbb{1}$).[10] To have this additional possibility to build invariant states there must be a $h \in S$ which exchanges inequivalent fixed points on the torus (corresponding to the exchange $\hat{g} \leftrightarrow h\hat{g}h^{-1}$). There is no such exchange for a SU(3) geometry (see App. A.1) and therefore no linear combinations of the described type appear in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold spectrum![11]

[10]The construction of these linear combinations is nicely explained in [20].

[11]These linear combinations only appear if there is a non-trivial subgroup of the point group, as it is the case for the point group $\mathbb{Z}_4$ with its subgroup $\mathbb{Z}_2$. In other words: Linear combinations of states appear only in the spectra of non-prime order orbifolds.

Chapter 3

$\mathbb{Z}_3 \times \mathbb{Z}_3$ models

After the general discussion of heterotic string theories on orbifolds in the last chapter, we will now turn to the explicit construction of models using the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.
The structure of the chapter is as follows. As a warming up, we first consider the most simple example of a model, the so-called standard embedding case. We will find out that there are slight mistakes in the literature [21], concerning the oscillator states of the untwisted sector. Then we review a three-generation model taken from the literature [22], discover some calculational mistakes and present the correct spectrum. After understanding the construction mechanism of $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold models we will continue with discussing constraints on the shifts, coming from modular invariance and will discover a new condition for the equivalence of two models. This condition leads us directly to an observation made by Vafa [4], called discrete torsion. We will finally conclude with some guidelines for $\mathbb{Z}_3 \times \mathbb{Z}_3$ model building in general and use them to construct two three-generation models, which are then used in the following chapter to discuss particular features of the spectrum, namely discrete family symmetries.
To calculate the various spectra, use was made of the two programs [23, 24].

3.1 Standard embedding

There are many different possibilities of how to embed the action of the spacetime twist into the gauge degrees of freedom. Each of them is characterised by a special choice of shift vectors V_1, V_2 which is constrained due to the modular invariance conditions Eq. (3.3). The easiest way to perform such an embedding is the *standard embedding.* Standard embedding means choosing an SU(3) subgroup of the first E_8 factor of the gauge group and taking the gauge twist action equal the action of the two spacetime $\mathbb{Z}_3$ twists.[1] Therefore

[1]In the Calabi-Yau picture this corresponds to equate the gauge connection with the spin connection

$$tr(R \wedge R) = \frac{1}{30} tr(F \wedge F) \ .$$

$q/\tilde{\beta}$	$q \cdot v_1$	$q \cdot v_2$
$(\pm 1, 0, 0, 0)$	0	0
$(0, \pm 1, 0, 0)$	$\pm\frac{1}{3}$	0
$(0, 0, \pm 1, 0)$	0	$\pm\frac{1}{3}$
$(0, 0, 0, \pm 1)$	$\mp\frac{1}{3}$	$\mp\frac{1}{3}$
$\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	0	0
$\pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$\pm\frac{1}{3}$	0
$\pm(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	0	$\pm\frac{1}{3}$
$\pm(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$\mp\frac{1}{3}$	$\mp\frac{1}{3}$
β^0_{-1}	0	0
β^1_{-1}	$\frac{1}{3}$	0
β^2_{-1}	0	$\frac{1}{3}$
β^3_{-1}	$-\frac{1}{3}$	$-\frac{1}{3}$

Table 3.1: Transformation phases of the SUGRA-multiplet state right movers and left moving oscillators.

the two standard embedding shifts for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold are given by

$$\begin{aligned} V_1 &= \tfrac{1}{3}(1, 0, -1, 0^5) \quad (0^8) \quad , \\ V_2 &= \tfrac{1}{3}(0, 1, -1, 0^5) \quad (0^8) \quad , \end{aligned}$$

where (0^8) means $(0, 0, 0, 0, 0, 0, 0, 0)$. In the following the explicit calculation of the spectrum in the untwisted and twisted sectors is presented.[2]

3.1.1 Untwisted sector

First we consider the states of the SUGRA-multiplet in $d = 10$ and show how we get the invariant states of the SUGRA-multiplet in $d = 4$ and the additional singlets. This should just be an illustration of the general arguments from App. A.1.
The SUGRA-multiplet in $d = 10$ written in complexified coordinates consists of states of the form $|q\rangle_R \otimes \tilde{\beta}^a_{-1}|0\rangle_L$. Tab. (3.1) shows the transformation phases of the right moving states corresponding to the SO(8) weight q and of the left moving oscillators. By trying to form invariant states for the SUGRA-multiplet in $d = 4$ (that means for the oscillator

[2]For more details and the explicit calculation for the simpler $\mathbb{Z}_3$ case see [9].

p_{hw}	$p_{hw} \cdot V_1$	$p_{hw} \cdot V_2$	q	rep under $\mathrm{G}_{unbroken}$
$(1,1,0^6)(0^8)$	$\frac{1}{3}$	$\frac{1}{3}$	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\mathbf{27};\mathbf{1})$
$(1,0,1,0^5)(0^8)$	0	$-\frac{1}{3}$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$(\mathbf{27};\mathbf{1})$
$(0,1,1,0^5)(0^8)$	$-\frac{1}{3}$	0	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\mathbf{27};\mathbf{1})$

Table 3.2: Charged matter representations in the untwisted sector.

$\tilde{\beta}^0_{-1}$) it becomes obvious that the first $\mathbb{Z}_3$ twist leaves invariant only the two gravitini (of each chirality)

$$|\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle_R \otimes \tilde{\beta}^{\mu}_{-1}|0\rangle_L \ ,$$
$$|\pm(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})\rangle_R \otimes \tilde{\beta}^{\mu}_{-1}|0\rangle_L \ ,$$

with $\mu = 0, \bar{0}$, corresponding to $N = 2$ SUSY in $d = 4$. The second $\mathbb{Z}_3$ twist now transforms the second gravitino nontrivially and therefore projects it out. In the end we are left with only one gravitino which corresponds to the wanted remaining $N = 1$ SUSY in $d = 4$. This is a general feature for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, where one twist alone only leads to $N = 2$ and the second one then to $N = 1$ SUSY.
The additional invariant singlet states of the form $|q\rangle_R \otimes \tilde{\beta}^a_{-1}|0\rangle_L$ will be omitted in the following.

The next step is to consider the gauge sector of states of the form $|q\rangle_R \otimes |p\rangle_L$ where p is an $\mathrm{E}_8 \times \mathrm{E}_8$ root. By computing the transformation phases $2\pi i p \cdot V_i$ (for $i = 1, 2$) of all the 240 + 240 roots, we find that there are 72 invariant roots of the first E_8 factor, while all roots of the second E_8 transform trivially. They correspond to the unbroken gauge group

$$\mathrm{G}_{unbroken} = \mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8 \ .$$

Of course this is not obvious. How the resulting gauge group and the charged matter representations are derived from the surviving $\mathrm{E}_8 \times \mathrm{E}_8$ roots is explained very detailed in [9].
Aside from the surviving roots of the unbroken gauge group, the other $\mathrm{E}_8 \times \mathrm{E}_8$ roots transform with nontrivial phases, but they can be combined with right moving states to form invariant tensor product states $|q\rangle_R \otimes |p\rangle_L$. Those invariant combinations, the charged matter, are shown in Tab. (3.2). In Tab. (3.2) for a whole representation only one weight vector p_{hw} is given. From this highest weight (hw) the whole representation can be obtained by subtracting simple roots [25, 26]. If the highest weight of some unbroken gauge group representation survives the orbifold projection, so does the whole representation. Only the spinorial q is given because it defines the chirality of the state (our convention is that left chiral states have a $-\frac{1}{2}$ in the first entry, right chiral ones a $\frac{1}{2}$). Of course the

superpartners with the appropriate vectorial weight are invariant under the space group too.
The breaking of the 248+248-dimensional adjoint representation of $\mathrm{E}_8 \times \mathrm{E}_8$ can be summarised as follows

$$(\mathbf{248};\mathbf{1}) + (\mathbf{1};\mathbf{248}) \longrightarrow (\mathbf{78};\mathbf{1}) + 2\,(\mathbf{1};\mathbf{1}) + 3\,(\mathbf{27};\mathbf{1}) + 3\,(\overline{\mathbf{27}};\mathbf{1}) + 6\,(\mathbf{1};\mathbf{1}) + (\mathbf{1};\mathbf{248})\ .$$

The $(\mathbf{78};\mathbf{1})$ constitutes the 78 dimensional adjoint representation of E_6, the $2\,(\mathbf{1};\mathbf{1})$ are the generators of the two U(1)'s and $(\mathbf{1};\mathbf{248})$ is the adjoint representation of the second E_8 factor. The states of $3\,(\overline{\mathbf{27}};\mathbf{1})$ are right chiral states and therefore describe the antiparticles of the $3\,(\mathbf{27};\mathbf{1})$ states. We count only the left chiral states as particles. The states of the $6\,(\mathbf{1};\mathbf{1})$ are projected out due to the lack of a proper right moving state and in the end from the untwisted sector we get the charged matter representations

$$3\,(\mathbf{27};\mathbf{1})\ .$$

3.1.2 Twisted sectors

In the eight twisted sectors we first have to solve the twisted sector mass equations for the right and left movers, respectively, and then project the resulting states at each fixed point onto centralizer-invariant states. Due to the absence of Wilson lines in the model at hand, we have the same mass equation for all fixed points of one sector and the more intricated procedure of calculating the centralizer and ensuring invariance under its action can be replaced by just assuring invariance under the two space group elements θ and ω, which correspond to the transformation phases

$$\begin{aligned} \theta &\leftrightarrow 2\pi i(q_{sh}\cdot v_1 - p_{sh}\cdot V_1)\ , \\ \omega &\leftrightarrow 2\pi i(q_{sh}\cdot v_2 - p_{sh}\cdot V_2)\ . \end{aligned}$$

This can be easily verified by considering the centralizers for the various fixed points in App. A.1. When we consider orbifolds with Wilson lines in later chapters we will have to calculate the centralizers of the constructing elements and use the projection conditions of Tab. (A.1) in App. A.1.
By calculating the δc-term for the different sectors we see that in the sectors $T_{(1,0)}$, $T_{(2,0)}$, $T_{(0,1)}$, $T_{(0,2)}$, $T_{(1,2)}$, $T_{(2,1)}$ we have the masslessness conditions

$$\begin{aligned} q_{sh}^2 &= \frac{5}{9}\ , \\ p_{sh}^2 &= \frac{14}{9} - 2\tilde{N}\ , \end{aligned}$$

while in $T_{(1,1)}$, $T_{(2,2)}$ due to the change in δc we are left with

$$\begin{aligned} q_{sh}^2 &= \frac{1}{3}\ , \\ p_{sh}^2 &= \frac{12}{9} - 2\tilde{N}\ . \end{aligned}$$

We first explicitly consider $T_{(1,0)}$ and then just state the results for the other sectors. In $T_{(1,0)}$ we have $q_{sh} = q + v_1$ and $p_{sh} = p + V_1$. There is one spinorial SO(8) weight q_{sh} which defines left chiral states and which solves the mass equation and the vectorial weight of the superpartner

$$\begin{aligned} q_{sh} &= (-\frac{1}{2}, -\frac{1}{6}, \frac{1}{2}, \frac{1}{6}) \ , \\ q_{sh} &= (0, -\frac{2}{3}, 0, -\frac{1}{3}) \ . \end{aligned}$$

To solve the mass equation for the left moving states we have to consider three possible values which $\tilde{N}$ can take, $\tilde{N} = 0, \frac{1}{3}, \frac{2}{3}$ (for $\tilde{N} > \frac{2}{3}$ the right hand side of the mass equation becomes negative):

- $\mathbf{\tilde{N} = 0}$: There are 56 $E_8 \times E_8$ roots which solve the mass equation $(p + V_1)^2 = \frac{14}{9}$. They constitute the following representations under $E_6 \times E_8$

$$(\mathbf{27}; \mathbf{1}) + (\overline{\mathbf{27}}; \mathbf{1}) + 2\,(\mathbf{1}; \mathbf{1}) \ .$$

  Tensoring together right and left moving states to form invariant combinations under the two independent $\mathbb{Z}_3$ twists θ and ω, we see that this is only possible for the $(\mathbf{27}; \mathbf{1})$ and the $(\overline{\mathbf{27}}; \mathbf{1})$.[3] The $(\overline{\mathbf{27}}; \mathbf{1})$ again are the antiparticles of the $(\mathbf{27}; \mathbf{1})$. Therefore the remaining matter content is $(\mathbf{27}; \mathbf{1})$, where the multiplicity 9 turns up due to the 9 fixed tori in the sector $T_{(1,0)}$.

- $\mathbf{\tilde{N} = \frac{1}{3}}$: Only one root, corresponding to a representation $(\mathbf{1}; \mathbf{1})$, solves the mass equation $(p + V_1)^2 = \frac{14}{9} - \frac{4}{3} = \frac{8}{9}$. The complete left moving state then consists of this root and an oscillator $\tilde{\beta}^a{}_{-\frac{1}{3}}$. From App. A.2 it can be derived that the only allowed oscillators in $T_{(1,0)}$ which give $\tilde{N} = \frac{1}{3}$ are $\tilde{\beta}^1{}_{-\frac{1}{3}}$ and $\tilde{\beta}^{\bar{3}}{}_{-\frac{1}{3}}$. Only with $\tilde{\beta}^{\bar{3}}{}_{-\frac{1}{3}}$ an invariant combination can be formed. Taking the fixed point multiplicity into account we end up with the charged matter $9\,(\mathbf{1}; \mathbf{1})$.

- $\mathbf{\tilde{N} = \frac{2}{3}}$: The mass equation $(p + V_1)^2 = \frac{2}{9}$ is again solved by only one root vector. But this time from the allowed oscillators $\tilde{\beta}^{\bar{1}}{}_{-\frac{2}{3}}$, $\tilde{\beta}^3{}_{-\frac{2}{3}}$, $\tilde{\beta}^1{}_{-\frac{1}{3}}\tilde{\beta}^1{}_{-\frac{1}{3}}$, $\tilde{\beta}^{\bar{3}}{}_{-\frac{1}{3}}\tilde{\beta}^{\bar{3}}{}_{-\frac{1}{3}}$ and $\tilde{\beta}^1{}_{-\frac{1}{3}}\tilde{\beta}^{\bar{3}}{}_{-\frac{1}{3}}$ two invariant combinations can be build with $\tilde{\beta}^{\bar{1}}{}_{-\frac{2}{3}}$ and $\tilde{\beta}^1{}_{-\frac{1}{3}}\tilde{\beta}^1{}_{-\frac{1}{3}}$. Therefore the matter representations are $18\,(\mathbf{1}; \mathbf{1})$.

Thus, in total, from $T_{(1,0)}$ we get the charged matter representations

$$3\,(\mathbf{27}; \mathbf{1}) + 27\,(\mathbf{1}; \mathbf{1}) \ ,$$

where all singlets are oscillator states. The explicit states and transformation phases are given in Tab. (3.3) and Tab. (3.4).

[3] These calculations must only be done for the highest weights because then they are also true for the whole representations.

massless lm state	rep	$p \cdot V_1$	$p \cdot V_2$	massless rm state	$q \cdot v_1$	$q \cdot v_2$
$\vert(-\frac{2}{3},-1,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{27};\mathbf{1})$	$-\frac{1}{9}$	$\frac{1}{9}$	$\vert(-\frac{1}{2},-\frac{1}{6},\frac{1}{2},\frac{1}{6})\rangle_R$	$-\frac{1}{9}$	$\frac{1}{9}$
$\tilde{\beta}^{1}_{-\frac{1}{3}}\vert(-\frac{2}{3},0,\frac{2}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$-\frac{2}{9}$			
$\tilde{\beta}^{\bar{3}}_{-\frac{1}{3}}\vert(-\frac{2}{3},0,\frac{2}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$\frac{1}{9}$			
$\tilde{\beta}^{\bar{1}}_{-\frac{2}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$\frac{1}{9}$			
$\tilde{\beta}^{3}_{-\frac{2}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$-\frac{2}{9}$			
$\tilde{\beta}^{1}_{-\frac{1}{3}}\tilde{\beta}^{1}_{-\frac{1}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$\frac{1}{9}$			
$\tilde{\beta}^{\bar{3}}_{-\frac{1}{3}}\tilde{\beta}^{\bar{3}}_{-\frac{1}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$-\frac{2}{9}$			
$\tilde{\beta}^{1}_{-\frac{1}{3}}\tilde{\beta}^{\bar{3}}_{-\frac{1}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$	$-\frac{1}{9}$	$\frac{4}{9}$			

Table 3.3: Massless right and left moving states in the twisted sector $T_{(1,0)}$.

invariant states in $T_{(1,0)}$	rep
$\vert(-\frac{1}{2},-\frac{1}{6},\frac{1}{2},\frac{1}{6})\rangle_R \otimes \vert(-\frac{2}{3},-1,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{27};\mathbf{1})$
$\vert(-\frac{1}{2},-\frac{1}{6},\frac{1}{2},\frac{1}{6})\rangle_R \otimes \tilde{\beta}^{\bar{3}}_{-\frac{1}{3}}\vert(-\frac{2}{3},0,\frac{2}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$
$\vert(-\frac{1}{2},-\frac{1}{6},\frac{1}{2},\frac{1}{6})\rangle_R \otimes \tilde{\beta}^{\bar{1}}_{-\frac{2}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$
$\vert(-\frac{1}{2},-\frac{1}{6},\frac{1}{2},\frac{1}{6})\rangle_R \otimes \tilde{\beta}^{1}_{-\frac{1}{3}}\tilde{\beta}^{1}_{-\frac{1}{3}}\vert(\frac{1}{3},0,-\frac{1}{3},0^5)(0^8)\rangle_L$	$(\mathbf{1};\mathbf{1})$

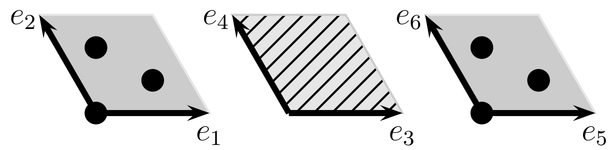
Table 3.4: Charged matter representations in the twisted sector $T_{(1,0)}$.

sector	charged matter
U	$3(\mathbf{27};\mathbf{1})$
$T_{(1,0)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(2,0)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(0,1)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(0,2)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(1,2)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(2,1)}$	$9\,(\mathbf{27};\mathbf{1})+27\,(\mathbf{1};\mathbf{1})$
$T_{(1,1)}$	$\emptyset$
$T_{(2,2)}$	$27\,(\mathbf{27};\mathbf{1})+81\,(\mathbf{1};\mathbf{1})$
Total	$84\,(\mathbf{27};\mathbf{1})+243\,(\mathbf{1};\mathbf{1})$

Table 3.5: Charged matter content of all sectors in the standard embedding case. Compared to [21] we computed a different total number of singlets. Presumably, in [21] the singlets from the untwisted sector are included, but they actually have to be projected out.

The procedure for the other seven twisted sectors is the same. Of course one should take into account the changed constants in the mass equations for $T_{(1,1)}$ and $T_{(2,2)}$.
The two sectors with the effective $\mathbb{Z}_3$ twist have the special feature that in $T_{(1,1)}$ there are only right chiral and in $T_{(2,2)}$ only left chiral states. There are three solutions to the left mover mass equation for $\tilde{N} = \frac{1}{3}$ and none for $\tilde{N} = \frac{2}{3}$. From these three solutions and the three allowed oscillators $\tilde{\beta}^{\bar{a}}_{-\frac{1}{3}}$ $(a = 1, 2, 3)$ three invariant combinations can be build.
The states in $T_{(2,2)}$ are localised at 27 proper fixed points and therefore the representations possess a multiplicity 27.
The full matter content of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ standard embedding is depicted in Tab. (3.5).

$T_{(2,0)}$:



$9\,(\mathbf{16}, \mathbf{1}; \mathbf{1})$

Figure 3.1: Localisation of the nine **16**-plets in the sector $T_{(2,0)}$.

3.2 Three-generation model from the literature

There is one three-generation $\mathbb{Z}_3 \times \mathbb{Z}_3$ model presented in the literature [22]. The calculations are presented here as a warm up for our further consideration of the structure of $\mathbb{Z}_3 \times \mathbb{Z}_3$ models and model building.
In the model of [22] three families are obtained by the following construction. First take two shifts V_1, V_2 which break the gauge group to some subgroup of $\mathrm{E}_8 \times \mathrm{E}_8$ with nine families in one of the twisted sectors with nine fixed tori. Then add a Wilson line A_1 in the first SU(3) torus. This Wilson line distinguishes between the fixed tori in such a way that they split as

$$9 \longrightarrow 3 + 3 + 3$$

into three copies of respectively three degenerate fixed tori. Three degenerate fixed tori will then inherit the three families.

- To proceed as described we take the two gauge shifts[4]

$$\begin{array}{rcl} V_1 &=& \frac{1}{3}(-4,1,1,0,0,0,0,0) \quad (1,1,0,0,0,0,0,0)\ , \\ V_2 &=& \frac{1}{3}(0,2,0,0,0,0,0,0) \quad (0,-1,4,4,1,0,0,0)\ , \end{array}$$

  which break the observable gauge group to SO(10) as

$$\mathrm{E}_8 \times \mathrm{E}_8 \longrightarrow \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(7) \times \mathrm{U}(1)^4\ .$$

  By calculating the full spectrum, one finds that there are nine generations of SO(10) (**16**-plets of SO(10)) in the sector $T_{(2,0)}$ as indicated in Fig. (3.1).

- The next step now is to add a Wilson line A_1 in the first torus which will differentiate between the three fixed points in this two dimensional space. Due to the presence of the Wilson line

$$A_1 \ = \ \tfrac{1}{3}(0,0,0,0,0,1,1,-2) \quad (0,0,0,0,0,-2,-2,-2)$$

[4]In [22] the shifts and the Wilson line are denoted

$$\begin{array}{rcl} V_1 &=& \frac{1}{3}(2,1,1,0,0,0,0,0) \quad (1,1,0,0,0,0,0,0)\ , \\ V_2 &=& \frac{1}{3}(0,2,0,0,0,0,0,0) \quad (0,-1,1,1,1,0,0,0)\ , \\ A_1 &=& \frac{1}{3}(0,0,0,0,0,1,1,-2) \quad (0,0,0,0,0,1,1,-2)\ . \end{array}$$

They are equivalent to the ones we have chosen, as can be seen from the discussion in Sec. 3.3.2.

$T_{(2,0)}$:

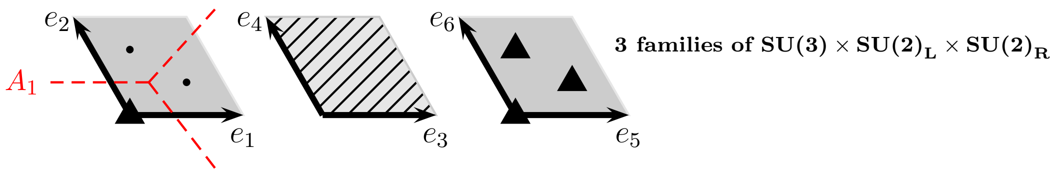


Figure 3.2: The Wilson line A_1 differentiates between the three fixed points in the first sublattice and leaves three generations $3\,(\mathbf{3},\mathbf{2},\mathbf{1})$ at the three 'red triangle fixed tori' in $T_{(2,0)}$.

the gauge group is further broken to

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1)^6 \ .$$

The three SO(10)-families now break into the representations of three $\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$-families[5] which are then located on three degenerate fixed tori of $T_{(2,0)}$ as indicated in Fig. (3.2).

The full matter content of this model is shown in Tab. (C.3) in App. C. In [22] there is a slight mistake in the sectors $T_{(0,1)}$ and $T_{(0,2)}$. The authors found the following representations additional to our result, located in these two sectors:

$$\begin{array}{ll} T_{(0,1)}: & 3[(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}),\, (\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}),\, (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3}),\, (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\bar{\mathbf{3}})] \\ T_{(0,2)}: & 3[(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}),\, (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}),\, (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\bar{\mathbf{3}}),\, (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})] \ . \end{array}$$

From consideration of Tab. (A.1) it is obvious that in the two sectors under consideration there cannot be states with multiplicity 3 due to the Wilson line located in the first torus and not differentiating between any of the nine fixed tori. Our calculation shows that these states are just absent.
Anyway this model shows how in $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold constructions three generations can arise very naturally from the fixed point structure of the underlying SU(3) tori.
A disadvantage of this model is that there are too many massless chiral states in the spectrum. Not all of them can become heavy by use of renormalizable and non-renormalizable Yukawa couplings.[6]

[5] The breaking is

$$(\mathbf{16},\mathbf{1};\mathbf{1}) \mapsto (\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) + (\bar{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1}) \ .$$

[6] The discussion of allowed and not allowed Yukawa couplings is postponed to chapter 4.

3.3 Modular invariance and equivalence of models

For a closed string theory, as the here considered heterotic string theory, there is the requirement of *modular invariance* which imposes strong constraints on the allowed models [2, 4]. In the first subsection we will briefly explain the origin of these constraints and give a handy formula, which the set of shifts and Wilson lines must fulfil.
We then notice that a particular orbifold model changes under a certain kind of transformations on the shifts. At first glance this is a rather astonishing fact but we will combine it with an old observation, namely the possibility of introducing an additional phase in the partition function [4]. This phase is called *discrete torsion* phase and it changes the model. At the end of this section it should be clarified that both concepts, that of transforming the shifts and that of discrete torsion, describe the same circumstances.

3.3.1 Modular invariance conditions for orbifolds

At one loop the vacuum-to-vacuum amplitude corresponds to a worldsheet of the shape of a two dimensional torus which is swept out by the considered closed string. In the complex plane $\mathbb{C}$ the two dimensional torus T^2 is described by the two complex numbers σ_1, σ_2. They define the torus lattice.[7] Now the question is when do two different sets $\{\sigma_1, \sigma_2\}$ and $\{\tilde{\sigma}_1, \tilde{\sigma}_2\}$ describe the same complex structure? This can be easily answered. They describe the same complex structure if they are related by

$$\begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{pmatrix} = A \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

via a 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \ .$$

To be in $SL(2, \mathbb{Z})$, of special linear form, means that A has integer entries and $\det A = ad - bc = 1$. For a proof see [13].
Because various sets of torus generators differing by $SL(2, \mathbb{Z})$ transformations (*modular transformations*) describe the same complex structure, the theory must be invariant under $SL(2, \mathbb{Z})$. Therefore invariance of the closed string partition function under $SL(2, \mathbb{Z})$, the so-called *modular group*, must be demanded.[8] To ensure this invariance the set of shifts and Wilson lines for a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold must obey the *weak modular invariance* condition of the form

$$N' \left[(kV_1 + lV_2 + n_\alpha A_\alpha)^2 - (kv_1 + lv_2)^2 \right] = 0 \mod 2 \ , \tag{3.1}$$

[7]Starting from σ_1, σ_2 the *modular parameter* τ can be introduced as $\tau \equiv \sigma_2/\sigma_1$, which specifies the complex structure of T^2. Then without loss of generality the two torus lattice generators can be taken as 1, τ.

[8]To be more precise we have to demand invariance under $PSL(2, \mathbb{Z})$. Here

$$PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/\mathbb{Z}_2$$

meaning that we take all invertible 2×2 matrices A with determinant $+1$ and identify A and $-A$.

where N' is the order of the respective twist $kv_1 + lv_2)^2$. For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold $k, l, n_\alpha \in \{0, 1, 2\}$. In general $\alpha = 1, ..., 6$ due to the six basis vectors $\{e_\alpha\}$ of the torus lattice, but the specific properties of the various orbifolds restrict the number of inequivalent Wilson lines. For $\mathbb{Z}_3 \times \mathbb{Z}_3$ we have 3 inequivalent Wilson lines. Eq. (3.1) is equivalent to the following set of equations

$$
\begin{aligned}
N'\,(V_i^2 - v_i^2) &= 0 \mod 2 \quad i = 1, 2\ , && (3.2a)\\
N'\,(V_1 \cdot V_2 - v_1 \cdot v_2) &= 0 \mod 1\ , && (3.2b)\\
N' A_\alpha \cdot (kV_1 + lV_2) &= 0 \mod 1\ , && (3.2c)\\
N' A_\alpha \cdot A_\beta &= 0 \mod 1 \quad \alpha \neq \beta\ , && (3.2d)\\
N' A_\alpha^2 &= 0 \mod 2\ . && (3.2e)
\end{aligned}
$$

For calculational reasons we bring the shifts and Wilson lines to *strong modular invariant* form, which means that they satisfy the even stronger condition

$$[(kV_1 + lV_2 + n_\alpha A_\alpha)^2 - (kv_1 + lv_2)^2] = 0 \mod 2\ . \qquad (3.3)$$

Analogously to the above case we can write this as

$$
\begin{aligned}
V_i^2 - v_i^2 &= 0 \mod 2 \quad i = 1, 2\ , && (3.4a)\\
V_1 \cdot V_2 - v_1 \cdot v_2 &= 0 \mod 1\ , && (3.4b)\\
A_\alpha \cdot (kV_1 + lV_2) &= 0 \mod 1\ , && (3.4c)\\
A_\alpha \cdot A_\beta &= 0 \mod 1 \quad \alpha \neq \beta\ , && (3.4d)\\
A_\alpha^2 &= 0 \mod 2\ . && (3.4e)
\end{aligned}
$$

It can be seen that the strong modular invariance condition follows directly from the argument that a closed string theory must be invariant under worldsheet translations $\sigma \mapsto \sigma + 2\pi$ and the generator of these being proportional to $L_0 - \tilde{L}_0$ [8].[9] It can be easily checked that all sets of shifts we used so far are of strong modular invariant form.
It should again be stressed that either the weak or the strong modular invariance conditions are strong constraints on the possible gauge embedding shifts V_1, V_2, A_α and they extremely restrict their number.

In addition to these, due to the gauge embedding being a group homomorphism, for the shifts and Wilson lines of a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold we have conditions

$$
\begin{aligned}
NV_1 &\in T_{\mathrm{E}_8\times\mathrm{E}_8}\ , && (3.5a)\\
MV_2 &\in T_{\mathrm{E}_8\times\mathrm{E}_8}\ , && (3.5b)\\
K_\alpha A_\alpha &\in T_{\mathrm{E}_8\times\mathrm{E}_8}\ , && (3.5c)
\end{aligned}
$$

with $\alpha = 1, ..., 6$, as is explained in [8, 9]. Here $T_{\mathrm{E}_8\times\mathrm{E}_8}$ denotes the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8$, N, M are the orders of the shifts and K_α of the Wilson lines. In the case of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold this means $3V_1, 3V_2, 3A_\alpha \in T_{\mathrm{E}_8\times\mathrm{E}_8}$. Again a simple check shows that all our previously used shifts satisfy these constraints.

[9]By L_0, $\tilde{L}_0$ we denote the zero modes of the Virasoro generators.

3.3.2 An additional condition for equivalent shift embeddings

Now the question arises, when do two different sets of shifts $\{V_1, V_2, A_\alpha\}$ and $\{\tilde{V}_1, \tilde{V}_2, \tilde{A}_\alpha\}$ describe the same model? This should mean that they lead to the same gauge group and charged matter representations.
Our first proposal would be that they describe the same model if

- they differ by $E_8 \times E_8$ roots.
- they can be transformed into each other by Weyl reflections.[10]

But by considering the transformation phase of the states under space group transformations we will find that this is not true.

We therefore assume that we have a set of strong modular invariant shifts $\{V_1, V_2, A_\alpha\}$ and change it to a new set of strong modular invariant shifts $\{\tilde{V}_1, \tilde{V}_2, \tilde{A}_\alpha\}$ by adding $E_8 \times E_8$ root vectors. Thus

$$\begin{aligned} \tilde{V}_1 &= V_1 + \Delta V_1 \ , \\ \tilde{V}_2 &= V_2 + \Delta V_2 \ , \\ \tilde{A}_\alpha &= A_\alpha + \Delta A_\alpha \ , \end{aligned}$$

with $\Delta V_1, \Delta V_2, \Delta A_\alpha \in T_{E_8 \times E_8}$.

Due to the transformed shift vectors the momenta of the states localised at the fixed point corresponding to the constructing element $\hat{g} = (\theta^t \omega^s, m_\alpha e_\alpha)$ must be shifted in the root lattice $p \mapsto \tilde{p} = p - t\Delta V_1 - s\Delta V_2 - m_\alpha \Delta A_\alpha$, in such a way that they still fulfil the masslessness condition. Therefore it is clear that $\tilde{p}_{sh} = p_{sh}$.
From Eq. (2.3) we see that for a state in the sector twisted by $\hat{g} = (\theta^t \omega^s, m_\alpha e_\alpha)$, the transformation phase ϕ under the space group element $(\theta^k \omega^l, n_\alpha e_\alpha)$ in the centralizer of the constructing element $\hat{g}$ reads

$$|q_{sh}\rangle \otimes (\tilde{\beta}^a_{n-\tilde{v}^a}|p_{sh}\rangle) \mapsto \exp(2\pi i\phi)|q_{sh}\rangle \otimes (\tilde{\beta}^a_{n-\tilde{v}^a}|p_{sh}\rangle) \ ,$$

where we keep in mind

$$p \mapsto p_{sh} = p + tV_1 + sV_2 + m_\alpha A_\alpha \ ,$$

$$\phi = q_{sh} \cdot (kv_1 + lv_2) - p_{sh} \cdot (kV_1 + lV_2 + n_\alpha A_\alpha) + \eta \ , \tag{3.6}$$

$$\tilde{v} = kv_1 + lv_2 \ .$$

Of course $k, l, n_\alpha \in \{0, 1, 2\}$ and η is the transformation phase of the possible oscillator

$$\tilde{\beta}^a_{n-\tilde{v}^a} \mapsto \exp(2\pi i\eta)\, \tilde{\beta}^a_{n-\tilde{v}^a} \ .$$

[10] Weyl reflections are certain automorphisms of the root lattice [27, 26, 25].

The transformation of the shift vectors causes a difference in the transformation phase of the states. Using Eq. (3.6)

$$\begin{aligned}
\Delta\phi &= \tilde{\phi} - \phi \\
&= [q_{sh} \cdot (kv_1 + lv_2) - \tilde{p}_{sh} \cdot (k\tilde{V}_1 + l\tilde{V}_2 + n_\alpha \tilde{A}_\alpha)] - [q_{sh} \cdot (kv_1 + lv_2) \\
&\quad - p_{sh} \cdot (kV_1 + lV_2 + n_\alpha A_\alpha)] \\
&= [q_{sh} \cdot (kv_1 + lv_2) - p_{sh} \cdot (k(V_1 + \Delta V_1) + l(V_2 + \Delta V_2) + n_\alpha (A_\alpha + \Delta A_\alpha))] \\
&\quad - [q_{sh} \cdot (kv_1 + lv_2) - p_{sh} \cdot (kV_1 + lV_2 + n_\alpha A_\alpha)] \\
&= -p_{sh} \cdot (k\Delta V_1 + l\Delta V_2 + n_\alpha \Delta A_\alpha) \\
&= (p + tV_1 + sV_2 + m_\alpha A_\alpha) \cdot (k\Delta V_1 + l\Delta V_2 + n_\alpha \Delta A_\alpha) \quad .
\end{aligned}$$

From the strong modular invariance conditions for both, the transformed and the untransformed shifts, we get

$$V_i \cdot \Delta V_i = 0 \mod 1 \qquad i = 1, 2 \ , \tag{3.7a}$$
$$V_1 \cdot \Delta V_2 + \Delta V_1 \cdot V_2 = 0 \mod 1 \ , \tag{3.7b}$$
$$V_i \cdot \Delta A_\alpha + \Delta V_i \cdot A_\alpha = 0 \mod 1 \ , \tag{3.7c}$$
$$A_\alpha \cdot \Delta A_\beta + \Delta A_\alpha \cdot A_\beta = 0 \mod 1 \qquad \alpha \neq \beta \ , \tag{3.7d}$$
$$A_\alpha \cdot \Delta A_\alpha = 0 \mod 1 \ . \tag{3.7e}$$

Using these constraint equations and the fact that scalar products between $E_8 \times E_8$ roots are always integer valued, we arrive at

$$\begin{aligned}
\Delta\phi = [&tlV_1 \cdot \Delta V_2 + skV_2 \cdot \Delta V_1 + tn_\alpha V_1 \cdot \Delta A_\alpha + sn_\alpha V_2 \cdot \Delta A_\alpha + km_\alpha A_\alpha \cdot \Delta V_1 \\
&+ lm_\alpha A_\alpha \cdot \Delta V_2 + m_\alpha n_\beta A_\alpha \cdot \Delta A_\beta] \mod 1 \ ,
\end{aligned}$$

where in the last summand $\alpha \neq \beta$. By using again the relations coming from the strong modular invariance conditions, we finally write the phase as

$$\Delta\phi = [(tl - sk)V_1 \cdot \Delta V_2 + (km_\alpha - tn_\alpha)A_\alpha \cdot \Delta V_1 + (lm_\alpha - sn_\alpha)A_\alpha \cdot \Delta V_2 + m_{[\alpha} n_{\beta]} A_\alpha \cdot \Delta A_\beta] \mod 1 \ , \tag{3.8}$$

where $m_{[\alpha} n_{\beta]}$ means that we restrict to antisymmetric combinations. The transformed shift vectors are only equivalent to the untransformed ones if we demand $\Delta\phi = 0$ and therefore the further conditions

$$\begin{aligned}
V_1 \cdot \Delta V_2 &= 0 \mod 1 \ , \\
A_\alpha \cdot \Delta V_i &= 0 \mod 1 \qquad i = 1, 2 \ , \\
A_\alpha \cdot \Delta A_\beta &= 0 \mod 1 \qquad \text{where } \alpha \neq \beta \ .
\end{aligned}$$

If this set of equations is not fulfilled, the spectra of the models corresponding to the shifts $\{V_i, A_\alpha\}$ and $\{\tilde{V}_i, \tilde{A}_\alpha\}$ are different in general. Whether they are different or not depends on

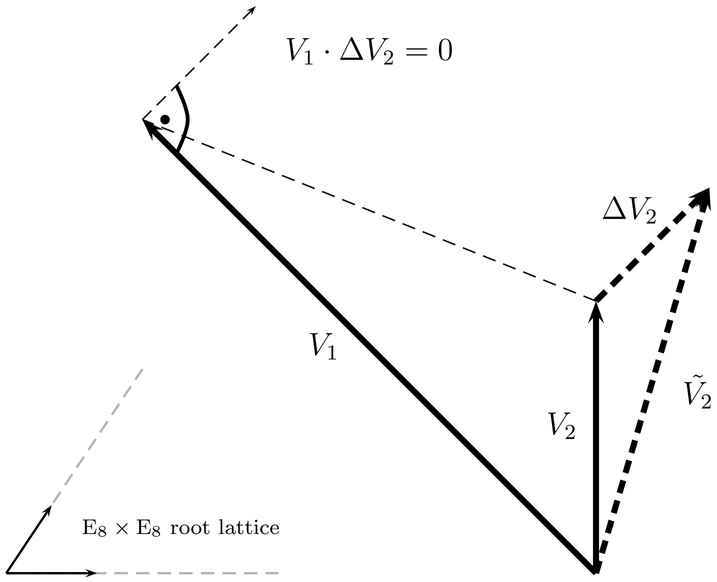


Figure 3.3: The figure shows an allowed equivalence transformation of the shifts. Transforming a shift vector by adding $E_8 \times E_8$ roots, not orthogonal to the other shift, can change the spectrum and thus the model.

the space group elements one has to project onto (the centralizers of the different sectors). In the case of the $\mathbb{Z}_3$ orbifold there is no change in the transformation phases and thus in the spectrum. The unbroken gauge group stays the same when we perform transformations of the discussed form on the shifts, because $\Delta\phi = 0$ is trivially fulfilled for the untwisted sector with the constructing element $\hat{g} = (\mathbb{1}, 0)$.
After the above considerations we are in position to give the proposed statement about the equivalence of orbifold models in a correct form.
Thus we conclude that the sets $\{V_i, A_\alpha\}$ and $\{\tilde{V}_i, \tilde{A}_\alpha\}$ lead to the same model, if

- they can be transformed into each other by Weyl reflections.
- they differ by $E_8 \times E_8$ roots, which are orthogonal to all other shifts respectively.

Fig. (3.3) illustrates how the relative position of the shift vectors influences the spectrum. A transformation by adding an $E_8 \times E_8$ root changes the angle between them and therefore causes the derived change in the transformation phase.

For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold it is easy to find two sets of shifts $\{V_1, V_2\}$ and $\{\tilde{V}_1, \tilde{V}_2\}$ which differ only by $E_8 \times E_8$ roots but lead to different massless spectra. We demonstrate this in the most simple case of the standard embedding:

$$\begin{aligned}
V_1 &= \tfrac{1}{3}(1,0,-1,0,0,0,0,0) \quad (0^8)\ ,\\
V_2 &= \tfrac{1}{3}(0,1,-1,0,0,0,0,0) \quad (0^8)\ ,\\
\\
\tilde{V}_1 &= \tfrac{1}{3}(1,-3,-1,3,0,0,0,0) \quad (0^8)\\
&= V_1 + (0,-1,0,1,0,0,0,0) \quad (0^8)\ ,\\
\tilde{V}_2 &= \tfrac{1}{3}(3,1,-1,0,3,0,0,0) \quad (0^8)\\
&= V_2 + (1,0,0,0,1,0,0,0) \quad (0^8)\ .
\end{aligned}$$

The spectra of the models corresponding to these two sets of shifts are given in App. C. We call the two models *model A* and *model A'*. From Tab. (C.1) and Tab. (C.2) it is obvious that the matter spectra of all twisted sectors differ, while the untwisted sector stays the same.

3.3.3 Discrete torsion phase for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds

In this section we explain the old idea of *discrete torsion* and will find out that it is related to the changed transformation phases, which we discussed above in Sec. 3.3.2.
The one-loop partition function Z for a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold, with N being a multiple of M, has the overall structure

$$Z = \frac{1}{NM} \sum_{\substack{g,h \\ [g,h]=0}} \varepsilon(g,h) Z(g,h)\ , \tag{3.9}$$

where the sum runs over pairs of commuting space group elements g, h and the $\varepsilon(g,h)$ are relative phases between the different terms in the partition function and thus between the different sectors.[11] Different assignments of phases will generate different orbifold models. At first glance, one would guess that the introduced phases $\varepsilon(g,h)$ are totally arbitrary. But this is not the case. Modular invariance demands invariance of the partition function under $PSL(2,\mathbb{Z})$ transformations. These transformations now can mix between different pairs of (g,h), between different sectors in the partition function. For example at one loop an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

maps a pair (g,h) onto some $(g^d h^{-c}, g^{-b} h^a)$ and therefore mixes the corresponding two sectors in the partition function.[12] From considering the case of more loops and demanding invariance of the partition function under modular transformations now the following

[11] Keeping g fixed and taking all summands with different h gives the contribution of the twisted sector T_g to the partition function.

[12] This is also an argument for the necessity to include all twisted sectors in the theory in order to retain modular invariance.

constraint equations on the phases ε arise [4]

$$\begin{aligned} \varepsilon(g_1 g_2, g_3) &= \varepsilon(g_1, g_3)\varepsilon(g_2, g_3) \ , && (3.10a) \\ \varepsilon(g_1, g_2) &= \varepsilon(g_2, g_1)^{-1} \ , && (3.10b) \\ \varepsilon(g, g) &= 1 \ . && (3.10c) \end{aligned}$$

We have to keep in mind that there is an additional condition from the requirement of factorizability of the two-loop partition function [13]

$$\varepsilon(g_1, h_1; g_2, h_2) = \varepsilon(g_1, h_1)\varepsilon(g_2, h_2) \ . \tag{3.11}$$

These equations are strong constraints on the form of the allowed phases.

Let us analyse them first in the case of a $\mathbb{Z}_N$ orbifold without Wilson lines. Each space group element g_i is given by $g_i = \theta^{k_i}$. Consideration of Eq. (3.10) leads to

$$\varepsilon(g_1, g_2) = \varepsilon(\theta^{k_1}, \theta^{k_2}) = (\varepsilon(\theta, \theta))^{k_1 + k_2} = 1 \ .$$

Therefore in the case of a $\mathbb{Z}_N$ orbifold there are no nontrivial relative phases allowed and thus no discrete torsion is possible.

In a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold without Wilson lines the situation is different because there are independent elements (such that the first element is not a power of the second) which commute with each other.
We denote a $g \in S$ as $g = \theta^{k_1}\omega^{k_2} = (k_1, k_2)$. By investigating the constraint equations we will now find that ε indeed does not necessarily equals one, but that its shape is totally determined. Therefore let us set $g_1 = (k_1, k_2)$, $g_2 = (l_1, l_2)$, $g_3 = (m_1, m_2)$ in the first of the constraint equations. Then Eq. (3.10a) reads

$$\varepsilon((k_1 + l_1, k_2 + l_2), (m_1, m_2)) = \varepsilon((k_1, k_2), (m_1, m_2))\varepsilon((l_1, l_2), (m_1, m_2))$$

and is nothing else than the defining equation for the exponential function. Thus the phase must read

$$\varepsilon((k_1, k_2), (l_1, l_2)) = e^{C\,(k_1 l_1 \pm k_1 l_2 \pm k_2 l_1 \pm k_2 l_2)} \ ,$$

with some constant C.
Now the second of the constraint equations must be imposed and restricts the sum of the products in the exponential to the antisymmetric combination $k_1 l_2 - k_2 l_1$. An overall $\pm$ can be absorbed in C. Therefore we have the unique allowed choice of the phase

$$\varepsilon(g_1, g_2) = \varepsilon((k_1, k_2), (l_1, l_2)) = e^{C(k_1 l_2 - k_2 l_1)} \ , \tag{3.12}$$

[13]The phase ε now depends on four space group elements the two-loop vacuum-to-vacuum diagram is characterised by four space group twists, corresponding to four non-contractible cycles. There is a fifth non-contractible cycle whose twisting space group element can be expressed through the other four [4].

which then obviously also fulfils the third constraint.
Again by use of Eq. (3.10) it is easy to show that the order of the phase must be N, which is the smaller of the two numbers N, M defining the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold

$$\varepsilon(g,h)^N = \varepsilon(g^N, h) = 1\,.$$

Therefore the discrete torsion phase must be a N-th root of unity, which means that in Eq. (3.12) $C = 2\pi i \frac{m}{N}$ (with $m \in \mathbb{Z}$) and finally

$$\varepsilon(g_1, g_2) = \varepsilon((k_1,k_2),(l_1,l_2)) = e^{\frac{2\pi i m}{N}(k_1 l_2 - k_2 l_1)} \equiv \varepsilon^{k_1 l_2 - k_2 l_1} \qquad m \in \mathbb{Z}\ . \tag{3.13}$$

Now let us try to connect the concept of a discrete torsion phase with our previous observation that transforming the shift vectors by adding $\mathrm{E}_8 \times \mathrm{E}_8$ roots will change the spectrum and thus the model under certain circumstances.
Assume that there are two models characterised by two shift vectors $\{V_1, V_2\}$ and $\{\tilde{V}_1, \tilde{V}_2\}$ respectively and $\tilde{V}_i = V_i + \Delta V_i$ with ΔV_i vectors of the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice. Both sets of shifts are assumed to be of strong modular invariant form, leading to the constraints of Eq. (3.7) on the ΔV_i.
For a state in the sector twisted by $\theta^{k_1}\omega^{k_2}$, the transformation of the shifts results in a change of the transformation phase under the nontrivial[14] centralizer element $\theta^{l_1}\omega^{l_2}$ which reads

$$\Delta\phi = -2\pi i (k_1 l_2 - k_2 l_1) V_1 \cdot \Delta V_2 \tag{3.14}$$

It is obvious from Eq. (3.7) that there is no phase change if both shift vectors are shifted by the same root. Therefore we take $\Delta V_1 \neq \Delta V_2$ and $V_1 = \frac{\Lambda_1}{N}$, with N being the smaller of the two numbers N, M defining the orbifold (this is without loss of generality) and $\Lambda_1 \in T_{\mathrm{E}_8\times\mathrm{E}_8}$. Then we have

$$V_1 \cdot \Delta V_2 = \frac{a}{N} \qquad a \in \mathbb{Z}\ ,$$

due to scalar products of $\mathrm{E}_8 \times \mathrm{E}_8$ roots being integer valued. Thus we just define

$$\varepsilon \equiv e^{-2\pi i V_1 \cdot \Delta V_2} = e^{-2\pi i a/N} \tag{3.15}$$

and have a nontrivial change of the projection phase by an N-th root of unity. The additional phase factor for the model $\{\tilde{V}_1, \tilde{V}_2\}$ therefore reads

$$e^{\Delta\phi} = \varepsilon^{k_1 l_2 - k_2 l_1}\ . \tag{3.16}$$

This phase obviously belongs to the pair of space group elements $(\theta^{k_1}\omega^{k_2}, \theta^{l_1}\omega^{l_2})$ and we could call it $\varepsilon(\theta^{k_1}\omega^{k_2}, \theta^{l_1}\omega^{l_2}) = \varepsilon(g_1, g_2)$. It is just of the form of Eq. (3.13).
Thus we can interprete the model $\{\tilde{V}_1, \tilde{V}_2\}$ as the model $\{V_1, V_2\}$ with a discrete torsion phase switched on. The conclusion is that we can generate models with a nontrivial discrete torsion phase just from the models with trivial discrete torsion phases by adding $\mathrm{E}_8 \times \mathrm{E}_8$ roots to the shift vectors.

[14]By 'nontrivial' we mean that the centralizer element is not a power of the constructing element $\theta^{k_1}\omega^{k_2}$.

Thus in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case there are always 3 very similar models, which differ just by discrete torsion phases in the partition function, or in other words by certain transformations on the shifts. The similarity of these models is obvious from the fact that they differ only by the transformation phase for states of the twisted sectors. Thus the gauge group and untwisted sector matter representations are always the same, while the twisted sector matter representations can be totally different. For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold the discrete torsion phase can take three different values. In fact two of these three models are the same after interchanging appropriate twisted sectors. To specify which of these three models is the one with no discrete torsion is just an arbitrary statement and makes not much sense after our discussion of the equivalent description of adding $\mathrm{E}_8 \times \mathrm{E}_8$ roots to the shifts. Therefore we will call the multiples of models with very similar features *brother models*,[15] to emphasise that they are all on the same level of importance.

[15] We prefer the name *brother models* rather than *parent model* for the model with trivial discrete torsion phase, as was introduced in [22].

3.4 Guidelines for $\mathbb{Z}_3 \times \mathbb{Z}_3$ model building

With the feeling for specific properties of $\mathbb{Z}_3 \times \mathbb{Z}_3$ models which we gained during the discussions in the first two sections of this chapter, we are now in position to make some more general statements about the construction of three-generation models. There are many examples of three-generation models in the literature, see e.g. [28] for $\mathbb{Z}_3$ and [29] for $\mathbb{Z}_2 \times \mathbb{Z}_2$ models. But for a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold there is only the one example of [22].
The most natural set-up for this kind of models with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ point group would be, to have the three matter generations localised at the three degenerate fixed points of a two dimensional SU(3) subtorus, and no Wilson line distinguishing between them. To construct such a model we can therefore use two properly chosen shift vectors and one or two Wilson lines. By switching on a third Wilson line we would destroy the equivalence of the fixed points at which the three generations should be located. Nevertheless it is also possible to construct three generation models with three Wilson lines, but then the generations will stem from different sectors. We will emphasise the first case, because in the next chapter we want to discuss a certain symmetry property of the generations.
Therefore first we have to choose a set of shifts $\{V_1, V_2\}$. There is a huge variety of allowed $\mathbb{Z}_3 \times \mathbb{Z}_3$ shifts with different unbroken gauge groups. In total there are more than hundred different models possible by only using these two shifts. Tab. (3.6) shows only a fraction of these, just to give a feeling for the variety of possible unbroken gauge groups.

In [30, 31] it was recently shown that for the aim of a three-generation model with standard model gauge group, the choice of an unbroken SO(10) with localised **16**-plets as the origin for the standard model generations is a good one. For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold the unique possibility to have properly localised left-handed states is in the $T_{(2,2)}$ sector. Now a problem arises. On the one hand there are a lot of inequivalent sets of shifts, breaking to SO(10) in the observable sector. But on the other hand due to the constructing twist $v = 2\,(v_1 + v_2) = 2 \cdot \frac{1}{3}(1, 1, -2)$ of $T_{(2,2)}$, the local shift $V = 2\,(V_1 + V_2)$ is a $\mathbb{Z}_3$ shift. From the discussions of the $\mathbb{Z}_3$ orbifold [2] we know, that for $\mathbb{Z}_3$ shifts there is no unbroken SO(10) and therefore it is not possible for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold to have SO(10) as local gauge group and the corresponding localised **16**-plets in $T_{(2,2)}$.
Thus the most promising ansatz would be to take two shifts with an unbroken SO(10) and 9 equivalent **16**-plets in one twisted sector $T_{(\tilde{k},\tilde{l})}$ with a bulk torus. There are many possibilities of allowed shifts for this set-up. With a first Wilson line the SO(10) can be broken to a left-right symmetric model (LR model)[16] with three generations in $T_{(\tilde{k},\tilde{l})}$, coming from three full **16**-plets. Then there is still the possibility of turning on a second Wilson line in the bulk torus of $T_{(\tilde{k},\tilde{l})}$. This Wilson line can be chosen such that it breaks

[16]Left-right symmetric means that the gauge group is

$$\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R\ .$$

To receive the standard model from a LR model the $\mathrm{SU}(2)_R$ will later be broken to $\mathrm{U}(1)_Y$. A generation of states for a LR model is constituted of the representations$(\mathbf{3},\mathbf{2},\mathbf{1}) + (\bar{\mathbf{3}},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2})$ and the SM Higgs is $(\mathbf{1},\mathbf{2},\mathbf{2})$.

shifts V_1, V_2	unbroken gauge group
$V_1 = \frac{1}{3}(2,1,1,0^5)(1,-2,3,0^5)$ $V_2 = \frac{1}{3}(1,2,-1,0^5)(3,1,2,0^5)$	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{E}_6 \times (\mathrm{U}(1))^3$
$V_1 = \frac{1}{3}(2,1,1,0^5)(1,-2,3,0^5)$ $V_2 = \frac{1}{3}(3,3,1,3,0^4)(3,3,1,3,0^4)$	$\mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SO}(10) \times \mathrm{SU}(2) \times (\mathrm{U}(1))^4$
$V_1 = \frac{1}{3}(1,1,0^6)(1,-2,3,0^5)$ $V_2 = \frac{1}{3}(3,3,0,2,1^4)(3,1,3,3,1,1,0^2)$	$\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{SO}(10) \times \mathrm{SU}(2) \times (\mathrm{U}(1))^3$
$V_1 = \frac{1}{3}(2,1,1,0^5)(1,-2,3,0^5)$ $V_2 = \frac{1}{3}(3,2,1,3,1^3,0)(3,1,3,3,1,1,0,0)$	$\mathrm{SU}(3)^3 \times \mathrm{SO}(10) \times \mathrm{SU}(2) \times (\mathrm{U}(1))^4$
$V_1 = \frac{1}{3}(2,1,1,0^5)(1,-2,3,0^5)$ $V_2 = \frac{1}{3}(3,3,1,3,1^4)(3,3,1,2,1,0^3)$	$\mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^3$
$V_1 = \frac{1}{3}(-1,3,0^6)(-1,3,0^6)$ $V_2 = \frac{1}{3}(3,-1,3^6)(-1,2,2,1,0^4)$	$\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \times (\mathrm{U}(1))^5$
$V_1 = \frac{1}{3}(-1,3,0^6)(-1,3,0^6)$ $V_2 = \frac{1}{3}(3,1,2^6)(-1,3,3,1^3,0^2)$	$\mathrm{SU}(7) \times \mathrm{SO}(8) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^4$
$V_1 = \frac{1}{3}(-1,3,0^6)(-1,3,0^6)$ $V_2 = \frac{1}{3}(0^8)(2,-2,3,1^3,0^2)$	$\mathrm{SO}(14) \times \mathrm{SU}(4) \times \mathrm{SU}(4) \times (\mathrm{U}(1))^3$

Table 3.6: A selection of different $\mathbb{Z}_3 \times \mathbb{Z}_3$ models with two shifts.

$\mathrm{SU}(2)_R$, leading finally to the standard model gauge group. The difficulty in this case will be that this Wilson line also projects out parts of the **16**. Therefore we can not obtain the full standard model generations from the **16**-plets. Of course there is the hope that the missing states accidentally come from other representations in the spectrum of the LR model. But it will remain accidental to get them with the right hypercharge quantum numbers. We therefore will not try to break the LR further to the standard model gauge group, but use the second Wilson line only to project out as many unwanted states as possible.

The described procedure is illustrated in the construction of two three-generation models, with respectively LR and SU(5) observable gauge group, in the next section.

3.4.1 Construction of a three-generation LR model

Now we closely follow the construction outline of the previous section and take the two shifts

$$\begin{aligned} V_1 &= \tfrac{1}{3}(2,1,1,0,0,0,0,0) \quad (1,-2,3,0,0,0,0,0) \;, \\ V_2 &= \tfrac{1}{3}(3,3,1,3,0,0,0,0) \quad (3,3,1,3,0,0,0,0) \;, \end{aligned}$$

which break the gauge group as follows

$$\mathrm{E}_8 \times \mathrm{E}_8 \to \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SO}(10) \times \mathrm{SU}(2) \times (\mathrm{U}(1))^4 \;.$$

From a technical point of view it is important to notice that the unbroken observable SO(10) lives in the last five entries of the eight dimensional root vectors of the first E_8.[17] Thus the simple SO(10) roots are $\{(0,0,0,1,-1,0,0,0)$, $(0,0,0,0,1,-1,0,0)$, $(0,0,0,0,0,1,-1,0)$, $(0,0,0,0,0,0,1,-1)$, $(0,0,0,0,0,0,1,1)\}$. The important observation now is that there are **16**-plets of SO(10) in the $T_{(1,0)}$ sector. Therefore the aim will be to place a Wilson line in the first subtorus $T^{(1)}$ which distinguishes between its three fixed points and provides us with three generations from three of the nine **16**-plets. This job will be done by

$$A_1 = \tfrac{1}{3}(3,3,3,0,0,1,1,1) \quad (2,-2,3,-2,1,1,1,0)$$

and the resulting unbroken gauge group is

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times (\mathrm{U}(1))^8 \;.$$

The full spectrum is given in Tab. (C.4) in App. C and we notice that we have the three LR generations in $T_{(1,0)}$ and four 'Higgs candidates'.[18] One of these stems from the untwisted sector U_3 and the other three from $T_{(2,0)}$. The situation is depicted in Fig. (3.4).
A shortcoming of this model is of course the chiral states $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\bar{\mathbf{4}})$, $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{4})$ which cannot be decoupled from the wanted matter content by letting some of the singlet fields acquire vevs.

With a second Wilson line in the bulk torus of $T_{(1,0)}$ we can now try to get rid of these unwanted states. We choose

$$A_2 = \tfrac{1}{3}(0,1,-1,0,0,0,0,0) \quad (5,-5,-5,-2,0,-3,3,9)$$

and notice that the unbroken observable gauge group is exactly the LR gauge group

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(3) \times (\mathrm{U}(1))^{10} \;.$$

But from the spectrum in Tab. (C.4) we see that there are still states, SU(2) dublets, which do not decouple but receive masses from the Higgs vevs.

[17] Due to this information and the special form of the Wilson line we can define the $B-L$ generator as

$$T_{B-L} \equiv (0,0,0,0,0,-2/3,-2/3,-2/3)(0^8) \;.$$

[18] There is also an additional electron corresponding to the representations $(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{-1}, (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_1$ in the untwisted sector.

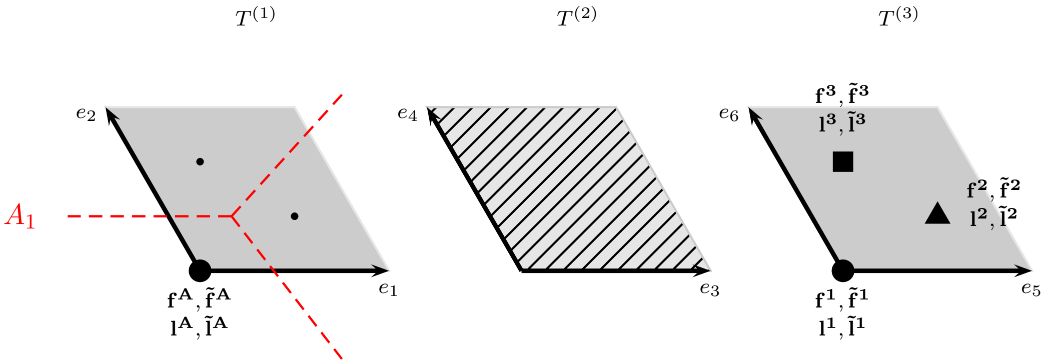


Figure 3.4: Assignments of the three families ($f^A = (\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1}; \mathbf{1})_{1/3}$, $\tilde{f}^A = (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{2}, \mathbf{1}; \mathbf{1})_{-1/3}$, $l^A = (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}; \mathbf{1})_{-1}$, $\tilde{l}^A = (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}; \mathbf{1})_{1}$) to the fixed points of $T_{(1,0)}$ in the case with one Wilson line A_1 in the first subtorus.

3.4.2 Construction of a three-generation SU(5) model

In a recent work [32] a discrete non-Abelian flavor symmetry S_3 was discussed, with its realisation in an SU(5) GUT. The aim of the next chapter is the discussion of discrete family symmetries. We now construct an SU(5) model to see later on, which family symmetry will appear. In spite of all their shortcomings, this SU(5) model will have the advantage of an increased clarity of the spectrum.
From Tab. (3.6) we take the two shifts

$$\begin{array}{rcll} V_1 & = & \frac{1}{3}(2,1,1,0,0,0,0,0) & (1,-2,3,0,0,0,0,0)\ , \\ V_2 & = & \frac{1}{3}(3,3,1,3,1,1,1,1) & (3,3,1,2,1,0,0,0)\ , \end{array}$$

which directly break $\mathrm{E}_8 \times \mathrm{E}_8$ to SU(5)

$$\mathrm{G}_{unbroken} = \mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^4\ .$$

We call it model (C) and present the matter content in Tab. (C.6) in App. C. From Tab. (C.6) we notice that the SU(5) families are distributed on the sectors $T_{(1,0)}$ and $T_{(2,0)}$.[19] The $\mathbf{10}$ is located on the nine fixed tori of $T_{(1,0)}$, the $\overline{\mathbf{5}}$ on those of $T_{(2,0)}$ and the higgs is the $\overline{\mathbf{5}}$ in U_2 for the d-type quarks and the $\mathbf{5}$ in $T_{(1,2)}$ for the u-type ones.
By adding two Wilson lines in $T^{(1)}$ and $T^{(2)}$

$$\begin{array}{rcll} A_1 & = & \frac{1}{3}(2,-2,-2,2,2,2,2,2) & (5,-5,-2,-2,3,-9,-3,3) \\ A_2 & = & \frac{1}{3}(4,4,0,0,0,0,0,0) & (2,-2,0,-2,-2,-2,-2,4) \end{array}$$

the gauge group further breaks to

$$\mathrm{G}_{unbroken} = \mathrm{SU}(5) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^{10}$$

[19] An SU(5) generation consists of $\overline{\mathbf{5}} + \mathbf{10}$.

and there are three remaining generations distributed on $T_{(1,0)}$ and $T_{(2,0)}$ (see Tab. (C.7). It is a nice feature of this model that all extra states are vectorlike and thus perhaps can be made heavy by giving vevs to some of the singlets.

In spite of all their shortcomings (e.g. the additional chiral states of model (C) and model (C')) the two constructed three-generation models are nice toy models to illustrate the specific features of the SU(3) tori with respect to discrete family symmetries. All the three generations come from the three degenerate fixed points of one such subtorus, and in chapter 4 it will be explained how due to this property a non-Abelian discrete family symmetry arises, which restricts the couplings between the states.

Chapter 4

Discrete symmetries

In order to get rid of certain unwanted operators, the standard model, the MSSM or a grand unified theory can be extended by an additional discrete symmetry group G. With a suitable chosen G this approach amongst others can lead to realistic proton decay rates and fermion mass matrices. The extended gauge group of the theory then reads

$$\text{(gauge group)} \times G \ .$$

To avoid contradictions with quantum gravity effects the discrete symmetry must stem from an enhanced gauge symmetry which is broken at a lower scale. Approaches to classify the allowed Abelian discrete gauge symmetries of this type can be found in [33, 34]. But particularly to receive realistic fermion mass matrices also non-Abelian discrete symmetries are considered in the literature.[1]

In string theory there exist some selection rules for the allowed couplings of states. They provide us with strong constraints on the renormalizable and non-renormalizable couplings and enormously reduce their number. In this way string theory determines the interactions of the massless spectrum states and therefore the phenomenological features of the particular model in great detail. These constraints can be interpreted as discrete symmetries which are imposed on the couplings, in addition to the existing gauge group. Thus there is a natural way how discrete symmetry groups arise and they do not have to be adopted artificially from the outside. In particular these discrete symmetries can be of non-Abelian type. We will especially analyse this case in the following.

But first we will explain the selection rules for heterotic string theories on orbifolds, specifying on the space group rule. We will conclude the chapter with the construction of realistic mass matrices for $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold models, in a similar approach to [36, 37, 38, 39, 40].

4.1 Selection rules for orbifolds

Interactions between the states of the massless spectrum have to be considered by investigating scattering amplitudes of these states. This is done by constructing the vertex

[1] For a comprehensive review of the recent work on this field see [35].

operators of the conformal field theory for the emission of the particular untwisted and twisted states. Out of these vertex operators the appropriate correlation functions are built and evaluated. By performing the calculations it is found, that some general statements can be made in order for the correlation functions, and thus the couplings, not to vanish. These statements are formulated in so-called *selection rules*. Details about the derivation of the rules can be found in [41, 42]. For a clear summary see [43]. In this chapter we will briefly explain their origin and then use them to investigate the couplings of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.

For the construction of the vertex operators it is important to notice that the fermionic fields of the theory have to be bosonized. For an introduction to this bosonization see [11]. The NSR fermions of the superstring are bosonized with momenta on the SO(10) weight lattice. Thus in the vertex operators there appear certain quantum numbers, as the SO(10) momenta of the right movers or the $\mathrm{E}_8 \times \mathrm{E}_8$ momenta of the left movers. Also the constructing space group elements of the involved states are important and conservation laws for the numbers n, describing the fixed points where the states are located, can be formulated. From the calculation of the correlation functions certain conservation laws on these quantum numbers arise, which have to be fulfilled in order for them, and thus the coupling not to vanish. They are called *space group*, *R-momentum* and *gauge momentum selection rule.*

Space group: The space group rule can easily be derived, without any nasty calculation, just from the argument that two interacting strings must join, for the time they interact [42]. This is pictorially shown in Fig. (4.1). There we have two interacting string states from the sectors T_{g_1} and T_{g_2}. While interacting the strings join and therefore constitute a string whose states correspond to the sector $T_{g_2 g_1}$. Finally they separate again into two strings with corresponding states in $T_{\tilde{g}_1}$ and $T_{\tilde{g}_2}$. Obviously, for this process to be allowed

$$g_2 g_1 = \tilde{g}_2 \tilde{g}_1$$

must hold. This can be generalised to more complicated interactions of m incoming and n outgoing strings

$$\prod_{i=1}^{m} g_i = \prod_{j=1}^{n} \tilde{g}_j \ .$$

We now bring the constructing elements of the outgoing states $\tilde{g}_j$ to the left hand side, what means that we view them as incoming states with opposite orientation (inverse twisting). Then the equation, constraining the space group elements corresponding to the interacting states, reads

$$\prod_{i=1}^{m+n} g_i = \mathbb{1} \ , \tag{4.1}$$

where we set $g_{m+j} = \tilde{g}_{n-j+1}$. The meaning of this selection rule is that n states can have a non-vanishing coupling only if the product of their corresponding constructing

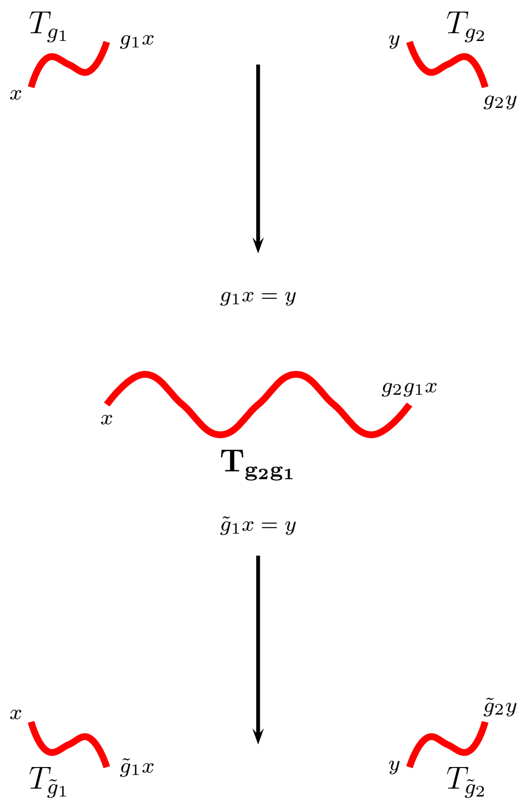


Figure 4.1: Interaction of two states, corresponding to the strings twisted by g_1 and g_2.

space group elements equals the identity $\mathbb{1}$.
By writing the space group elements as $(\theta^{(i)}, n_je_j)$ or equivalently $(\theta^{(i)}, (\mathbb{1} - \theta^{(i)})f)$ (with f the appropriate fixed point coordinate), we can separate Eq. (4.1) into two parts, corresponding to the two factors of the semi-direct product. In the literature it is sometimes referred to them as *space group* and *point group* selection rule. We denote the product of the involved space group elements by

$$\prod_{i=1}^{n}(\theta^{(i)}, n_k^{(i)}e_k) = (\prod_{i=1}^{n}\theta^{(i)}, v)$$

and thus can state the two rules as

$$\prod_{i=1}^{n}\theta^{(i)} = \mathbb{1} \qquad (\textit{point group rule}) \;, \tag{4.2a}$$

$$v \in \{\sum_{i,k} a_k^{(i)}(1-\theta^{(i)})e_k |\, a_k^{(i)} \in \mathbb{Z}\} \qquad (\textit{space group rule}) \;. \tag{4.2b}$$

From this form of the space group selection rule it is obvious that it demands nothing but that the six dimensional real vector v is constrained to lie on a lattice which is spanned by the vectors $\{(1-\theta^{(i)})e_k\}$ where $i = 1, ..., n$ and $k = 1, ..., 6$.

R-momentum: In the literature the conservation law of the SO(10) momenta $q_{sh} = q + v$ (the *H-momenta*) is usually called H-momentum selection rule. It restricts the sum of the SO(10) momenta of the states in a certain coupling to vanish. To state this rule completely correct, also in the case of couplings of excited states (oscillators involved in the coupling), we stick to [3] and define the *R-momenta*

$$r^a = q^a_{sh} - \underline{N} + \underline{\bar{N}} \ , \tag{4.3}$$

with $a = 1, ..., 3$. Here $\underline{N}$ and $\underline{\bar{N}}$ are not the number operators defined in App. A.2 but operators, which just count the numbers how often the creation operators $\tilde{\beta}^a$ and $\tilde{\beta}^{\bar{a}}$ appear in the state. Therefore they are always integer valued. The general selection rule for each component of these R-momenta reads

$$\sum_i r^a_{(i)} = -1 \quad \mathrm{mod}\ N^{(a)} \ , \tag{4.4}$$

where (i) again labels the states involved in the considered coupling and $N^{(a)}$ is the order of the spacetime twist in the torus $T^{(a)}$.
In the untwisted sector there are three possible R-momenta for left-handed non-oscillator states (thus H-momenta), given by $(-1, 0, 0), (0, -1, 0), (0, 0, -1)$, where we denote only the last three entries of the SO(10) momenta. Therefore we think of the untwisted sector as to be divided into three subsectors U_1, U_2, U_3 to which the untwisted states with respectively one of the three possible H-momenta belong. Due to Eq. (4.4) the renormalizable coupling $U_1U_2U_3$ is always allowed.

Gauge momentum: The gauge momenta $p_{sh} = p + V$ ($E_8 \times E_8$ roots) of the states in a given correlation function must add up to zero in order for it not to vanish. This rule means nothing else than that the allowed couplings must be gauge invariant.

But these rules only tell whether a coupling is allowed or vanishes. If it is allowed it appears with a certain coupling strength, which can be determined by calculating the appropriate correlation function. These coupling strengths are calculated e.g. in [42, 44]. It is found that for a coupling of states at various fixed points the strength of the coupling decreases exponentially with increasing distance between them. Thus couplings between states, localised at distinct fixed points, are exponentially suppressed.

The selection rules and this exponential suppression together can lead to interesting mass hierarchies for the fermions in the theory. They determine which couplings appear in the superpotential and strongly constrain it.

4.1.1 $\mathbb{Z}_m$ symmetries from the space group

It is instructive to think of the selection rules, like the R-momentum or the space group rule, as discrete symmetries which are acting on the states and under whose action the couplings must be invariant. The selection rules are formulated as constraint equations on some discrete quantum numbers. Thus e.g. the point group rule Eq. (4.2a) can be written as an equation for the exponents k_j of the spacetime twists, which enter a certain coupling $\sum_j k_j = 0 \mod N$ and thus just states that for a $\mathbb{Z}_N$ orbifold there is a $\mathbb{Z}_N$ symmetry constraining the couplings between states of different sectors.
It is a bit less straightforward to see that also the space group rule provides us with a $\mathbb{Z}_m$ symmetry group. Nevertheless, we will concentrate on these discrete symmetries from the space group and in this section, as a warming up for the following, explicitly show how they arise in the case of a $\mathbb{Z}_3$ and a $\mathbb{Z}_4$ orbifold. The starting point for these considerations will always be the space group rule in the form of Eq. (4.2b), and we will only consider renormalizable couplings.

$\mathbb{Z}_3$ symmetry for the $\mathbb{Z}_3$ orbifold

In the case of the $\mathbb{Z}_3$ orbifold[2] the only renormalizable Yukawa coupling which is allowed by the point group and the R-momentum selection rule is of the form[3]

$$T_1\,T_1\,T_1\ .$$

Additionally imposing the space group selection rule means that we have to consider the expression $g_1 g_2 g_3$ where all three construction elements correspond to the first twisted sector. We do the following consideration only for the first torus spanned by $\{e_1, e_2\}$ (but it can be analogously done for the other tori) and thus set $g_1 = (\theta, n_1^{(1)}e_1 + n_2^{(1)}e_2)$, $g_2 = (\theta, n_1^{(2)}e_1 + n_2^{(2)}e_2)$, $g_3 = (\theta, n_1^{(3)}e_1 + n_2^{(3)}e_2)$ (where $n_1^{(1)}, n_2^{(1)}, n_1^{(2)}, n_2^{(2)}, n_1^{(3)}, n_2^{(3)} \in \{0, 1\}$) and calculate this product by using the orbifold twist properties $\theta e_1 = e_2$, $\theta e_2 = -e_1 - e_2$:

$$\begin{aligned} g_1 g_2 g_3 &= (\theta, n_1^{(1)}e_1 + n_2^{(1)}e_2)(\theta^2, n_1^{(3)}e_2 - n_2^{(3)}e_1 - n_2^{(3)}e_2 + n_1^{(2)}e_1 + n_2^{(2)}e_2) \\ &= (\mathbb{1}, (n_1^{(1)} + n_2^{(3)} - n_1^{3} - n_2^{(3)})e_1 + (n_2^{(1)} + n_1^{(2)} - n_2^{(2)} - n_1^{(3)})e_2) \quad . \end{aligned}$$

The lattice on which the resulting shift must lie is spanned by $(1-\theta)e_1 = e_1 - e_2$, $(1-\theta)e_2 = 2e_2 + e_1$. This lattice is obviously coarser than the one spanned by $\{e_1,\, e_2\}$. It can be written as $\{\, ae_1 \,+\, be_2 |\, a, b \in \mathbb{Z},\, a + b = 0 \mod 3\,\}$. Thus we find that the space group selection rule Eq. (4.2b) reads

$$v \in \{\, ae_1 \,+\, be_2 |\, a, b \in \mathbb{Z},\, a + b = 0 \mod 3\,\}\ .$$

[2] For details on the $\mathbb{Z}_3$ orbifold see [9, 1, 2].

[3] There would be another allowed coupling of the form $T_2T_2T_2$, but we already stated that in T_2 there are just the antiparticles of the states in T_1.

From the explicit form of v we find $a = n_1^{(1)} + n_2^{(3)} - n_1^{(3)} - n_2^{(2)}$ and $b = n_2^{(1)} + n_1^{(2)} - n_2^{(2)} - n_1^{(3)}$ and further rewrite the rule

$$n_1^{(1)} + n_2^{(1)} + n_2^{(3)} + n_1^{(2)} - 2(n_2^{(2)} + n_1^{(3)}) = 0 \mod 3 \ .$$

By redefining $n^{(i)} \equiv n_1^{(i)} + n_2^{(i)}$ we arrive at

$$\sum_{i=1}^{3} n^{(i)} = 0 \mod 3 \ . \tag{4.5}$$

Here $n^{(i)} \in \{0, 1, 2\}$ and it is obvious from the geometry of the SU(3) root lattice that each fixpoint is uniquely characterised by one number with that range of values.
The interesting point is that the constraint Eq. (4.5) can be interpreted as a $\mathbb{Z}_3$ symmetry which has to be imposed on the Yukawa couplings. To define the action of this discrete symmetry on the states, we label a twisted sector state by the number n which indicates its localisation and denote it $|\Phi_n\rangle$. The space group $\mathbb{Z}_3$ symmetry then acts on it as $|\Phi_n\rangle \mapsto \exp(\frac{2\pi i n}{3})|\Phi_n\rangle$. Thus the whole Yukawa coupling transforms as

$$|\Phi_{n_1}\rangle|\Phi_{n_2}\rangle|\Phi_{n_3}\rangle \mapsto e^{\frac{2\pi i}{3}(n_1+n_2+n_3)}|\Phi_{n_1}\rangle|\Phi_{n_2}\rangle|\Phi_{n_3}\rangle \stackrel{!}{=} 1 \cdot |\Phi_{n_1}\rangle|\Phi_{n_2}\rangle|\Phi_{n_3}\rangle \ .$$

$\mathbb{Z}_2$ symmetries for the $\mathbb{Z}_4$ orbifold

This section is inserted for the reader, who knows about the details of the $\mathbb{Z}_4$ orbifold. It shows that the construction of the $\mathbb{Z}_3$ case can be easily generalised to other orbifolds. It can be skipped, because in later sections we will not refer to it.
In the case of the $\mathbb{Z}_4$ orbifold[4] the investigation of the space group selection rule reveals some additional difficulties due to the existence of the $\mathbb{Z}_2$ torus. Thus we must consider the $\mathbb{Z}_2$ and the two Z_4 tori separately.
The point group and R-momentum selection rules allow Yukawa couplings of three types:

$$U_1 U_2 U_3 \quad T_1 T_1 T_2 \quad T_2 T_2 U_1 \ .$$

Due to the choice of the shift $v = \frac{1}{4}(-2, 1, 1)$ in the first torus $T^{(1)}$ a $\mathbb{Z}_2$ symmetry is moded out while in $T^{(2)}$ and $T^{(3)}$ it is a $\mathbb{Z}_4$ symmetry. Now the same calculation as in the $\mathbb{Z}_3$ case will be performed for each of the three tori $T^{(i)}$ $(i = 1, 2, 3)$ and each of the three types of couplings. The calculations for the second and the third torus lead to the same result.

$\mathbf{T^{(1)}}$: The first torus is spanned by the basis vectors $\{e_1,\ e_2\}$. The coupling of type $U_1 U_2 U_3$ and $T_2 T_2 U_1$ trivially fulfil the space group selection rule, because for T_2 the first torus is a bulk torus. Therefore only the case $T_1 T_1 T_2$ has to be investigated. The corresponding space group elements are written as $g_1 = (\theta, n_1^{(1)} e_1 + n_2^{(1)} e_2)$, $g_2 =$

[4] For a detailed discussion of the $\mathbb{Z}_4$ orbifold see [45, 8, 46].

$(\theta, n_1^{(2)} e_1 + n_2^{(2)} e_2)$ and $g_3 = (\theta^2, 0)$ (where $n_1^{(1)}, n_2^{(1)}, n_1^{(2)}, n_2^{(2)} \in \{0, 1\}$) and use will be made of $\theta e_1 = -e_1$, $\theta e_2 = -e_2$:

$$\begin{aligned} g_1 g_2 g_3 &= (\theta, n_1^{(1)} e_1 + n_2^{(1)} e_2)(\theta^3, n_1^{(2)} e_1 + n_2^{(2)} e_2) \\ &= (\mathbb{1}, (n_1^{(1)} - n_1^{(2)}) e_1 + (n_2^{(1)} - n_2^{(2)}) e_2) \; . \end{aligned}$$

From $(1-\theta)e_1 = 2e_1$, $(1-\theta)e_2 = 2e_2$ and $(1-\theta^2)e_1 = 0$, $(1-\theta^2)e_2 = 0$ it can be seen that the lattice on which v is constrained to lie is spanned by $\{2e_1, \, 2e_2\}$ and thus has a double gridwidth compared to the $\{e_1, \, e_2\}$-lattice. Therefore it can be written as $\{ae_1 + be_2 | \, a, b \in \mathbb{Z}, \, a = 0 \mod 2, \, b = 0 \mod 2\}$ and the space group selection rule reads

$$v \in \{ae_1 + be_2 | \, a, b \in \mathbb{Z}, \, a = 0 \mod 2, \, b = 0 \mod 2\} \; .$$

By looking at the explicit form of v we find

$$\begin{aligned} n_1^{(1)} + n_1^{(2)} &= 0 \mod 2 \; , \\ n_2^{(1)} + n_2^{(2)} &= 0 \mod 2 \; . \end{aligned}$$

The meaning of those two equations is nothing but that the Yukawa couplings are constrained by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry which acts on the states $|\phi_{(n_1,n_2)}\rangle$ sitting at fixpoints in $T^{(1)}$ characterised by (n_1, n_2) as $|\phi_{(n_1,n_2)}\rangle \mapsto (\exp(\pi i n_1), \exp(\pi i n_2)) |\phi_{(n_1,n_2)}\rangle$.[5]

$\mathbf{T^{(2)}}$: The second torus is spanned by $\{e_3, \, e_4\}$. Here the situation is different compared to the previous one due to the appearing linear combinations of states at inequivalent fixed points[6] on the torus. Both types of couplings $T_1 T_1 T_2$ and $T_2 T_2 U_1$ must be considered.

$T_1 T_1 T_2$: With the space group elements $g_1 = (\theta, n^{(1)} e_3)$, $g_2 = (\theta, n^{(2)} e_3)$ and $g_3 = (\theta^2, n_3^{(3)} e_3 + n_4^{(3)} e_4)$ we calculate (using $\theta e_3 = e_4$, $\theta e_4 = -e_3$)

$$\begin{aligned} g_1 g_2 g_3 &= (\theta, n^{(1)} e_3)(\theta^3, n_3^{(3)} e_4 - n_4^{(3)} e_3 + n^{(2)} e_3) \\ &= (\mathbb{1}, (n^{(1)} - n_3^{(3)}) e_3 + (n^{(2)} - n_4^{(3)}) e_4) \; . \end{aligned}$$

Due to $(1-\theta)e_3 = e_3 - e_4$ and $(1-\theta)e_4 = e_3 + e_4$ the constraining lattice for v can be written as $\{ae_3 + be_4 | \, a, b \in \mathbb{Z}, \, a + b = 0 \mod 2\}$. Thus v is constrained to satisfy

$$v \in \{ae_3 + be_4 | \, a, b \in \mathbb{Z}, \, a + b = 0 \mod 2\} \; ,$$

which means

$$n^{(1)} + n^{(2)} + n_3^{(3)} + n_4^{(3)} = 0 \mod 2 \; .$$

[5]With the notation (a, b) we want to emphasise that we mean the direct product of two $\mathbb{Z}_2$ groups.

[6]*Inequivalent fixed points* here means that there are two fixed points in the second twisted sector T_2 of the $\mathbb{Z}_4$ orbifold, which are interchanged under the action of the $\mathbb{Z}_4$ twist.

By defining the discrete charge $n^{(3)} \equiv n_3^{(3)} + n_4^{(3)}$ this can be understood as a $\mathbb{Z}_2$ symmetry which acts on it.

$T_2T_2U_1$: The calculation for $g_1 = (\theta^2, n_3^{(1)}e_3 + n_4^{(1)}e_4)$ and $g_2 = (\theta^2, n_3^{(2)}e_3 + n_4^{(2)}e_4)$ is the same as for the first torus and leads to

$$n_i^{(1)} + n_i^{(2)} = 0 \mod 2 \quad i \in \{3, 4\}$$

and thus to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acting on the charges (n_3, n_4).

To summarise the previous considerations, we label a twisted sector state $|\Phi\rangle$ by the order of the twisted sector k and the discrete charges (n_1, n_2), (n_3, n_4), (n_5, n_6) which indicate the corresponding fixpoint in the three tori $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ and thus write

$$|\Phi_{k\,(n_1,n_2)(n_3,n_4)(n_5,n_6)}\rangle \ .$$

Then Yukawa couplings of the form $T_1T_1T_2$ have to be invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acting on (n_1, n_2), a $\mathbb{Z}_2$ acting on $\tilde{n}_3 \equiv n_3 + n_4$ and a $\mathbb{Z}_2$ acting on $\tilde{n}_4 \equiv n_5 + n_6$. Couplings of the form $T_2T_2U_3$ have to be invariant under two $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries acting on (n_3, n_4) and (n_5, n_6).

remark: For the $\mathbb{Z}_4$ orbifold, as for any other orbifold with linear combinations of states,[7] there is an additional selection rule arising from the demand, that the product of the space group elements must include the identity, the γ-*rule*. For each linear combination state there is a new label γ, which is just the eigenvalue under the twist which exchanges the fixed points at which the state is localised. Then the rule reads

$$\prod_i \gamma^{(i)} = 1 \tag{4.6}$$

and means that the coupling should be invariant under the exchanging space group element.

[7]For non-prime orbifolds, with the point group having a non-trivial subgroup, fixed points of a certain twisted sector are exchanged under a twist of lower order than the constructing twist. Then the physical meaningful states are linear combinations of the states localised at the exchanged fixed points, rather than states located exactly at one single point.

4.2 Renormalizable couplings for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold

The next step is to use the selection rules which were discussed in Sec. 4.1 to derive the allowed couplings for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold. There are neither linear nor quadratic couplings allowed because linear ones as terms of the superpotential would break supersymmetry and quadratic ones would generate masses for the states (the states under consideration are all massless). We therefore consider the next order, renormalizable Yukawa couplings of the type

$$\text{(sector A)(sector B)(sector C)} .$$

There will be allowed non-renormalizable couplings involving more states as well. They are important to derive realistic structures of the mass matrices. We will not consider them yet.

Step by step we investigate the R-momentum, the point group, the gauge momentum and the space group rule to see which couplings can appear to give masses to the states.
The R-momentum $r_{(k,l)}$ (with k, l labelling the sector) for the non-oscillator states of a certain sector is found by solving the right mover masslessness condition which gives the appropriate q_{sh}.
As an example, in $T_{(1,0)}$ we have $q_{sh} = \big((0,0,0,1),(0,-1,0,0),(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})\big)$ $+\ v_1$. We count only left chiral states and therefore have $q_{sh} = [(0,-1,0,0)+v_1] = \frac{1}{3}(0,-2,0,-1)$ and the corresponding spinorial q_{sh}. The R-momenta are then calculated with Eq. (4.3) and they are listed for all sectors in Tab. (4.1), consistent with [22].

Investigation of the rule $\sum_{(k,l)} r^i_{(k,l)} = -1 \mod 3$ for each component of $r_{(k,l)}$ (the summation index means that we just sum over the three sectors (k,l) which are involved in the given coupling) provides us with the set of allowed renormalizable couplings given in Tab. (4.2).[8] In Tab. (4.2) we considered only couplings between non-oscillator states.

It can be checked easily that for the R-momentum allowed couplings the point group selection rule is automatically satisfied. Therefore it does not restrict the number of allowed couplings any further and Tab. (4.2) gives all couplings allowed by these two selection rules.
If excited states are considered (oscillator states) the R-momentum rule constrains the properly defined R-momenta due to Eq. (4.4).

The selection rule concerning the gauge momenta p means nothing else than that the couplings must be gauge invariant.

Finally it is very illuminating to consider the restrictions coming from the space group selection rule more precisely. To do so we could investigate the space group rule for each of the 10 couplings of Tab. (4.2). We would then find that in all 10 cases it dictates the same

[8] If we considered right chiral states the sector $T_{(2,2)}$ would be replaced with $T_{(1,1)}$ and the other sectors in the couplings with $T_{(2,2)}$ by their conjugated sectors, meaning e.g. $T_{(1,0)} \leftrightarrow T_{(2,0)}$.

sector	H-momentum $r_{(k,l)}$ (right chiral)	H-momentum $r_{(k,l)}$ (left chiral)
U_1	$(1,0,0)$	$(-1,0,0)$
U_2	$(0,1,0)$	$(0,-1,0)$
U_3	$(0,0,1)$	$(0,0,-1)$
$T_{(1,0)}$	$\frac{1}{3}(1,0,2)$	$\frac{1}{3}(-2,0,-1)$
$T_{(2,0)}$	$\frac{1}{3}(2,0,1)$	$\frac{1}{3}(-1,0,-2)$
$T_{(0,1)}$	$\frac{1}{3}(0,1,2)$	$\frac{1}{3}(0,-2,-1)$
$T_{(0,2)}$	$\frac{1}{3}(0,2,1)$	$\frac{1}{3}(0,-1,-2)$
$T_{(1,2)}$	$\frac{1}{3}(1,2,0)$	$\frac{1}{3}(-2,-1,0)$
$T_{(2,1)}$	$\frac{1}{3}(2,1,0)$	$\frac{1}{3}(-1,-2,0)$
$T_{(1,1)}$	$\frac{1}{3}(1,1,1)$	-
$T_{(2,2)}$	-	$\frac{1}{3}(-1,-1,-1)$

Table 4.1: R-momenta of the different sectors for non-oscillator states.

$\mathbb{Z}_3$ symmetry concerning the quantum numbers $n = 0, 1, 2$ corresponding to the localisation of the fixed points in the twisted sectors of the states involved in the coupling. But it is also possible to prove this $\mathbb{Z}_3$ symmetry in general for an arbitrary number of coupled states from the sectors of an orbifold with an underlying SU(3) root lattice structure. Because it is more elegant, the second proof is presented here.

$\mathbb{Z}_3$ symmetry from the space group: Assume that there is a coupling involving n states, corresponding to the twisted sectors $T_{(l)}$ $(l = 1, ..., n)$

$$(T_{(1)})(T_{(2)})...(T_{(n)}) \ . \tag{4.7}$$

The point group of the underlying orbifold is $\mathbb{Z}_3$, acting on the SU(3) root lattice (a two dimensional torus with the basis vector $\{e_1, e_2\}$). What are the restrictions on the couplings due to the space group?

Multiplying the space group elements g_i (i=1,...,n) corresponding to the coupled

$U_1U_2U_3$	$T_{(1,0)}T_{(0,2)}T_{(2,1)}$	$T_{(2,2)}T_{(2,2)}T_{(2,2)}$
$U_1T_{(0,1)}T_{(0,2)}$	$T_{(2,0)}T_{(2,1)}T_{(2,2)}$	
$U_2T_{(1,0)}T_{(2,0)}$	$T_{(1,0)}T_{(0,1)}T_{(2,2)}$	
$U_3T_{(1,2)}T_{(2,1)}$	$T_{(0,1)}T_{(2,0)}T_{(1,2)}$	

Table 4.2: Renormalizable Yukawa-couplings for non-oscillator states allowed by the R-momentum selection rule (and point group selection rule) in accordance with [47].

states as constructing elements yields

$$\begin{aligned}\prod_{i=1}^{n} g_{(l)} &= (\theta^{(n)}, n_1^{(n)}e_1 + n_2^{(n)}e_2)...(\theta^{(2)}, n_1^{(2)}e_1 + n_2^{(2)}e_2)(\theta^{(1)}, n_1^{(1)}e_1 + n_2^{(1)}e_2) \\ &= (\theta^{(n)}, n_1^{(n)}e_1 + n_2^{(n)}e_2)...(\theta^{(2)}\theta^{(1)}, n_1^{(1)}\theta^{(2)}e_1 + n_2^{(1)}\theta^{(2)}e_2 + n_1^{(2)}e_1 + n_2^{(2)}e_2) \\ &= \Bigg(\prod_{i=1}^{n}\theta^{(i)}, n_1^{(n)}(\theta^{(n-1)}...\theta^{(1)})e_1 + n_2^{(n)}(\theta^{(n-1)}...\theta^{(1)})e_2 \\ &\quad + n^{(n-1)(1)}(\theta^{(n-2)}...\theta^{(1)})e_{(1)} + ... + n_2^{(2)}\theta^1 e_2 + n_1^{(1)}e_1 + n_2^{(1)}e_2\Bigg) \ .\end{aligned}$$

There are $2n$ summands in the right hand part corresponding to the torus translations. Each $\theta^{(i)}$ acts either as $e_1 \to e_2$, $e_2 \to -e_1 - e_2$ or as $e_1 \to -e_1 - e_2$, $e_2 \to e_1$. The constraint $v \in \{\sum_{i,k} a_k^{(i)}(1-\theta^{(i)})e_k \,|\, a_k^{(i)} \in \mathbb{Z}\}$ now can be rewritten as $v \in \{ae_1 + be_2 \,|\, a, b \in \mathbb{Z};\ a + b = 0 \mod 3\}$. Therefore the prefactors in front of the $(-e_1 - e_2)$-terms always sum up to yield $-2n_k^{(i)} = n_k^{(i)} \mod 3$ and we get the space group constraint

$$\sum_{i=1}^{n}(n_1^{(i)} + n_2^{(i)}) = 0 \mod 3 \ .$$

By defining $n^{(i)} \equiv n_1^{(i)} + n_2^{(i)}$, the constraint equation is written in the nicest form

$$\sum_{i=1}^{n} n^{(i)} = 0 \mod 3 \ , \tag{4.8}$$

meaning that the space group selection rule can be understood as a $\mathbb{Z}_3$ symmetry acting on the quantum numbers $n^{(i)}$ which label the localisation of the states. The allowed couplings then must be invariant under this discrete symmetry. There is such a $\mathbb{Z}_3$ symmetry for the localisations in each of the three two dimensional tori of the orbifold, resulting in an over-all $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry which restricts the allowed couplings.

We can illuminate this a bit further by labelling a general state $|\Phi_{n_1n_2n_3}\rangle$ by the three quantum numbers n_i, now corresponding to the three two dimensional tori. The derived $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry then acts as

$$|\Phi_{n_1n_2n_3}\rangle \mapsto (e^{2\pi i n_1/3}, e^{2\pi i n_2/3}, e^{2\pi i n_3/3})|\Phi_{n_1n_2n_3}\rangle \ . \tag{4.9}$$

Eq. (4.8) must hold for each of the three tori, i.e. for each of the n_i. For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold we know already that we have renormalizable couplings of the form $UT\tilde{T}$ and $T\tilde{T}\tilde{\tilde{T}}$ involving states of the twisted sectors. Fig. (4.2) shows the allowed localisations in one of the three tori for each case respectively.
It is obvious from the picture that in subtori where all states are localised they must either be localised at the same fixed point or all at different ones (there is only one of these possibilities depicted in the picture), in order to receive an allowed coupling. If in two of the three considered tori the states are bulk states, the third localised state must live at the origin of the third torus.
In a torus with the situation depicted in the second picture of Fig. (4.2) (one bulk state) the states of the two twisted sectors must be localised at the same point in the orbifold space. To convince oneself of this fact, it is important to notice that the fixed points of T are labelled in a different way than those of $\tilde{T}$. This is due to the fact that the constructing element of the T fixed points is the inverse of the constructing element of the fixed points of the $\tilde{T}$ states. Therefore the fixed points with $n = 1, 2$ of $\tilde{T}$ are exchanged, compared with those of T and the situation is really the one shown in the picture.

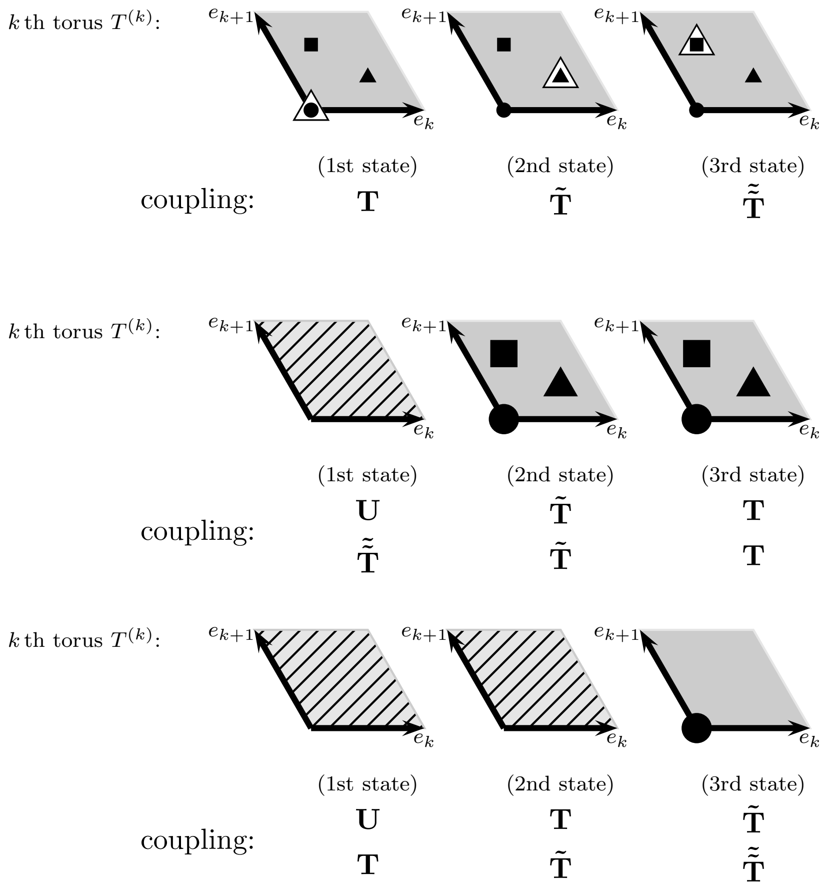


Figure 4.2: Space group allowed localisations for states in the couplings $UT\tilde{T}$ and $T\tilde{T}\tilde{\tilde{T}}$. Only couplings between states, localised at fixed points marked with the same symbol, are allowed. The picture differentiates between three cases with respectively no, one or two bulk tori. The first case obviously is only possible for couplings of the type $T\tilde{T}\tilde{\tilde{T}}$ while the others can appear both for $UT\tilde{T}$ and $T\tilde{T}\tilde{\tilde{T}}$ couplings.

4.3 Non-Abelian family symmetries

There is a particular way how non-Abelian discrete symmetry groups arise in orbifold models. An example of an appearing dihedral group D_4 of order eight can be found in [3]. We will be more general and investigate the type of these symmetry groups.
Up to now we only found Abelian symmetries emerging from the selection rules. We assume that we already imposed the selection rules due to H-momentum, gauge invariance, oscillator couplings and point group on the couplings and are thus left with a certain $\mathbb{Z}_m$ symmetry from the space group rule, as explained in Sec. 4.1. Now the non-Abelian structure appears due to some permutation symmetry and the non-Abelian type of the permutation groups.

To investigate this mechanism, we assume that we have the following set-up of an orbifold model, arising from the construction described in Sec. 2.3: Some families of states are located at the fixed points of a certain twisted sector. If no Wilson line distinguishes between some of these fixed points, the same representations appear at each of them. Thus the whole theory must be invariant under permutations of the states at the degenerate fixed points. Therefore we have one family of states at each of the, say n degenerate fixed points and these families must fall into n dimensional representations of the permutation group of n objects, S_n.
For this reason the couplings of the theory must be invariant under the two discrete groups $\mathbb{Z}_m$ and S_n and the unique discrete symmetry group G, constraining the couplings, is the multiplicative closure of these

$$\mathrm{G} = \mathrm{S}_n \cup \mathbb{Z}_m \equiv \{ s \cdot z \,|\, s \in \mathrm{S}_n,\; z \in \mathbb{Z}_m \} \;. \tag{4.10}$$

For a simplified notation we defined the symbol $\cup$ as the multiplicative closure of the two groups. The emerging G is obviously finite, discrete and non-Abelian.
In the following subsections we will derive G for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, investigate its properties and comment on the general case, which of course also includes the D_4 of [3].

4.3.1 A naturally arising family symmetry $\Delta(54)$

In the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, as in all other orbifolds with an SU(3) subtorus, there is a very intuitive setting for the three families of fermions. This is the case when the triplification of families is a consequence of the existence of three degenerate fixed points in one two dimensional subtorus. Then the families are located at the three degenerate fixed points of one particular twisted sector. From the space group rule there is a $\mathbb{Z}_3$ and from the possible permutations of the three families a S_3 group under which the families are charged. Following the line of argumentation of the previous section we have to construct the discrete symmetry group

$$\mathrm{S}_3 \cup \mathbb{Z}_3 \;,$$

meaning that we assign the families to a three dimensional representation of both groups. The $\mathbb{Z}_3$ from the space group is then generated by

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix} , \qquad \omega = e^{2\pi i/3} , \tag{4.11}$$

and the S_3 by

$$g_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{4.12}$$

By calculating the closure of these three 3×3-matrix generators one finds that they close in a group of order 54. For the moment we will call it G_{54},

$$\mathrm{G}_{54} = \mathrm{S}_3 \cup \mathbb{Z}_3 .$$

We now further investigate this group to see which of the known groups it is and finally to be able to handle it. First it is worth noticing that the S_3 could have analogously been generated by the two matrices g_2 and $-g_3$.[9] Then all generators of the resulting 54 dimensional group are unitary (as they were already before) with determinant $+1$. And thus G_{54} is isomorphic to a subgroup of SU(3). How does it arranges into the scheme of classified subgroups of SU(3) proposed by the authors of [48]?

From Tab. (B.1) we see that there is one SU(3) subgroup of order 54, namely $\Delta(54)$ and we will show that G_{54} is really equal to it. To do so we write G_{54} as a semi-direct product, with the explicit derivation given in Sec. B.2.1.
The result is that G_{54} can be written as

$$\mathrm{G}_{54} = \mathrm{S}_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \Delta(54) \tag{4.13}$$

and therefore, adopting the notation of [49], we finally arrive at a discrete symmetry group $\Delta(54)$. This group does not appear in the table of classified SU(3) subgroups of [48]. But there it is a subgroup of $\Delta(216)$.[10] Due to $216 = 2 \cdot 3 \cdot 4 \cdot 9$, the smallest permutation group which has $\Delta(216)$ as a subgroup is S_9. Therefore we have the inclusion

$$\Delta(54) \subset \Delta(216) \subset \mathrm{S}_9 .$$

It is interesting to notice that S_9 would be the arising discrete permutation symmetry group for a $\mathbb{Z}_3 \times \mathbb{Z}_3$ (or a $\mathbb{Z}_3$) orbifold in the case without any Wilson lines in any of the three

[9] It can easily be seen that the group table of S_3 does not change under this change of sign.

[10] This can easily be seen by noticing

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix} , \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in \Delta(216) .$$

two dimensional tori and the families located in one of the completely localised sectors. In the presence of Wilson lines this is broken down to $\Delta(54)$.

The argumentation of Sec. B.2.1 also holds for the general case of the multiplicative closure of an S_n and a $\mathbb{Z}_m$ group. We would now take matrices for an n dimensional representation, especially the $\mathbb{Z}_m$ generator as $\mathrm{diag}(1, e^{\frac{2\pi i}{m}\cdot 1}, e^{\frac{2\pi i}{m}\cdot 2}, ..., e^{\frac{2\pi i}{m}\cdot (m-1)})$ and the whole derivation stays almost unchanged. Therefore we have

$$\mathrm{S}_n \cup \mathbb{Z}_m = \mathrm{S}_n \ltimes (\mathbb{Z}_{m_1} \times ... \times \mathbb{Z}_{m_k}) \ . \tag{4.14}$$

The m_i are integer numbers, determined by n and m. The order of these groups can be read off from the orders of the permutation group and the direct product of the cyclic groups to be $n! \cdot \prod_{i=1}^{k} m_i$.

remark: in the case of two families localised at two degenerate fixed points of a SO(4) root lattice we thus derive the symmetry group

$$\mathrm{D}_4 = \mathrm{S}_2 \cup \mathbb{Z}_2 = \mathrm{S}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \ , \tag{4.15}$$

which is just the D_4 already mentioned in [3].

Thus finally we have derived a handy formula, namely Eq. (4.14), for that class of non-Abelian discrete family symmetries arising in the described way, from a $\mathbb{Z}_m$ of the space group and a S_n of the permutations of degenerate fixed points. Some properties of these groups, like character tables and products of irreducible representations (irreps) are given in Sec. B.2.2.
It should be noticed that this argumentation changes slightly, if we take into account the effect of different couplings strengths for the various couplings. Then it is not the whole permutation symmetry under which the couplings are invariant, but only the subgroup, which leaves the geometric configuration of fixed points, there distances and angles, invariant.

Of course, already many attempts have been made in order to classify the possible non-Abelian discrete family symmetries [50, 35]. To our knowledge only groups up to order 31 have been studied systematically. The number of groups for a given order increases enormously if the order equals 2^k. Thus considering groups of order ≥ 32 will make the classification more complex.
Also in most explicit models the introduced non-Abelian family symmetry groups are of order ≤ 32. There are only a few attempts made to construct (supersymmetric) GUTs based on SU(3) subgroups of higher orders. See e.g. [51] and [52] for examples concerning $\Delta(75)$ and $\Delta(48)$.
We now found out that discrete symmetry groups of much higher orders naturally appear from string theory (in our case from heterotic orbifold models). In those models these groups do not have to be imposed artificially, they just arise from the construction. This should be enough argument to consider and classify also groups of higher orders than 31 with respect to the question, which amongst them can lead to realistic mass matrices.

4.4 Realisation of the $\Delta(54)$ symmetry

The aim of the present chapter is to explain the properties of the $\Delta(54)$ group as a symmetry group of an explicit $\mathbb{Z}_3 \times \mathbb{Z}_3$ model. For this reason we take model (D'), the three-generation SU(5) model with two Wilson lines from Sec. 3.4.2. In spite of all shortcomings of an SU(5) GUT it has the advantage of less representations and thus an increased clarity of the spectrum, compared to the LR models.
Model (D') is a three generation model in which the group $\Delta(54)$ is realised as a discrete non-Abelian family symmetry. By discussing the allowed renormalizable couplings we will find out that in this model no hierarchy of the quark masses is possible. But this is not a generic feature of $\Delta(54)$ as the following discussion in Sec. 4.4.1 shows. We will finally overcome the shortcomings of model (D') and construct a set-up of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ model which would provide us with non-diagonal quark mass matrices at the level of order six non-renormalizable couplings. In this toy model the generations are located in different sectors as it is the case for model (D') and it should be possible to realise it by finding appropriate shifts and Wilson lines.

The three generations of model (D') are located at the three degenerate fixed points in the third SU(3) subtorus of the sectors $T_{(1,0)}$ and $T_{(2,0)}$, with the $(\mathbf{10};\mathbf{1})$ in $T_{(1,0)}$ and the $(\overline{\mathbf{5}};\mathbf{1})$ in $T_{(2,0)}$. We assume that the two higgses come from U_2 and $T_{(1,2)}$, with $h_d = (\overline{\mathbf{5}},\mathbf{1}) \in U_2$ and $h_u = (\mathbf{5},\mathbf{1}) \in T_{(1,2)}$ as displayed in Tab. (C.7) in App. C. In addition, there are 244 singlets distributed over all sectors and additional $(\mathbf{5},\mathbf{1})$, $(\overline{\mathbf{5}},\mathbf{1})$, $(\mathbf{1},\mathbf{3})$ and $(\mathbf{1},\overline{\mathbf{3}})$ representations. The nice feature of this model is that there is an equal number of additional $(\mathbf{5},\mathbf{1})$ and $(\overline{\mathbf{5}},\mathbf{1})$ as well as $(\mathbf{1},\mathbf{3})$ and $(\mathbf{1},\overline{\mathbf{3}})$ representations, they are vectorlike. Thus eventually they can be made heavy by letting some of the singlets acquire vevs and decouple from the observable matter.

To discuss the features of $\Delta(54)$ we will first consider how masses for the down-type quarks arise. These masses are generated via couplings of the type[11]

$$f^A \tilde{f}^B h_d = (\mathbf{10};\mathbf{1})(\overline{\mathbf{5}},\mathbf{1})(\overline{\mathbf{5}};\mathbf{1}) \ , \tag{4.16}$$

which are R-momentum allowed renormalizable couplings of the form $U_2 T_{(1,0)} T_{(2,0)}$, as can be read off from Tab. (4.2). Now the question is which irreps of $\Delta(54)$ do the gauge group representations constitute? They are located at the fixed points of the third SU(3) subtorus. In this subspace the constructing spacetime twists of the two sectors $T_{(1,0)}$ and $T_{(2,0)}$ are the inverse of each other. Thus the fixed points of $T_{(2,0)}$ with the space group quantum numbers $n = 1, 2$ are interchanged compared to those of $T_{(1,0)}$. In terms of $\Delta(54)$

[11]The reduction of the relevant SU(5) representations is

$$\overline{\mathbf{5}} \times \mathbf{10} = \mathbf{5} + \mathbf{45}$$

$$\mathbf{10} \times \mathbf{10} = \overline{\mathbf{5}} + \overline{\mathbf{45}} + \overline{\mathbf{50}} \ .$$

sector	irrep	sector	irrep
U_i	$\mathbf{1_1}$	$T_{(2,2)}$	$\mathbf{3}$
$T_{(1,0)}$	$\mathbf{3}$	$T_{(2,0)}$	$\overline{\mathbf{3}}$
$T_{(0,1)}$	$\mathbf{3}$	$T_{(0,2)}$	$\overline{\mathbf{3}}$
$T_{(1,2)}$	$\mathbf{1_1}$	$T_{(2,1)}$	$\mathbf{1_1}$

Table 4.3: Assignment of $\Delta(54)$ irreps to the various twisted sectors in the case of the components of the $\mathbf{3}$ and $\overline{\mathbf{3}}$ arising from the three degenerate fixed points of the third SU(3) subtorus (two Wilson lines in $T^{(1)}$ and $T^{(2)}$).

representations this means that if there are two sectors in which the twist acts inversely (we already called such sectors *conjugated sectors*) and we assign the components of a $\mathbf{3}$ to the fixed points of the one, then we have to assign the components of a $\overline{\mathbf{3}}$ to the fixed points of the other sector. This is depicted in Fig. (4.3). In general, to each twisted sector there corresponds a $\Delta(54)$ irrep, due to the number of degenerate fixed points. This is depicted in Tab. (4.3).

Therefore in the coupling of Eq. (4.16) we take the $(\mathbf{10};\mathbf{1})$ as a $\mathbf{3}$ and the matter $(\overline{\mathbf{5}};\mathbf{1})$ as a $\overline{\mathbf{3}}$ while the higgs $(\overline{\mathbf{5}};\mathbf{1})$ of course must be a $\mathbf{1_1}$, because it comes from the untwisted sector. With this choice the coupling $f^A\tilde{f}^A h_d$ in terms of $\Delta(54)$ irreps reads

$$\mathbf{3}\times\overline{\mathbf{3}}\times\mathbf{1_1}=\mathbf{1_1}+\mathbf{2_1}+\mathbf{2_2}+\mathbf{2_3}+\mathbf{2_4}\ ,$$

where we used the results of Tab. (B.3). Due to the appearance of the $\mathbf{1_1}$ on the right hand side we see that the coupling is $\Delta(54)$ allowed.

The form of this $\mathbf{1_1}$ in terms of the basis functions of the $\mathbf{3}$, $\overline{\mathbf{3}}$ and $\mathbf{1_1}$ can easily be deduced. With the notation defined in Fig. (4.3) the invariant term from $x\times\overline{y}$ reads $\mathbf{1_1}=\sum_{i=1}^3 x_i\overline{y}_i$. In terms of the matter representations we have

$$\mathbf{1_1}=\left(\sum_{A=1}^{3}f_A\tilde{f}_A\right)h_d=f_1\tilde{f}_1h_d+f_2\tilde{f}_2h_d+f_3\tilde{f}_3h_d$$

as an allowed term of the superpotential. It obviously generates same masses for all three families.

remark: The invariance of terms of the form $x_A\overline{y}_A$ under the space group $\mathbb{Z}_3$ of Eq. (4.8) can be read off from the localisation of the components of the $\Delta(54)$ irreps in Fig. (4.3). In the third SU(3) subtorus the difference between the fixed points of $T_{(1,0)}$ and $T_{(2,0)}$ is that the $n=1,2$ values are interchanged. Thus e.g. in the term $x_2\overline{y}_2$ the x_2 is located at the $n=1$ fixed point of $T_{(1,0)}$ and the $\overline{y}_2$ at the $n=2$ fixed points of $T_{(2,0)}$. Therefore the whole term $x_2\overline{y}_2$ is invariant under $\mathbb{Z}_3$ and the same holds for $x_3\overline{y}_3$, while for $x_1\overline{y}_1$ the invariance is trivial, because both factors are located at the $n=0$ fixed points and thus are invariant for themselves.

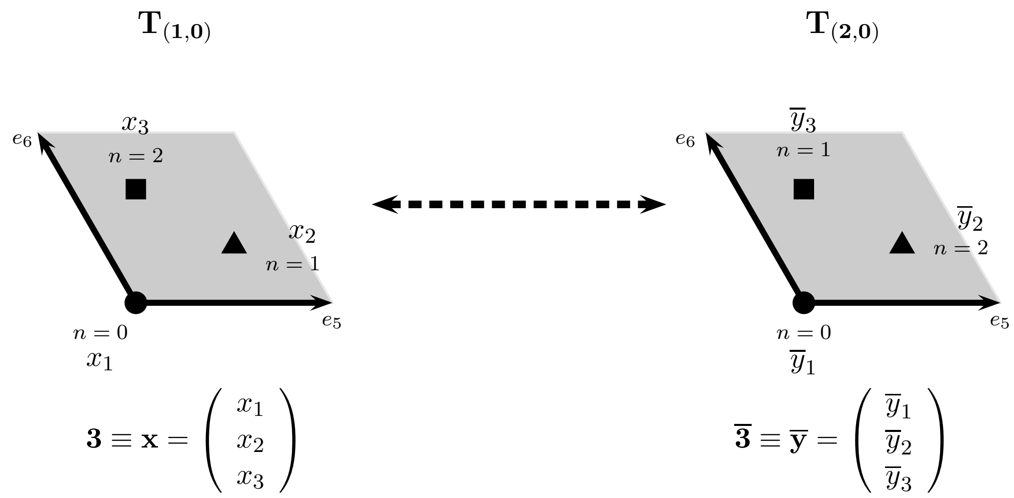


Figure 4.3: Assignment of the components of the $\Delta(54)$ irreps to the fixed points of the two sectors $T_{(1,0)}$ and $T_{(2,0)}$. The two sectors can be replaced by each pair of conjugated sectors, as explained. It is important to notice that the fixed points with $n = 1, 2$ are interchanged, as points in space.

The next step is to consider the additional $(\mathbf{5};\mathbf{1})$ and $(\mathbf{1};\mathbf{3})$ representations (and their complex conjugates) and to see whether they can acquire large masses from couplings to singlets, to decouple from the wanted matter. We consider only renormalizable couplings and turn to the $(\mathbf{5};\mathbf{1})$ first.
The additional $(\mathbf{5};\mathbf{1})$ and $(\overline{\mathbf{5}};\mathbf{1})$ come from the sectors U_3, $T_{(1,0)}$, $T_{(2,0)}$, $T_{(0,2)}$, $T_{(1,2)}$ and $T_{(2,1)}$. Thus we can use the couplings $U_2T_{(1,0)}T_{(2,0)}$, $U_3T_{(1,2)}T_{(2,1)}$ and $T_{(1,0)}T_{(2,1)}T_{(0,2)}$ from Tab. (4.2). The assignments of the states of the various sectors to $\Delta(54)$ irreps can be read off Tab. (4.3). E.g. the $\tilde{\phi}^{1,2,3}$, which are defined in Tab. (4.3), constitute a $\mathbf{3}$, the $\phi^{2,3,4}$ a $\overline{\mathbf{3}}$ and s^2 a $\mathbf{1_1}$. The three irreps form an invariant via $U_2T_{(1,0)}T_{(2,0)}$ as

$$\mathbf{3} \times \overline{\mathbf{3}} \times \mathbf{1_1} = \mathbf{1_1} + \ldots ,$$

with the invariant on the right hand side

$$\mathbf{1_1} = \left(\sum_{i=1}^{3} \phi^{i+1} \tilde{\phi}^i \right) s^2 .$$

Thus the corresponding 3×3 submatrix of the 10×10 matrix of the couplings of all 10 additional $(\mathbf{5};\mathbf{1})$ to singlets is diagonal. In this way the full 10×10 matrix of all renormalizable couplings can be determined. It turns out that it is of rank six, meaning that there are four additional vectorlike representations which stay massless.
By calculating the 14×14 matrix of the renormalizable couplings of the additional $(\mathbf{1};\mathbf{3})$

to singlets, we find that it is of rank six too. Thus at the level of renormalizable couplings there are four $(\mathbf{5};\mathbf{1})$, $(\overline{\mathbf{5}};\mathbf{1})$ and eight $(\mathbf{1};\mathbf{3})$, $(\mathbf{1};\overline{\mathbf{3}})$ which stay massless in addition to the three generations and the higgses.

Perhaps it will be possible to decouple the remaining additional states by use of non-renormalizable couplings to singlets. But the investigation of these higher order couplings is rather more involved. Already at the next order of R-momentum allowed couplings, couplings of order six, we have to handle 65 different ones.[12] So far we did not consider flat directions of the potential for the singlets and thus do not know whether it is really possible to let the appropriate singlets acquire vevs.
One severe problem will remain for models similar to model (D'). Due to assigning f to a $\mathbf{3}$ and $\tilde{f}$ to a $\overline{\mathbf{3}}$ the masses of the three generations will always be equal and there are no non-diagonal entries which could give rise to a nontrivial CKM phase [36]. A hierarchy in the quark mass matrices could only be realised by breaking $\Delta(54)$ to a subgroup. In the next section we will overcome this problem by taking the generations and the higgs from other sectors.

4.4.1 General remarks on mass matrices

In the last section we found out that one of the main problems of model (D') is that it does not causes hierarchical mass matrices. To show that this is just a shortcoming of model (D') and not of the discrete symmetry $\Delta(54)$ we now construct a $\mathbb{Z}_3 \times \mathbb{Z}_3$ set-up which brings about a hierarchy of quark masses. We will not give an explicit model, but just assume that the families and the higgses are localised in certain sectors.
First we will consider only renormalizable couplings and then take into account also non-renormalizable couplings to singlets. For the sake of clarity of our discussion, we assume that there are two Wilson lines in the first two subtori $T^{(1)}$ and $T^{(2)}$. Thus the three degenerate fixed points of $T^{(3)}$ give rise to a multiplicity three of the states localised at them.[13] Then all ten possibilities for renormalizable couplings in terms of $\Delta(54)$ irreps are listed in Tab. (4.4).
Now it depends on the appropriate model to which of the $\Delta(54)$ irreps the families and the higgses $(f^A, \tilde{f}^A, h^A)$ are assigned. From Tab. (4.4) we see that there are three different possible types of couplings[14]

$$\begin{array}{ll} (1) & \mathbf{1_1} \times \mathbf{1_1} \times \mathbf{1_1} \ , \\ (2) & \mathbf{1_1} \times \mathbf{3} \times \overline{\mathbf{3}} \ , \\ (2') & \mathbf{1_1} \times \mathbf{3} \times \overline{\mathbf{3}} \ , \\ (3) & \mathbf{3} \times \mathbf{3} \times \mathbf{3} \ , \end{array}$$

[12] The number of renormalizable couplings is 10, due to Tab. (4.2).

[13] The two Wilson lines can also be chosen to belong to two other subtori. Then the whole discussion stays the same if we appropriately change the assignments of irreps to sectors in Tab. (4.4). A third Wilson line would destroy the permutation symmetry and thus break $\Delta(54)$.

[14] The complex conjugated couplings lead to the same results.

coupling	$\Delta(54)$ irreps
$U_1U_2U_3$	$\mathbf{1_1} \times \mathbf{1_1} \times \mathbf{1_1}$
$U_1T_{(0,1)}T_{(0,2)}$	$\mathbf{1_1} \times \mathbf{3} \times \overline{\mathbf{3}}$
$U_2T_{(1,0)}T_{(2,0)}$	$\mathbf{1_1} \times \mathbf{3} \times \overline{\mathbf{3}}$
$U_3T_{(1,2)}T_{(2,1)}$	$\mathbf{1_1} \times \mathbf{3} \times \overline{\mathbf{3}}$
$T_{(1,0)}T_{(0,2)}T_{(2,1)}$	$\mathbf{3} \times \overline{\mathbf{3}} \times \mathbf{1_1}$
$T_{(2,0)}T_{(2,1)}T_{(2,2)}$	$\overline{\mathbf{3}} \times \mathbf{1_1} \times \mathbf{3}$
$T_{(1,0)}T_{(0,1)}T_{(2,2)}$	$\mathbf{3} \times \mathbf{3} \times \mathbf{3}$
$T_{(0,1)}T_{(2,0)}T_{(1,2)}$	$\mathbf{3} \times \overline{\mathbf{3}} \times \mathbf{1_1}$
$T_{(2,2)}T_{(2,2)}T_{(2,2)}$	$\mathbf{3} \times \mathbf{3} \times \mathbf{3}$

Table 4.4: All ten renormalizable couplings for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold in terms of the $\Delta(54)$ irreps. The assignment of $\mathbf{3}$ and $\overline{\mathbf{3}}$ to the various sectors is due to our choice from the last section. They can be interchanged.

which are all $\Delta(54)$ allowed, because they include a $\mathbf{1_1}$, as can be seen by use of Tab. (B.3). The assignment of the $f^A, \tilde{f}^A, h^A$ to the irreps further distinguishes between case (2) with $h = \mathbf{1_1}$ and (2') where $h = \overline{\mathbf{3}}$.[15]
The next step would be to discuss the possible mass matrices due to the coupling $f^A \tilde{f}^B h^c$ for each of these four cases. Case (1) leads to a democratic mass matrix, (2) to a diagonal matrix with equal diagonal entries (this is the case of model (D') in Sec. 4.4) and (2') to a matrix with equal columns.

However, we will specify on case (3) and show, how an interesting mass matrix arises from it. For (3) we have to assign $h = \mathbf{3}$, $f = \mathbf{3}$ and $\tilde{f} = \mathbf{3}$. This can e.g be realised in a model with all $h, f, \tilde{f}$ from $T_{(2,2)}$ or in a model with one of them from $T_{(1,0)}$, one from $T_{(0,1)}$ and one from $T_{(2,2)}$.[16] It should be possible to construct models of this type. In terms of $\Delta(54)$ irreps the coupling between the families and the higgses reads

$$h^A f^B \tilde{f}^C = \mathbf{3} \times \mathbf{3} \times \mathbf{3} = 2 \cdot \mathbf{1_1} + ... \ ,$$

with the two invariants on the right hand side of the form

$$\mathbf{1_1} = \begin{cases} \sum\limits_{A=1}^{3} f^A \tilde{f}^A h^A & \text{(A)} \ , \\ f^1 \tilde{f}^2 h^3 + f^1 \tilde{f}^3 h^2 + f^2 \tilde{f}^1 h^3 + f^2 \tilde{f}^3 h^1 + f^3 \tilde{f}^1 h^2 + f^3 \tilde{f}^2 h^1 & \text{(B)} \ . \end{cases}$$

[15] Assigning h to $\mathbf{3}$ would lead to the same matrix as case (2').
[16] In the first case the masses arise from couplings of the type $T_{(2,2)}T_{(2,2)}T_{(2,2)}$, in the second from $T_{(1,0)}T_{(0,1)}T_{(2,2)}$.

It is important to notice that the states in the couplings of case (B) are located at three distinct fixed points in the third SU(3) subtorus. Therefore these couplings are exponentially suppressed, compared to those of case (A). This exponential suppression factor Y depends on the distance between the fixed points at which the states are localised. For the derivation of it and its particular form for the TTT coupling of the $\mathbb{Z}_3$ orbifold, which is just the case we need here, see [44, 36].
Therefore the mass matrix due to renormalisable couplings of the families to the three higgses reads

$$M_{(3)} = \begin{pmatrix} \langle h^1\rangle & Y\langle h^3\rangle & Y\langle h^2\rangle \\ Y\langle h^3\rangle & \langle h^2\rangle & Y\langle h^1\rangle \\ Y\langle h^2\rangle & Y\langle h^1\rangle & \langle h^3\rangle \end{pmatrix} . \tag{4.17}$$

An interesting structure of the matrix $M_{(3)}$ can arise if only the third component of the higgs h^3 acquires a nonvanishing vev and the masses of the two other generations are generated by non-renormalizable couplings to singlets.
The next order of R-momentum allowed couplings for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold beyond the renormalizable level is six. These couplings can be used to couple the families to singlets s, which then acquire vevs and generate mass terms[17]

$$f\tilde{f} \times ssss\ . \tag{4.18}$$

For simplicity, we assume that one of these singlets constitutes a $\mathbf{3}$ and the other three singlets three $\mathbf{1_1}$ irreps of $\Delta(54)$.[18] Then Eq. (4.18) in terms of $\Delta(54)$ reads

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} \times \mathbf{1_1} \times \mathbf{1_1} \times \mathbf{1_1} = 2 \cdot \mathbf{1_1} + \ldots\ .$$

If it is allowed for these singlets to acquire nonvanishing vevs $\langle s\rangle \neq 0$, these vevs will constitute a mass matrix of an analogous form as in Eq. (4.17). We write the vevs of the three singlets which constitute the $\mathbf{3}$ as $\langle s^i\rangle$ and the product of the three other singlet vevs as $\langle s\rangle^3$ and define $\varepsilon_i \equiv \langle s^i\rangle\langle s\rangle^3$. Then after setting $\langle h^3\rangle = 1$, $\langle h^1\rangle = 0$, $\langle h^2\rangle = 0$ the full mass matrix at the level of order six couplings reads

$$M = \begin{pmatrix} \varepsilon_3 & Y + Y\varepsilon_1 & Y\varepsilon_2 \\ Y + Y\varepsilon_1 & \varepsilon_2 & Y\varepsilon_3 \\ Y\varepsilon_2 & Y\varepsilon_3 & 1 + \varepsilon_1 \end{pmatrix} . \tag{4.19}$$

It is a symmetric matrix due to f and $\tilde{f}$ both being a $\mathbf{3}$ of $\Delta(54)$. All Y and ε entries in that matrix are parameters which depend on the size and shape of the orbifold (due to Y depending on the T moduli [36]) and the particular model (values of the singlet vevs). A realistic pattern arises if ε_1 and ε_3 vanish. This would be the case if there are no appropriate singlets in the model under consideration. The mass matrix then reads

$$M = \begin{pmatrix} 0 & Y & Y\varepsilon_2 \\ Y & \varepsilon_2 & 0 \\ Y\varepsilon_2 & 0 & 1 \end{pmatrix} .$$

[17] Again we assume that there are singlets in the appropriate sectors, such that the couplings are allowed under all selection rules except the space group ($\Delta(54)$) rule.

[18] There are four other possible assignments of the four singlets to $\Delta(54)$ irreps.

This matrix already strongly looks like some of the symmetric quark mass matrices which were proposed in [53].[19] If the angles and radii of the orbifold and the singlet vevs of the model allow to have $\varepsilon_2 \approx \lambda^4$ and $Y \approx \lambda^6$, with $\lambda \approx 0,22$ being the $(1,2)$ element of the CKM matrix as explained in [53], the matrix becomes

$$M = \begin{pmatrix} 0 & \lambda^6 & \lambda^{10} \\ \lambda^6 & \lambda^4 & 0 \\ \lambda^{10} & 0 & 1 \end{pmatrix} . \tag{4.20}$$

By neglecting the order ten term $\lambda^{10} \approx 0$ this matrix is just of the form of solution (1) for Y_u of [53]. Solution (4) and (5) for Y_d can be obtained from Eq. (4.19) in the same manner.

[19] The authors of [53] investigated symmetric mass matrices with a maximum of six texture zeros, which could lead to interesting hierarchical structures, in general.

Chapter 5

Conclusion and Outlook

There are two major conclusions which can be drawn from this study. The first concerns discrete torsion. The discussion in Sec. 3.3 reveals that discrete torsion has not to be seen exclusively as some strange degree of freedom which can only be understood by considering the transformation behaviour of the partition function under modular transformations. Instead we derived an equivalent description in terms of transformations of the embedding shifts. Therefore we conclude that introducing a discrete torsion phase is equivalent to modify the shift embedding in an appropriate way. This information, concerning the equivalence of different embedding shifts, will be important for classifications of orbifold models.

The second conclusion is that there is a certain class of non-Abelian discrete family symmetries emerging from the fixed point structure of orbifolds. These symmetries are constructed in Sec. 4.3 and App. B.2.1. In this construction the space group selection rule translates between the localisation of the states and the formulation in terms of discrete groups, thus between geometry and symmetry. This general construction allows to arrange the particular symmetry for orbifolds with SU(3) tori into the classification of finite discrete subgroups of SU(3). It applies to recent results as well [3] as to other orbifolds with different fixed point structures. Expressing the selection rules in terms of discrete symmetries provides us with an intuitive understanding of which couplings are allowed, based on the fixed point structure of the orbifold. As a result the texture of mass matrices can be easily read off the fixed point structure. Therefore model (D') can be ruled out from being realistic. But in Sec. 4.4.1 it is shown that also for a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold the construction of models with realistic mass matrices should be possible.
To summarise we state that the use of an appropriate discrete symmetry facilitates the construction of models with realistic mass matrices. Therefore investigation of fixed point structures leading to discrete symmetries, as is done in this thesis, could be an important step in the direction of selecting a good orbifold with realistic mass patterns.

Appendix A

Explicit calculations for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold

A.1 Additional projection conditions for $\mathbb{Z}_3 \times \mathbb{Z}_3$

As explained in Sec. 2.3.3 for each fixed point, described by the constructing element $\hat{g}$, we have to find the set of commuting elements, the centralizer

$$Z_{\hat{g}} = \{g = (X, y) \in S \,|[g, \hat{g}] = 0\} \tag{A.1}$$

and then project onto centralizer invariant states. Therefore the additional projection conditions are found by deriving the centralizer for each constructing element. The centralizer can of course be derived very easily making use of a computer. Nevertheless we do the calculation by hand for instructive reasons.
For the following calculation we label the fixed points by $n_\alpha = 0, 1, 2$ with $\alpha = 1, 2, 3$. For deriving projection conditions we are only interested in the corresponding action on the gauge degrees of freedom for each space group element. We know $\theta e_1 = e_2$, $\theta e_2 = -e_1 - e_2$ and that the order of the three possible Wilson lines is three. Therefore we can now use only one torus shift e_α for each of the three tori for the calculation instead of both e_1^α, e_2^α. In this new notation the corresponding construction elements for each of the three fixed points in each of the three tori are given in Fig. (A.1) for the sector $T_{(1,0)}$.
We now consider the particular fixed point with $\hat{g} = (\theta^k \omega^l, n_\alpha e_\alpha)$ and derive $Z_{\hat{g}}$

$$\begin{aligned} [\hat{g}, g] &= [(\theta^k \omega^l, n_\alpha e_\alpha), (X, y)] \\ &= ([\theta^k \omega^l, X], \theta^k \omega^l y - n_\alpha X e_\alpha + n_\alpha e_\alpha - y) \ . \end{aligned}$$

Demanding the vanishing of the commutator and using the fact that the point group was chosen to be Abelian yields the equation

$$(\theta^k \omega^l - 1) y = n_\alpha (X - 1) e_\alpha \tag{A.2}$$

for elements in the centralizer. Eq. (A.2) must now be solved for every fixed point in the various twisted sectors.

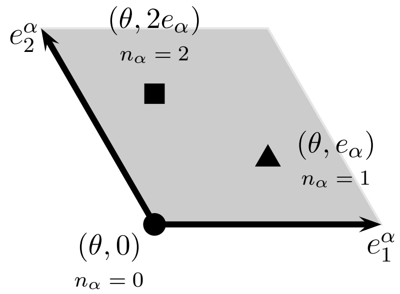


Figure A.1: The constructing elements for the fixed points in $T_{(1,0)}$.

For an example how this can be done we consider $T_{(1,0)}$ (which means $\theta^k\omega^l \to \theta$) and particular fixed points of it:

- $\hat{\boldsymbol{g}} = (\boldsymbol{\theta}, \mathbf{0})$ means that the states are localised at the origin. Eq. (A.2) now reads

$$(\theta - 1)y = 0 \ .$$

θ only rotates the first and the third torus and therefore $(\mathbb{1}, e_2)$ is obviously commuting with $\hat{g}$ and being an element in the centralizer. Also the elements $(\theta, 0)$, $(\omega, 0)$ are in $Z_{\hat{g}}$. All other elements which commute with $\hat{g}$ decompose into products of those three elements and therefore are no additional projection elements. For space group elements including e_1, e_3, it is obvious that the left hand side of Eq. (A.2) does not vanish due to the torus lattice vectors being rotated by θ. Thus

$$Z_{\hat{g}} = \{(\mathbb{1}, e_2), (\theta, 0), (\omega, 0)\} \ .$$

By use of the mass equation and the modular invariance condition, it is not hard to show that the transformation phase of the states under the space group element $\hat{g}$ vanishes in general. Therefore the constructing element always leads to a trivial projection condition which we will not quote any further. But we have to bear it in mind because other commuting elements could be written as products of $\hat{g}$ with other elements in the centralizer and therefore also lead only to trivial projection conditions. The non trivial elements in the centralizer are

$$Z_{\hat{g}} = \{(\mathbb{1}, e_2), (\omega, 0)\} \ . \tag{A.3}$$

The corresponding actions on the gauge degrees of freedom then are

$$\{(\mathbb{1}; A_2), (\mathbb{1}; V_2)\} \ .$$

Thus for the states which are localised at the particular fixed point and labelled by the shifted $E_8 \times E_8$ roots p_{sh} we have the left mover projection phases

$$p_{sh} \cdot A_2 \ , \tag{A.4}$$

$$p_{sh} \cdot V_2 \ . \tag{A.5}$$

The sum of these left moving phases and the right moving ones must equal zero in order for the states to survive the orbifold projection.

- $\boldsymbol{\hat{g} = (\theta, n_3 e_3)}$ meaning that in the first torus the states are localised at the origin. Eq. (A.2) in this case reads

$$(\theta - 1)y = n_3(X - 1)e_3$$

and again there are $\hat{g}, (\mathbb{1}, e_2) \in Z_{\hat{g}}$. There are other commuting elements whose point group part rotates e_3 in the same manner as θ does. The appropriate point group elements which act on e_3 as θ are ω and $\theta^2\omega^2$, leading to $(\omega, n_3e_3), (\theta^2\omega^2, n_3e_3) \in Z_{\hat{g}}$. By writing $(\theta^2\omega^2, n_3e_3) = (\theta, n_3e_3)^2(\omega, n_3e_3)^{-1}$, we see that one of the two resulting projection conditions will be trivially fulfilled if the projection condition for the other element is satisfied.[1] Therefore one choice of the nontrivial part of the centralizer is

$$Z_{\hat{g}} = \{(\mathbb{1}, e_2), (\omega, n_3e_3)\} \ , \tag{A.6}$$

with the corresponding projection phases

$$p_{sh} \cdot A_2 \ , \tag{A.7}$$

$$p_{sh} \cdot (V_2 + n_3 A_3) \ . \tag{A.8}$$

The above considerations must now be repeated for all fixed points and fixed tori of each sector. The result is presented in Tab. (A.1).
From the tabular it can be seen that in the sectors with one bulk torus one has always to project onto the Wilson line in the bulk torus and onto one additional space group element which is characterised through the action of its point group part on the translational part of the particular constructing element. This action must equal the action of the point group part of the constructing element. In the totally localised sectors one finds only one nontrivial projection element.

For orbifolds with non-prime order point groups, which allow the exchange of inequivalent fixed points, the physical meaningful states are linear combinations of states localised at the exchanged fixed points, as was explained in Sec. 2.3.3. For a $\mathbb{Z}_3$ point group acting

[1] The inverse for a space group element (X, y) is given by

$$(X^{-1}, -X^{-1}y) \ .$$

on an SU(3) root lattice, this exchange of inequivalent fixed points does not happen, as can be seen easily. Therefore take e.g. the fixed point corresponding to the constructing element (θ, e_1) and show that it cannot be transformed into one of the fixed points which correspond to $(\theta, 0)$ and $(\theta, e_1 + e_2)$. If this exchange was possible, the conjugation of (θ, e_1) with some other space group element would yield the constructing element of the exchanged fixed point. Therefore calculate for an arbitrary $(X, x) \in S$

$$(X, x)(\theta, e_1)(X, x)^{-1} = (\theta, -\theta x + X e_1 + x)$$

and consider the expression

$$v \equiv (1 - \theta)x + X e_1 \ . \tag{A.9}$$

Due to the $\mathbb{Z}_3$ point group there are three possible choices for X. For each of these choices $X \in \{\mathbb{1}, \theta, \theta^2\}$ Eq. (A.9) restricts the vector v to lie on the lattice $\Lambda = \{ae_1 + be_2 |\, a, b \in \mathbb{Z},\, a + b = 1 \mod 3\}$ as can be seen by inserting e_1 and e_2 for x. Thus with the above v the constructing elements $(\theta, 0)$ and $(\theta, e_1 + e_2)$ cannot be generated. Therefore for a $\mathbb{Z}_3$ geometry there is no exchange of inequivalent fixed points and no possibility of constructing linear combinations.

sector	fixed point/torus	centralizer $Z_{\hat{g}}$	lm projection phases
$T_{(1,0)}$	$(n_1, -, n_3)$	$\{(\mathbb{1}, e_2), (\omega, n_3 e_3)\}$	$p_{sh} \cdot A_2\,;\, p_{sh} \cdot (V_2 + n_3 A_3)$
$T_{(2,0)}$	$(n_1, -, n_3)$	$\{(\mathbb{1}, e_2), (\omega^2, n_3 e_3)\}$	$p_{sh} \cdot A_2\,;\, p_{sh} \cdot (2V_2 + n_3 A_3)$
$T_{(0,1)}$	$(-, n_2, n_3)$	$\{(\mathbb{1}, e_1), (\theta, n_3 e_3)\}$	$p_{sh} \cdot A_1\,;\, p_{sh} \cdot (V_1 + n_3 A_3)$
$T_{(0,2)}$	$(-, n_2, n_3)$	$\{(\mathbb{1}, e_1), (\theta^2, n_3 e_3)\}$	$p_{sh} \cdot A_1\,;\, p_{sh} \cdot (2V_1 + n_3 A_3)$
$T_{(2,1)}$	$(n_1, n_2, -)$	$\{(\mathbb{1}, e_3), (\omega, n_2 e_2)\}$	$p_{sh} \cdot A_3\,;\, p_{sh} \cdot (V_2 + n_2 A_2)$
$T_{(1,2)}$	$(n_1, n_2, -)$	$\{(\mathbb{1}, e_3), (\omega^2, n_2 e_2)\}$	$p_{sh} \cdot A_3\,;\, p_{sh} \cdot (2V_2 + n_2 A_2)$
$T_{(1,1)}$	$(0, 0, 0)$	$\{(\theta, 0)\}$	$p_{sh} \cdot V_1$
	$(0, 0, n_3)$	$\{(\theta^2\omega, 0)\}$	$p_{sh} \cdot (2V_1 + V_2)$
	$(0, n_2, 0)$	$\{(\theta, 0)\}$	$p_{sh} \cdot V_1$
	$(n_1, 0, 0)$	$\{(\omega, 0)\}$	$p_{sh} \cdot V_2$
	$(0, n_2, n_3)$	$\{(\theta^2, n_3 e_3)\}$	$p_{sh} \cdot (2V_1 + n_3 A_3)$
	$(n_1, 0, n_3)$	$\{(\omega^2, n_3 e_3)\}$	$p_{sh} \cdot (2V_2 + n_3 A_3)$
	$(n_1, n_2, 0)$	$\{(\omega, n_2 e_2)\}$	$p_{sh} \cdot (V_2 + n_2 A_2)$
	(n_1, n_2, n_3)	$\emptyset$	-
$T_{(2,2)}$	$(0, 0, 0)$	$\{(\theta, 0)\}$	$p_{sh} \cdot V_1$
	$(0, 0, n_3)$	$\{(\theta^2\omega, 0)\}$	$p_{sh} \cdot (2V_1 + V_2)$
	$(0, n_2, 0)$	$\{(\theta, 0)\}$	$p_{sh} \cdot V_1$
	$(n_1, 0, 0)$	$\{(\omega, 0)\}$	$p_{sh} \cdot V_2$
	$(0, n_2, n_3)$	$\{(\theta, n_3 e_3)\}$	$p_{sh} \cdot (V_1 + n_3 A_3)$
	$(n_1, 0, n_3)$	$\{(\omega, n_3 e_3)\}$	$p_{sh} \cdot (V_2 + n_3 A_3)$
	$(n_1, n_2, 0)$	$\{(\omega^2, n_2 e_2)\}$	$p_{sh} \cdot (2V_2 + n_2 A_2)$
	(n_1, n_2, n_3)	$\emptyset$	-

Table A.1: The centralizers and corresponding projection conditions for all fixed points/tori in the twisted sectors of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold. The complete transformation phase of the state is build from the left mover (lm) projection phases by subtracting them from the right mover phases due to Eq. (2.3).

A.2 Mode expansions and number operators in the twisted sectors

It is important to realise that the mode expansions in the twisted sectors change and that the twisted oscillators get a fractional moding due to the new boundary condition. How this arises will be shown explicitly in the following.
From the coordinates of the heterotic string X^i with $i = 4, ..., 9$ we defined the complex coordinates of the compactified dimensions as

$$Z^a \equiv \frac{1}{\sqrt{2}} \left(X^{2a+2} + iX^{2a+3} \right) , \tag{A.10a}$$

$$Z^{\bar{a}} = \frac{1}{\sqrt{2}} \left(X^{2a+2} - iX^{2a+3} \right) , \tag{A.10b}$$

where $a = 1, ..., 3$.
We consider the twisted sector due to the twist $\exp(2\pi i v^a)$ (to generalise to the other twisted sectors just replace v^a with the appropriate twist $\tilde{v}^a = kv_1^a + lv_2^a$ in the calculation below). Therefore the twisted boundary conditions for the complex coordinates are

$$Z^a(\tau, \sigma + \pi) = e^{2\pi i v^a} Z^a(\tau, \sigma) , \tag{A.11}$$

$$Z^{\bar{a}}(\tau, \sigma + \pi) = e^{-2\pi i v^a} Z^{\bar{a}}(\tau, \sigma) . \tag{A.12}$$

We first consider the left movers. By inserting the formula for the real fields

$$X^i(\tau, \sigma) = x^i + p^i \tau + \frac{i}{\sqrt{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \tilde{\alpha}_n^i \, e^{-2in(\tau+\sigma)}$$

into Eq. (A.10a) we obtain

$$Z^a(\tau, \sigma) = z^a + p^a \tau + \frac{i}{\sqrt{2}} \sum_{\substack{n=-\infty \\ n \neq 0}} \frac{1}{n} \tilde{\beta}_n^a \, e^{-2in(\tau+\sigma)} ,$$

where we defined $z^a := \frac{1}{\sqrt{2}}(x^{2a+2} + ix^{2a+3})$, $p^a := \frac{1}{\sqrt{2}}(p^{2a+2} + ip^{2a+3})$, and the complex oscillator $\tilde{\beta}_n^a := \frac{1}{\sqrt{2}}(\tilde{\alpha}_n^{2a+2} + i\tilde{\alpha}_n^{2a+3})$.

By imposing the twisted boundary condition Eq. (A.11) we find

$$z^a + p^a \tau + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\beta}_n^a \, e^{-2in(\tau+\sigma+\pi)} = e^{2\pi i v^a} z^a + e^{2\pi i v^a} p^a \tau + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\beta}_n^a \, e^{-2in(\tau+\sigma)+2\pi i v^a} .$$

In order that this condition holds for all (τ, σ) the above defined complex quantities must satisfy

$$z^a = e^{2\pi i v^a} z^a , \tag{A.13a}$$

$$p^a = 0 , \tag{A.13b}$$

$$-2\pi i n = 2\pi i v^a - 2\pi i m \quad m \in \mathbb{Z} . \tag{A.13c}$$

From Eq. (A.13a) we conclude that the twisted string (more precisely its centre of mass coordinate) is located at a fixed point. Eq. (A.13b) only holds for directions where the twist acts nontrivially. The last equation finally reads

$$n = m - v^a$$

and forces the oscillators to be fractionally moded due to the given value of v^a.

The calculation for $Z^{\bar{a}}$ runs analogously with the only difference that it transforms with a negative phase and thus yields the oscillator modes

$$n = m + v^a \ .$$

Using $(\tilde{\alpha}^i_n)^* = \tilde{\alpha}^i_{-n}$ in Eq. (A.10b) the oscillator of the complex conjugated coordinate is given by

$$\tilde{\beta}^{\bar{a}}_{n+v^a} = \frac{1}{\sqrt{2}}\left(\tilde{\alpha}^{2a+2}_{-(n+v^\alpha)} - i\tilde{\alpha}^{2a+3}_{-(n+v^\alpha)}\right)$$

and we have the twisted mode expansions

$$\begin{aligned} Z^a(\tau,\sigma) &= z^a_f + p^a\,\tau + \tfrac{i}{\sqrt{2}}\sum_{m=-\infty}^{\infty} \tfrac{1}{m-v^a}\tilde{\beta}^a_{m-v^a}\, e^{-2i(m-v^a)(\tau+\sigma)} \ , \\ Z^{\bar{a}}(\tau,\sigma) &= z^{\bar{a}}_f + p^{\bar{a}}\,\tau + \tfrac{i}{\sqrt{2}}\sum_{m=-\infty}^{\infty} \tfrac{1}{m+v^a}\tilde{\beta}^{\bar{a}}_{m+v^a}\, e^{-2i(m+v^a)(\tau+\sigma)} \ , \end{aligned}$$

where the subscript on z^a_f indicates the coordinate of a fixed point.
It is important to notice that in the twisted sectors there are only the oscillators

$$\tilde{\beta}^a_{m-v^a}\,,\ \tilde{\beta}^{\bar{a}}_{m+v^a} \qquad m \in \mathbb{N}\ . \tag{A.14}$$

To write down the appropriate number operator we need to know the commutation relations of the fractionally moded oscillators. Due to the commutation relations of the real oscillators, $[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = -m\,\delta_{m+n,0}\,\eta^{\mu\nu}$, they can be calculated

$$\begin{aligned} [\tilde{\beta}^a_{n-v^a}, \tilde{\beta}^{\bar{b}}_{m+v^b}] &= \frac{1}{2}[\tilde{\alpha}^{2a+2}_{n-v^a} + i\tilde{\alpha}^{2a+3}_{n-v^a}, \tilde{\alpha}^{2b+2}_{m+v^b} - i\tilde{\alpha}^{2b+3}_{m+v^b}] \\ &= \frac{1}{2}\left([\tilde{\alpha}^{2a+2}_{n-v^a}, \tilde{\alpha}^{2b+2}_{m+v^b}] - i([\tilde{\alpha}^{2a+2}_{n-v^a}, \tilde{\alpha}^{2b+3}_{m+v^b}] - [\tilde{\alpha}^{2a+3}_{n-v^a}, \tilde{\alpha}^{2b+2}_{m+v^b}]) + [\tilde{\alpha}^{2a+3}_{n-v^a}, \tilde{\alpha}^{2b+3}_{m+v^b}]\right) \\ &= \frac{1}{2}\left((n-v^a)\,\delta_{n+m,0}\,\delta^{ab} - i(0-0) + (n-v^a)\,\delta_{n+m,0}\,\delta^{ab}\right) \\ &= (n-v^a)\,\delta_{m+n,0}\,\delta^{ab}\ . \end{aligned}$$

By a very similar calculation we find

$$\begin{aligned} [\tilde{\beta}^a_{n-v^a}, \tilde{\beta}^b_{m-v^b}] &= \frac{1}{2}[\tilde{\alpha}^{2a+2}_{n-v^a} + i\tilde{\alpha}^{2a+3}_{n-v^a}, \tilde{\alpha}^{2b+2}_{m-v^b} - i\tilde{\alpha}^{2b+3}_{m-v^b}] \\ &= \frac{1}{2}\left((n-v^a)\,\delta_{n+m-2v^a,0}\,\delta^{ab} + i(0+0) - (n-v^a)\,\delta_{n+m-2v^a,0}\,\delta^{ab}\right) \\ &= 0\ . \end{aligned}$$

and analogously for the complex conjugated oscillators. Thus finally we arrive at the commutation relations

$$[\tilde{\beta}^a_{n-v^a}, \tilde{\beta}^{\bar{b}}_{m+v^b}] = (n - v^a)\delta^{ab}\delta_{m+n,0} \ , \tag{A.15}$$

$$[\tilde{\beta}^a_{n-v^a}, \tilde{\beta}^{b}_{m+v^b}] = [\tilde{\beta}^{\bar{a}}_{n-v^a}, \tilde{\beta}^{\bar{b}}_{m+v^b}] = 0 \ . \tag{A.16}$$

From $[\tilde{\beta}^a_{n-v^a}, \tilde{\beta}^{\bar{a}}_{-(n-v^a)}] = (n - v^a)$ it is obvious that we have to distinguish between two cases:

$\boldsymbol{(n - v^a) > 0}$ **:** The $\tilde{\beta}^{\bar{a}}_{-(n-v^a)}$ are creation and the $\tilde{\beta}^{a}_{(n-v^a)}$ the corresponding annihilation operators. Due to the commutation relations the number operator then can be written as $\tilde{N} = \sum\limits_{n,a} \tilde{\beta}^{\bar{a}}_{-(n-v^a)}\tilde{\beta}^a_{n-v^a}$.

$\boldsymbol{(n - v^a) < 0}$ **:** In this case the creation operators are the $\tilde{\beta}^{a}_{(n-v^a)}$ and the annihilation operators the $\tilde{\beta}^{\bar{a}}_{-(n-v^a)}$. Now we have $\tilde{N} = \sum\limits_{n,a} \tilde{\beta}^a_{n-v^a}\tilde{\beta}^{\bar{a}}_{-(n-v^a)}$.

Therefore the complete number operator for the left movers reads

$$\tilde{N} \equiv \sum_{n-v^a>0} \tilde{\beta}^{\bar{a}}_{-(n-v^a)}\tilde{\beta}^a_{n-v^a} + \sum_{n-v^a<0} \tilde{\beta}^a_{n-v^a}\tilde{\beta}^{\bar{a}}_{-(n-v^a)} \ . \tag{A.17}$$

The oscillator states and $\tilde{N}$-eigenvalues for the two different cases are

$$\begin{array}{llcll} \tilde{\beta}^{\bar{a}}_{-(n-v^a)}|k\rangle & \text{with} \ \ \tilde{N}\,(\tilde{\beta}^{\bar{a}}_{-(n-v^a)}|k\rangle) & = & (n - v^a)\,(\tilde{\beta}^{\bar{a}}_{-(n-v^a)}|k\rangle) & (n - v^a > 0) \ , \\ \tilde{\beta}^{\bar{a}}_{n-v^a}|k\rangle & \text{with} \ \ \tilde{N}\,(\tilde{\beta}^{\bar{a}}_{n-v^a}|k\rangle) & = & (n - v^a)\,(\tilde{\beta}^{\bar{a}}_{n-v^a}|k\rangle) & (n - v^a < 0) \ . \end{array}$$

For completeness the results for the right movers are given. They follow from the same calculations as above.
The mode expansion for the right movers reads

$$Z^a(\tau, \sigma) = z^a + q^a\,\tau + \frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \frac{1}{n + v^a}\beta_{n+v^a}\, e^{-2i(n+v^a)(\tau-\sigma)} \ ,$$

with $\beta^a_{n+v^a} = \frac{1}{\sqrt{2}}(\alpha^{2a+2}_{n+v^a} + i\alpha^{2a+3}_{n+v^a})$ and $\beta^{\bar{a}}_{n-v^a} = \frac{1}{\sqrt{2}}(\alpha^{2a+2}_{-(n-v^a)} - i\alpha^{2a+3}_{-(n-v^a)})$. The only non-vanishing commutation relations are

$$[\beta^a_{n+v^a}, \beta^{\bar{b}}_{m-v^b}] = (n + v^a)\delta^{ab}\delta_{m+n,0} \ .$$

The number operator therefore finally can be written as

$$N \equiv \sum_{n+v^a>0} \beta^{\bar{a}}_{-(n+v^a)}\beta^a_{n+v^a} + \sum_{n+v^a<0} \beta^a_{n+v^a}\beta^{\bar{a}}_{-(n+v^a)} \ .$$

Appendix B

Group theory

B.1 Excerpts from finite group theory

Direct and semi-direct products: Suppose that there are two groups A and B and for every element $a \in A$ there is an automorphism $\hat{a}$ of B

$$a \mapsto \hat{a} \in \mathrm{Aut}(B) \qquad \forall\, a \in A\ .$$

Then the set of all ordered pairs $\{(a, b)|\, a \in A,\, b \in B\}$ with the product

$$(a, b)(a', b') = (aa', \hat{a}'(b)\, b')$$

forms a group. This group is called the *semi-direct product* of A and B, relative to the homomorphism $a \mapsto \hat{a}$ and is denoted by

$$A \ltimes B\ . \tag{B.1}$$

If $\hat{a} = \mathbb{1} \quad \forall a \in A$, the semi-direct product is nothing but the *direct product*, denoted by

$$A \times B\ . \tag{B.2}$$

For the construction in Sec. 4.3.1 we take the automorphism as the conjugation, and

therefore define the semi-direct product by

$$(a, b)(a', b') = (aa', a'^{-1} b\, a'\, b')\ .$$

Theorem: G is the semi-direct product of S and N if, and only if[1]

- N is a normal subgroup of G
- S is a subgroup of G

[1]For a simplified notation we define $A \cup B$ of the two groups A and B as the multiplicative closure.

- $N \cap S = \mathbb{1}$
- $S \cup N = G$

Character tables: The character χ_ρ of a certain matrix representation $\rho(g)$ of the finite group G, with $g \in G$, is defined as the trace

$$\chi_\rho(g) \equiv \mathrm{tr}\rho(g) \qquad g \in G \; .$$

Then it is easy to show that $\chi_\rho(g_1 g_2) = \chi_\rho(g_1)\chi_\rho(g_2)$ and that the characters are constant on conjugacy classes.

In a character table the rows denote the irreducible representations (irreps), the columns the conjugacy classes of G. The elements of the character table are the characters of the elements of the certain conjugacy classes in the appropriate representations. The first column therefore gives the dimensions of the irreps.
Obviously, there are as many irreps as there are conjugacy classes. The content of irreps of a certain general representation can be determined due to the following formula

$$n^{\rho_\alpha} = \frac{1}{|G|} \sum_{[g] \in G} |[g]| \, \chi_{\rho_\alpha}(g)^* \chi_\rho(g) \; , \tag{B.3}$$

where $|G|$ denotes the order of the group G, $[g]$ denotes a representative of the conjugacy class of the element $g \in G$, $|[g]|$ the number of elements contained in it and $\chi_\rho(g)$ and $\chi_{\rho_\alpha}(g)$ are the characters of the element $g \in G$ in the representations ρ and ρ_α, respectively. The summation is to be understood to run over conjugacy classes, which makes sense because the summands of all elements of one class are the same. Then n^{ρ_α} denotes the number, how often the irrep ρ_α is contained in ρ.
Thus Eq. (B.3) can be used to reduce a general representation, meaning to decompose it into a sum of irreps.

B.2 Finite discrete Subgroups of SU(3)

There is not as much literature available about discrete finite subgroups of SU(3) as about those of SU(2) and the overview of this section is mainly based only on [54, 55, 49].[2]
In [55] those subgroups of SU(3) which are not subgroups of SU(2) are classified into 10 *crystal-like* (Σ) and the two infinite series of the *dihedral-like* (Δ) subgroups. We will briefly summarise the most important facts for us.

Crystal-like groups Σ: The so-called crystal-like subgroups and their double groups are denoted by $\Sigma(n\phi)$ where $n\phi$ is the order of the group. The parameter ϕ with values 1 or 3 is introduced to distinguish between subgroups of SU(3) and of SU(3)/C.[3] For $\phi = 1$ the particular group is a subgroup of SU(3)/C while for $\phi = 3$ it is one of SU(3). $\Sigma(60)$ and $\Sigma(168)$ are subgroups of both. There are the following crystal-like subgroups

$$\Sigma(60),\ \Sigma(168),\ \Sigma(36\phi),\ \Sigma(72\phi),\ \Sigma(216\phi),\ \Sigma(360\phi)\ . \tag{B.4}$$

It can be shown that $\Sigma(60) = A_5$ with A_5 the alternating group on 5 letters[4] and that it is isomorphic to the icosahedral group. Furthermore $\Sigma(168) \subset A_7$ and $\Sigma(360) = A_6 \subset S_6$. The corresponding character tables and generators are given in [55].
For the groups $\Sigma(36), \Sigma(72), \Sigma(216)$ it can be shown that they are all subgroups of S_9 with the additional relation

$$\Sigma(36) \subset \Sigma(72) \subset \Sigma(216) \subset S_9\ .$$

$\Sigma(216)$ is also called *Hessian group*. Again the character tables and generators are listed in [55].

Dihedral-like groups Δ: In addition, there are the two infinite series of the dihedral-like subgroups

$$\Delta(3n^2),\ \ \Delta(6n^2) \qquad n \in \mathbb{Z}\ . \tag{B.5}$$

They are analogs of the dihedral groups. In [55] their generators are chosen such that the orders of the subgroups are $|\Delta(3n^2)| = 3n^2$ and $|\Delta(6n^2)| = 6n^2$ only for even n while for odd n the group of order $6(2 \cdot n)^2$ is generated. We will come back to this point later.
The classified SU(3) subgroups are shown in Tab. (B.1).

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[2]There is an additional comment in [48].

[3]The fractional linear group of SU(3) is defined by dividing out the centre C of SU(3)

$$C = \{\mathbb{1}, \omega\mathbb{1}, \omega^2\mathbb{1}\} \qquad \omega = e^{2\pi i/3}\ .$$

This corresponds to the construction SU(2)/C in the SU(2) case, where the parameter analogous to ϕ would take the values $1, 2$.

[4]A_5 is the subgroup of the symmetric group on 5 letters S_5, consisting only of the even permutations.

- From [55] we take the generators of the $\Delta(3n^2)$ to be $A(\frac{2\pi}{n}j, \frac{2\pi}{n}k)$ and $E(0,0)$ with $j,k \in \mathbb{Z}$ and

$$A(\frac{2\pi}{n}j, \frac{2\pi}{n}k) = \begin{pmatrix} e^{\frac{2\pi i}{n}j} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{n}k} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{n}(j+k)} \end{pmatrix} , \tag{B.6a}$$

$$E(0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \tag{B.6b}$$

There is an abstract way of writing the $\Delta(3n^2)$ groups as semi-direct products

$$\Delta(3n^2) = \mathbb{Z}_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \ , \tag{B.7}$$

which is used in [49].
The number of conjugacy classes is

$$\#(\text{classes}) = \begin{cases} \frac{1}{3}(8+n^2) & \frac{n}{3} \notin \mathbb{Z} \\ 8 + \frac{1}{3}n^2 & \frac{n}{3} \in \mathbb{Z} \end{cases}$$

and there are the following numbers of one and three dimensional irreps

$$\begin{array}{lcl} \text{1-dim:} & 3 & \frac{n}{3} \notin \mathbb{Z} \ , \\ \text{3-dim:} & \frac{1}{3}(n^2-1) \ , & \end{array}$$

$$\begin{array}{lcl} \text{1-dim:} & 9 & \frac{n}{3} \in \mathbb{Z} \ , \\ \text{3-dim:} & \frac{1}{3}(n^2-3) \ . & \end{array}$$

- To generate the $\Delta(6n^2)$ subgroups in addition to the two matrices of Eq. (B.6), in [55] a third generator is introduced

$$B(\frac{2\pi i}{n}j, \frac{2\pi}{n}k) = \begin{pmatrix} e^{\frac{2\pi i}{n}j} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi}{n}k} \\ 0 & -e^{-\frac{2\pi i}{n}(j+k)} & 0 \end{pmatrix} . \tag{B.8}$$

Like in the previous case also the $\Delta(6n^2)$ subgroups can be written as semi-direct products [49]

$$\Delta(6n^2) = S_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \ . \tag{B.9}$$

In this notation also the $\Delta(6n^2)$ groups for n odd are included (e.g. $\Delta(54)$ for $n=3$), as can easily be seen by calculating the orders from the order 6 and n of respectively the S_3 and the two $\mathbb{Z}_n$ groups to be $6 \cdot n \cdot n = 6n^2$ (independently of whether n is even or odd).[5] in this work, for simplified notation, we think of

[5]There is no contradiction to [55]. There, due to the particular choice of the generators, the $\Delta(6n^2)$ groups appear just as subgroups of the $\Delta(6(2\cdot n)^2)$ and are not seen as own Δ subgroups. In the semi-direct product notation, it is obvious that $S_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is a subgroup of $S_3 \ltimes (\mathbb{Z}_{2n} \times \mathbb{Z}_{2n})$.

the Δ subgroups as defined by Eq. (B.9), keeping in mind that this does not contradict [55].
The number of conjugacy classes for these groups is given by

$$\#(\text{classes}) = \begin{cases} \frac{1}{6}(8+9n+n^2) & \frac{n}{3} \notin \mathbb{Z} \\ \frac{1}{6}(24+9n+n^2) & \frac{n}{3} \in \mathbb{Z} \end{cases}$$

and there are one, two, three and six dimensional irreps of numbers

$$\begin{array}{lcl} \text{1-dim:} & 2 & \frac{n}{3} \in \mathbb{Z}\ , \\ \text{2-dim:} & 4\ , & \\ \text{3-dim:} & 2(n-1)\ , & \\ \text{6-dim:} & \frac{1}{6}n(n-3)\ , & \\ & & \\ \text{1-dim:} & 2 & \frac{n}{3} \notin \mathbb{Z}\ , \\ \text{2-dim:} & 1\ , & \\ \text{3-dim:} & 2(n-1)\ , & \\ \text{6-dim:} & \frac{1}{6}(n-2)(n-1)\ . & \end{array}$$

These formulae are taken from [49].

crystal-like	order	dihedral-like	n	order
$\Sigma(60)$	60	$\Delta(3n^2)$	1	3
$\Sigma(168)$	168		2	12
$\Sigma(36\phi)$	36, 108		3	27
$\Sigma(72\phi)$	72, 216		4	48
$\Sigma(216\phi)$	216, 648		5	75
$\Sigma(360\phi)$	360, 1080		6	108
			7	147
			8	192
			⋮	⋮
		$\Delta(6n^2)$	1	6, (24)
			2	24
			3	54, (216)
			4	96
			5	150, (600)
			6	216
			7	294, (1176)
			8	384
			⋮	⋮

Table B.1: Crystal-like and dihedral-like subgroups of SU(3). The numbers in brackets indicate the orders one would get by using the generators of [55] to generate the corresponding subgroup (as explained in the previous section).

B.2.1 G_{54} as a classified SU(3) subgroup

To show how G_{54} fits into the classification scheme of subgroups of SU(3) we construct an invariant subgroup and then write it as a semi-direct product due to the theorem of App. B.1.

There is an invariant subgroup N of G_{54} consisting of the matrices

$$N = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \bar{\omega} & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar\omega \end{pmatrix}, \begin{pmatrix} \bar\omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} \bar\omega & 0 & 0 \\ 0 & \bar\omega & 0 \\ 0 & 0 & \bar\omega \end{pmatrix} \Bigg\}$$

and thus of order 9. That N is indeed normal can be seen by noticing that the S_3 matrices only permute the diagonal entries. We also notice that N is isomorphic to a direct product of two $\mathbb{Z}_3$ groups, because the mapping to the elements (z_1, z_2) of the abstract direct product

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar\omega \end{pmatrix} \mapsto (z_1, 1)$$

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \bar\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (1, z_2)$$

does not change the group table.[6] Thus we have

$$N = \mathbb{Z}_3 \times \mathbb{Z}_3 \ .$$

Now we are in position to apply the theorem of App. B.1 to G_{54}:

- $N = \mathbb{Z}_3 \times \mathbb{Z}_3$ is an invariant subgroup of G_{54}
- $\mathrm{S}_3 \subset \mathrm{G}_{54}$ is a subgroup
- $\mathrm{S}_3 \cap (\mathbb{Z}_3 \times \mathbb{Z}_3) = 1$
- $\mathrm{S}_3 \cup (\mathbb{Z}_3 \times \mathbb{Z}_3) = \mathrm{G}_{54}$

The theorem then tells us that G_{54} can be written as the semi-direct product

$$\mathrm{G}_{54} = \mathrm{S}_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \Delta(54) \ , \tag{B.10}$$

where the last equality is due to [49] as explained in App. B.2.

remark: With the multiplication in the semi-direct product defined in App. B.1 the assignment between G_{54} and $\mathrm{S}_3 \ltimes \mathbb{Z}_3 \times \mathbb{Z}_3$ is the following. For an element (s, z) (with $s \in \mathrm{S}_3$, $z \in N$) in the semi-direct product we get the corresponding element of G_{54} by

$$(s, z) \mapsto s\, z \ .$$

This respects the multiplicative structure and therefore is an isomorphism

$$(s, z)(s', z') = sz\, s'z' = ss'(s'^{-1}zs')z' = (ss', s'^{-1}zs'z') \ .$$

[6] That the mapping

$$z_1 z_2 \mapsto (z_1, z_2)$$

is a group homomorphism means that it respects the multiplicative structure, is easy to show, because N is Abelian

$$(z_1, z_2)(z_1', z_2') = z_1 z_2 z_1' z_2' = z_1 z_1' z_2 z_2' = (z_1 z_1', z_2 z_2') \ .$$

B.2.2 Properties of $\Delta(54)$

In this section we consider the specific properties of $\Delta(54)$ which we need to know for explicit model building. In particular we are interested in the irreducible representations into which the states will fall. The formulae for the numbers of conjugacy classes and irreps are given in Sec. B.2.
For the purpose of reducing products of representations, we denote the character table of $\Delta(54)$ in Tab. (B.2). From the table we read off the existence of the following 10 irreps

$$\begin{array}{rc} \text{four 3-dim irreps} & \mathbf{3},\, \bar{\mathbf{3}},\, \mathbf{3}',\, \bar{\mathbf{3}}' \;, \\ \text{four 2-dim irreps} & \mathbf{2}_1,\, \mathbf{2}_2,\, \mathbf{2}_3,\, \mathbf{2}_4 \;, \\ \text{two 1-dim irreps} & \mathbf{1}_1,\, \mathbf{1}_2 \;. \end{array}$$

With help of Eq. (B.3) from App. B.1 all various products of these representations can be reduced. E.g. in the product representation $\mathbf{3} \times \mathbf{3}$ there are one $\bar{\mathbf{3}}'$ and two $\bar{\mathbf{3}}$ contained and thus

$$\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}}' + 2 \cdot \bar{\mathbf{3}} \;.$$

Doing so for all possible products we finally derive Tab. (B.3). Many of the group theoretical calculations were done by use of the program GAP [56].

irrep	1a	6a	6b	3a	3b	3c	2a	3d	3e	3f
	(1)	(9)	(9)	(6)	(6)	(6)	(9)	(6)	(1)	(1)
$\mathbf{1}_1$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_2$	1	-1	-1	1	1	1	-1	1	1	1
$\mathbf{2}_1$	2	0	0	2	-1	-1	0	-1	2	2
$\mathbf{2}_2$	2	0	0	-1	-1	-1	0	2	2	2
$\mathbf{2}_3$	2	0	0	-1	-1	2	0	-1	2	2
$\mathbf{2}_4$	2	0	0	-1	2	-1	0	-1	2	2
$\mathbf{3}'$	3	$-\bar{\omega}$	$-\omega$	0	0	0	-1	0	$3\,\bar{\omega}$	$3\,\omega$
$\bar{\mathbf{3}}'$	3	$-\omega$	$-\bar{\omega}$	0	0	0	-1	0	$3\,\omega$	$3\,\bar{\omega}$
$\bar{\mathbf{3}}$	3	ω	$\bar{\omega}$	0	0	0	1	0	$3\,\omega$	$3\,\bar{\omega}$
$\mathbf{3}$	3	$\bar{\omega}$	ω	0	0	0	1	0	$3\,\bar{\omega}$	$3\,\omega$

Table B.2: Character table of the group $\Delta(54)$ (with $\omega = e^{2\pi i/3}$). The second row gives the number of elements in the certain class and the second column the dimension of the representation.

irrep	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}'$	$\mathbf{3}'$	$\mathbf{2}_4$	$\mathbf{2}_3$	$\mathbf{2}_2$	$\mathbf{2}_1$	$\mathbf{1}_2$	$\mathbf{1}_1$
$\mathbf{3}$	$\bar{\mathbf{3}}'+2\cdot\bar{\mathbf{3}}$	$\mathbf{1}_1+$ $\mathbf{2}_1+\mathbf{2}_2+\mathbf{2}_3+\mathbf{2}_4$	$\mathbf{1}_2+$ $\mathbf{2}_1+\mathbf{2}_2+\mathbf{2}_3+\mathbf{2}_4$	$\bar{\mathbf{3}}+2\cdot\bar{\mathbf{3}}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{3}$
$\bar{\mathbf{3}}$		$\mathbf{3}'+2\cdot\mathbf{3}$	$\mathbf{3}+2\cdot\mathbf{3}'$	$\mathbf{1}_2+$ $\mathbf{2}_1+\mathbf{2}_2+\mathbf{2}_3+\mathbf{2}_4$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}$
$\bar{\mathbf{3}}'$			$\mathbf{3}'+2\cdot\mathbf{3}$	$\mathbf{1}_1+$ $\mathbf{2}_1+\mathbf{2}_2+\mathbf{2}_3+\mathbf{2}_4$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}+\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}'$	$\bar{\mathbf{3}}$
$\mathbf{3}'$				$\bar{\mathbf{3}}'+2\cdot\bar{\mathbf{3}}$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}+\mathbf{3}'$	$\mathbf{3}$	$\mathbf{3}'$
$\mathbf{2}_4$					$\mathbf{1}_1+\mathbf{1}_2+$ $\mathbf{2}_4$	$\mathbf{2}_1+\mathbf{2}_2$	$\mathbf{2}_1+\mathbf{2}_3$	$\mathbf{2}_2+\mathbf{2}_3$	$\mathbf{2}_4$	$\mathbf{2}_4$
$\mathbf{2}_3$						$\mathbf{1}_1+\mathbf{1}_2+$ $\mathbf{2}_3$	$\mathbf{2}_1+\mathbf{2}_4$	$\mathbf{2}_2+\mathbf{2}_2$	$\mathbf{2}_3$	$\mathbf{2}_3$
$\mathbf{2}_2$							$\mathbf{1}_1+\mathbf{1}_2+$ $\mathbf{2}_2$	$\mathbf{2}_3+\mathbf{2}_4$	$\mathbf{2}_2$	$\mathbf{2}_2$
$\mathbf{2}_1$								$\mathbf{1}_1+\mathbf{1}_2+$ $\mathbf{2}_1$	$\mathbf{2}_1$	$\mathbf{2}_1$
$\mathbf{1}_2$									$\mathbf{1}_1$	$\mathbf{1}_2$
$\mathbf{1}_1$										$\mathbf{1}_1$

Table B.3: Reducing all possible products of irreps of $\Delta(54)$.

Appendix C

Matter content of models

In this section we display the full matter content of those models we referred to in this work.
To explain the notation, with $(\mathbf{X};\mathbf{Y})_Q$ we denote an X representation under the observable part of the gauge group and a Y representation under the hidden part, with $\mathrm{U}(1)_Y$ or $\mathrm{U}(1)_{B-L}$ charge Q for a standard model or a LR gauge group respectively. With f^A, $\tilde{f}^A$ we denote the quark doublets and singlets (for a LR model the representations $(\mathbf{3},\mathbf{2},\mathbf{1})$ and $(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})$, for an SU(5) model the matter $\mathbf{10}$ and $\bar{\mathbf{5}}$), with $l^A,\tilde{l}^A$ the lepton doublets and singlets, with s^A the singlets and with h^A the potential higgses of the certain model.

In this work we considered different models. They are labelled as follows:

model (A)	Standard embedding,
model (A')	Standard embedding transformed,
model (B)	LR model (this is the model of [22]),
model (C)	LR model with one Wilson line,
model (C')	LR model with two Wilson lines,
model (D)	SU(5) model without any Wilson lines,
model (D')	SU(5) model with two Wilson lines .

The matter content of these models is given in Tab. (C.1),...,Tab. (C.7), respectively. The third column of each table gives the multiplicity of the states, the fourth their interpretation as known particles, if possible.

Model (A): • Shifts:

$$\begin{array}{rcl} V_1 & = & \frac{1}{3}(1,0,-1,0,0,0,0,0) \quad (0,0,0,0,0,0,0,0) \\ V_2 & = & \frac{1}{3}(0,1,-1,0,0,0,0,0) \quad (0,0,0,0,0,0,0,0) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{E}_6 \times \mathrm{E}_8 \times (\mathrm{U}(1))^2$$

- Matter content given in Tab. (C.1).

Model (A'): • Shifts:

$$\begin{array}{rcl} V_1 & = & \frac{1}{3}(1,-3,-1,3,0,0,0,0) \quad (0,0,0,0,0,0,0,0) \\ V_2 & = & \frac{1}{3}(3,1,-1,0,3,0,0,0) \quad (0,0,0,0,0,0,0,0) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{E}_6 \times \mathrm{E}_8 \times (\mathrm{U}(1))^2$$

- Matter content given in Tab. (C.2).

Model (B): • Shifts:

$$\begin{array}{rcl} V_1 & = & \frac{1}{3}(-4,1,1,0,0,0,0,0) \quad (1,1,0,0,0,0,0,0) \\ V_2 & = & \frac{1}{3}(0,2,0,0,0,0,0,0) \quad (0,-1,4,4,1,0,0,0) \\ A_1 & = & \frac{1}{3}(0,0,0,0,0,1,1,-2) \quad (0,0,0,0,0,-2,-2,-2) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1)^6$$

- Matter content given in Tab. (C.3).

Model (C): • Shifts:

$$\begin{array}{rcll} V_1 &=& \frac{1}{3}(2,1,1,0,0,0,0,0) & (1,-2,3,0,0,0,0,0) \\ V_2 &=& \frac{1}{3}(3,3,1,3,0,0,0,0) & (3,3,1,3,0,0,0,0) \\ A_1 &=& \frac{1}{3}(3,3,3,0,0,1,1,1) & (2,-2,3,-2,1,1,1,0) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times (\mathrm{U}(1))^8$$

- Matter content given in Tab. (C.4).

Model (C'): • Shifts:

$$\begin{array}{rcll} V_1 &=& \frac{1}{3}(2,1,1,0,0,0,0,0) & (1,-2,3,0,0,0,0,0) \\ V_2 &=& \frac{1}{3}(3,3,1,3,0,0,0,0) & (3,3,1,3,0,0,0,0) \\ A_1 &=& \frac{1}{3}(3,3,3,0,0,1,1,1) & (2,-2,3,-2,1,1,1,0) \\ A_2 &=& \frac{1}{3}(0,1,-1,0,0,0,0,0) & (5,-5,-5,-2,0,-3,3,9) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(3) \times (\mathrm{U}(1))^{10}$$

- Matter content given in Tab. (C.5).

Model (D): • Shifts:

$$\begin{array}{rcll} V_1 &=& \frac{1}{3}(2,1,1,0,0,0,0,0) & (1,-2,3,0,0,0,0,0) \\ V_2 &=& \frac{1}{3}(3,3,1,3,1,1,1,1) & (3,3,1,2,1,0,0,0) \end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^3$$

- Matter content given in Tab. (C.6).

Model (D'): • Shifts:

$$\begin{array}{rcrl}
V_1 & = & \frac{1}{3}(2,1,1,0,0,0,0,0) & (1,-2,3,0,0,0,0,0) \\
V_2 & = & \frac{1}{3}(3,3,1,3,1,1,1,1) & (3,3,1,2,1,0,0,0) \\
A_1 & = & \frac{1}{3}(2,-2,-2,2,2,2,2,2) & (5,-5,-2,-2,3,-9,-3,3) \\
A_2 & = & \frac{1}{3}(4,4,0,0,0,0,0,0) & (2,-2,0,-2,-2,-2,-2,4)
\end{array}$$

- Unbroken gauge group:

$$\mathrm{G}_{unbroken} = \mathrm{SU}(5) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^{10}$$

- Matter content given in Tab. (C.7).

sector	representation			sector	representation		
U_1	$(\overline{\mathbf{27}};\mathbf{1})$	1		$T_{(2,2)}$	$(\overline{\mathbf{27}};\mathbf{1})$	27	
U_2	$(\overline{\mathbf{27}};\mathbf{1})$	1			$(\mathbf{1};\mathbf{1})$	81	
U_3	$(\overline{\mathbf{27}};\mathbf{1})$	1					
$T_{(1,0)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9		$T_{(2,0)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9	
	$(\mathbf{1};\mathbf{1})$	27			$(\mathbf{1};\mathbf{1})$	27	
$T_{(0,1)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9		$T_{(0,2)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9	
	$(\mathbf{1};\mathbf{1})$	27			$(\mathbf{1};\mathbf{1})$	27	
$T_{(1,2)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9		$T_{(2,1)}$	$(\overline{\mathbf{27}};\mathbf{1})$	9	
	$(\mathbf{1};\mathbf{1})$	27			$(\mathbf{1};\mathbf{1})$	27	

Table C.1: Matter content of the various sectors of the standard embedding model of Sec. 3.1.

sector	representation			sector	representation		
U_1	$(\overline{\mathbf{27}};\mathbf{1})$	1		$T_{(2,2)}$	$(\mathbf{1};\mathbf{1})$	81	
U_2	$(\overline{\mathbf{27}};\mathbf{1})$	1					
U_3	$(\overline{\mathbf{27}};\mathbf{1})$	1					
$T_{(1,0)}$	$(\mathbf{27};\mathbf{1})$	9		$T_{(2,0)}$	$(\mathbf{1};\mathbf{1})$	27	
	$(\mathbf{1};\mathbf{1})$	27					
$T_{(0,1)}$	$(\mathbf{1};\mathbf{1})$	27		$T_{(0,2)}$	$(\mathbf{27};\mathbf{1})$	9	
					$(\mathbf{1};\mathbf{1})$	27	
$T_{(1,2)}$	$(\mathbf{1};\mathbf{1})$	27		$T_{(2,1)}$	$(\mathbf{27};\mathbf{1})$	9	
					$(\mathbf{1};\mathbf{1})$	27	

Table C.2: Matter content of the various sectors of the brother model of the standard embedding model, which was obtained by adding appropriate $\mathrm{E}_8\times\mathrm{E}_8$ roots to the standard embedding shifts.

sector	representation			sector	representation		
U_1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	s^A	$T_{(2,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	18	s^A
	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$	1			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	18	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1})$	1					
U_2	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	s^A				
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})$	1					
U_3	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	s^A				
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})$	1					
$T_{(1,0)}$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	3	h^A	$T_{(2,0)}$	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3	f^A
	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3			$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	3	$\tilde{f}^A$
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3	l^A
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	3	$\tilde{l}^A$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	3			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3	s^A		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	12	s^A
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	3			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{3})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}})$	3			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	3			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})$	3					
$T_{(0,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	9	s^A	$T_{(0,2)}$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	9	h^A
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	9					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	9					
$T_{(1,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	6	s^A	$T_{(2,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	12	s^A
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	6			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	6	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	6			$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	3	
	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3			$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}})$	3			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	3			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{3})$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})$	3			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$	3	
					$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\overline{\mathbf{4}},\mathbf{1})$	3	

Table C.3: Matter content of the various sectors of model (B), the LR model of [22].

sector	representation			sector	representation		
U_1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	2	s^A	$T_{(2,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	18	s^A
	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2};\mathbf{1})_0$	1			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	18	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4})_0$	1			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{1/3}$	9	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}})_0$	1			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	9	
U_2	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{-1}$	1	l^4				
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_1$	1	$\tilde{l}^4$				
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_0$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	4	s^A				
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4})_0$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}})_0$	1					
U_3	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1};\mathbf{1})_0$	1	h^1				
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2};\mathbf{1})_{-1}$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2};\mathbf{1})_1$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{6})_0$	1					
$T_{(1,0)}$	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{1/3}$	3	f^A	$T_{(2,0)}$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1};\mathbf{1})_0$	3	h^A
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	3	$\tilde{f}^A$		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3	s^A
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{-1}$	3	l^A		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_0$	6	
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_1$	3	$\tilde{l}^A$		$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_0$	3			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{-1/3}$	6	
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	6			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	6	
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{1/3}$	6			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	6	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	6			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})_{-2/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	3			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{3}})_{2/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4})_{-2/3}$	3			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{4})_{2/3}$	3	
$T_{(0,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}})_0$	9		$T_{(0,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	9	
$T_{(2,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	9	s^A	$T_{(1,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	12	s^A
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{-2/3}$	3			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{-2/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	3			$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	6			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{1/3}$	6	
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{2/3}$	6			$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{1})_{-1/3}$	6	
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3			$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3	
	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	6			$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{4})_0$	3	
					$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{4}})_0$	3	
					$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{6})_0$	3	

Table C.4: Matter content of the various sectors of model (C), the LR model of Sec. 3.4.1 with one Wilson line.

sector	representation			sector	representation		
U_1	$(\mathbf{1},\mathbf{2},\mathbf{2};\mathbf{1})_0$	1	h^A	$T_{(2,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	21	s^A
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	1	s^A		$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	9	
U_2	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1}$	1	l^4		$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{-1/3}$	9	
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_1$	1	$\tilde{l}^4$		$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	12	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{3}})_0$	1			$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	12	
U_3	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1}$	1	l^5				
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_1$	1	$\tilde{l}^5$				
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})_0$	1					
$T_{(1,0)}$	$(\mathbf{3},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	3	f^A	$T_{(2,0)}$	$(\mathbf{1},\mathbf{2},\mathbf{2};\mathbf{1})_0$	3	h^A
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2};\mathbf{1})_{-1/3}$	3	$\tilde{f}^A$		$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	6	s^A
	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1}$	3	l^A		$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_1$	3	$\tilde{l}^A$		$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{1/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3	s^A		$(\mathbf{3},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	3			$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	3					
$T_{(0,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	33	s^A	$T_{(0,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	6	s^A
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})_0$	3					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{3}})_0$	3					
$T_{(1,2)}$	$(\mathbf{1},\mathbf{2},\mathbf{2};\mathbf{1})_0$	1	h^A	$T_{(2,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	15	s^A
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_0$	8	s^A		$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	3	
	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	2			$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{-1/3}$	5	
	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})_{1/3}$	4			$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{1/3}$	5	
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{-1/3}$	4			$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{-1/3}$	3	
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})_{1/3}$	2			$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	5	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	6			$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	5	
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	6			$(\mathbf{3},\mathbf{1},\mathbf{1};\mathbf{1})_0$	1	
	$(\mathbf{3},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3			$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1};\mathbf{1})_0$	1	
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1};\mathbf{1})_0$	3			$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})_0$	4	
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1};\mathbf{1})_{2/3}$	1			$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{3}})_0$	4	
	$(\mathbf{3},\mathbf{1},\mathbf{1};\mathbf{1})_{-2/3}$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})_{-2/3}$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{3}})_{2/3}$	1					

Table C.5: Matter content of the various sectors of model (C'), the LR model of Sec. 3.4.1 with two Wilson lines.

sector	representation			sector	representation		
U_1	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	s^A	$T_{(2,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1})$	27	
	$(\mathbf{5},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\overline{\mathbf{5}},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{20},\mathbf{1})$	1					
U_2	$(\overline{\mathbf{5}},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	h				
	$(\overline{\mathbf{10}},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{15}},\mathbf{3})$	1					
U_3	$(\mathbf{5},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	1	h				
	$(\mathbf{10},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	1					
	$(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{6}},\overline{\mathbf{3}})$	1					
$T_{(1,0)}$	$(\mathbf{10},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	9	f^A	$T_{(2,0)}$	$(\overline{\mathbf{5}},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	9	$\tilde{f}^A$
	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	9			$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$	9	
					$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	18	
					$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	9	s^A
$T_{(0,1)}$	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{3})$	9		$T_{(0,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\overline{\mathbf{3}})$	9	
$T_{(1,2)}$	$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}})$	9		$T_{(2,1)}$	$(\mathbf{1},\mathbf{2},\mathbf{1};\overline{\mathbf{6}},\mathbf{1})$	9	

Table C.6: Matter content of the various sectors of the SU(5) model of Sec. 3.4.2 without Wilson lines ($\mathrm{G}_{unbroken} = \mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{SU}(3) \times (\mathrm{U}(1))^4$).

sector	representation			sector	representation		
U_1	$(\mathbf{1};\mathbf{1})$	3	s^1	$T_{(2,2)}$	$(\mathbf{1};\mathbf{1})$	87	$s^{158,\ldots,244}$
U_2	$(\overline{\mathbf{5}};\mathbf{1})$	1	$\boldsymbol{h_d}$		$(\mathbf{1};\mathbf{3})$	3	$\chi^{12,13,14}$
	$(\mathbf{1};\mathbf{1})$	1	s^2		$(\mathbf{1};\overline{\mathbf{3}})$	6	$\tilde{\chi}^{9,\ldots,14}$
	$(\mathbf{1};\mathbf{3})$	1	χ^1				
	$(\mathbf{1};\overline{\mathbf{3}})$	1	$\tilde{\chi}^1$				
U_3	$(\mathbf{5};\mathbf{1})$	1	ϕ^1				
	$(\mathbf{1};\mathbf{1})$	1	s^3				
	$(\mathbf{1};\mathbf{3})$	1	χ^2				
$T_{(1,0)}$	$(\mathbf{10};\mathbf{1})$	3	$\boldsymbol{f^A}$	$T_{(2,0)}$	$(\overline{\mathbf{5}};\mathbf{1})$	3	$\boldsymbol{\tilde{f}^A}$
	$(\overline{\mathbf{5}};\mathbf{1})$	3	$\tilde{\phi}^{1,2,3}$		$(\mathbf{5};\mathbf{1})$	3	$\phi^{2,3,4}$
	$(\mathbf{1};\mathbf{1})$	9	$s^{4,\ldots,12}$		$(\mathbf{1};\mathbf{1})$	18	$s^{13,\ldots,30}$
$T_{(0,1)}$	$(\mathbf{1};\mathbf{1})$	24	$s^{31,\ldots,54}$	$T_{(0,2)}$	$(\mathbf{1};\mathbf{1})$	21	$s^{55,\ldots,75}$
					$(\overline{\mathbf{5}};\mathbf{1})$	3	$\tilde{\phi}^{4,5,6}$
$T_{(1,2)}$	$(\mathbf{1};\mathbf{1})$	36	$s^{76,\ldots,111}$	$T_{(2,1)}$	$(\mathbf{1};\mathbf{1})$	46	$s^{112,\ldots,157}$
	$(\mathbf{5};\mathbf{1})$	4	$\boldsymbol{h_u}$, $\phi^{5,6,7}$		$(\mathbf{5};\mathbf{1})$	3	$\phi^{8,9,10}$
	$(\mathbf{1};\mathbf{3})$	3	$\chi^{3,4,5}$		$(\mathbf{1};\mathbf{3})$	6	$\chi^{6,\ldots,11}$
	$(\mathbf{1};\overline{\mathbf{3}})$	3	$\tilde{\chi}^{2,3,4}$		$(\mathbf{1};\overline{\mathbf{3}})$	4	$\tilde{\chi}^{5,\ldots,8}$
	$(\overline{\mathbf{5}};\mathbf{1})$	1	$\tilde{\phi}^7$		$(\overline{\mathbf{5}};\mathbf{1})$	3	$\tilde{\phi}^{8,9,10}$

Table C.7: Matter content of the various sectors of the SU(5) model of Sec. 3.4.2, with two Wilson lines. We numbered the singlets and additional non-singlet representations.

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