Geometrical Aspects of Heterotic Orbifolds

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Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Referent:	Prof. Dr. Hans Peter Nilles
Korreferent :	Priv. Doz. Dr. Stefan Förste

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Contents

Chapter 1

Introduction

The Standard Model of particle physics with three generations of quarks and leptons has been successfully tested up to energies of ~ 100 GeV. Its local $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge invariance explains the strong, the weak and the electromagnetic interactions. The origin of mass is described by the Higgs mechanism, where the Higgs particle is as yet undiscovered.

Despite of its success, some unsatisfactory aspects indicate the need to go beyond the framework of the Standard Model. Among the most prominent suggestions are:

- Supersymmetry.
  Supersymmetric models have many benefits, one of them is that the gauge couplings of the strong, the weak and the electromagnetic interactions unify at a high scale at 3×10^{16} GeV, suggesting the idea of grand unification.

- Grand Unified Theories (GUTs).
  The main idea of grand unification is that at high energies $\sim 3 \times 10^{16}$ GeV a larger gauge symmetry is realized, which describes the strong, the weak and the electromagnetic interactions in a unified framework. Possible GUT gauge groups are for example SU(5) and SO(10).

- Extra dimensions.
  Theories with extra dimensions can, for example, provide a nice setting for GUTs, i.e. the GUT gauge group is realized when the extra dimensions are resolved at high energies. Additionally, this setup allows for a very appealing solution of the doublet-triplet splitting problem.

In general, superstring theories incorporate all features that are listed above. Therefore, they are promising candidates for a unified description of all fundamental interactions, including gravity. From the five consistent string theories, the ten-dimensional $\mathrm{E}_8 \times \mathrm{E}_8'$ heterotic string theory is considered in this work. This theory has $\mathcal{N} = 1$ supersymmetry in $d = 10$ and an $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge group. Furthermore, the spectrum of massless particles contains a spin-2 particle, which can be identified as the graviton.

However, the most obvious difference between the heterotic string theory and the observable world is that we live in four-dimensional space-time and not in ten. Therefore, a compactification scheme is needed that effectively reduces the number of dimensions from ten to four, while preserving $\mathcal{N} = 1$ SUSY in $d = 4$. One method is the compactification on a Calabi-Yau manifold. But only very few explicit examples are known and up to now an easy method for model building is missing. A second possibility is the compactification on an orbifold, which can often be viewed as a special limit of Calabi-Yau manifolds where the curvature is concentrated at isolated points. One of the main advantages of this simpler setup is that orbifolds are flat except at the singularities and thus allow for explicit calculations. By introducing background fields, such as Wilson lines, realistic models can be constructed.

Overview

The outline of this work is as follows: Chapter 2 begins with a brief discussion of the heterotic string and fixes the notation. Next, we consider compactification on a six-torus and describe the problems with this approach. Orbifolds are presented as a possible candidates for constructing phenomenologically viable models. The rules and the consistency conditions for orbifold constructions are explained in some detail, where special emphasis is given on the geometry. In chapter 3, the general rules for orbifold compactifications, as described in the previous chapter, are illustrated using the simplest setup, namely the $\mathbb{Z}_3$ orbifold. Chapters 4 and 5 are the main parts of this work. In chapter 4, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is explained, which involves many additional features not present in the context of prime orbifolds. In particular, its geometry has a richer structure, allowing for states living in four as well as in six dimensions. This will ultimately lead us to the picture of branes embedded in ten-dimensional space. Furthermore, it is shown that Wilson lines are an effective mechanism to break the gauge group and, more importantly, to reduce the number of chiral generations. In chapter 5, using the insight gained from the previous chapters, the first three generations model in the framework of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds is constructed.

Chapter 2

Orbifold Constructions

This chapter starts with a brief review of the ten-dimensional heterotic string. Later, it is shown that the attempt to compactify six dimensions on a six-torus leads to problems and it is described how the compactification on orbifolds can solve them. Furthermore, all tools to calculate the massless spectrum are given and special emphasis is given on the geometry.

2.1 The Heterotic String

The starting point for orbifold compactifications is the heterotic string in ten dimensions [1]. We use the bosonic construction with a ten dimensional right moving superstring and a 26 dimensional left moving bosonic string. 16 left moving bosonic coordinates are compactified on the torus $T_{\mathrm{E}_8\times\mathrm{E}'_8}$, which is defined to be the root lattice of the group $\mathrm{E}_8 \times \mathrm{E}'_8$[1]. The heterotic string is a theory of closed strings, propagating in ten dimensions. The coordinates of the string are mappings from the world sheet coordinates τ (proper time) and σ (space-like parameter) into the ten-dimensional target space. However, not all ten coordinates are independent. Therefore, in light cone coordinates, two components of the left and right movers are gauge fixed. We will denote the remaining physical degrees of freedom of the right moving superstring by X_R^i and Ψ_R^i (for $i = 1, \ldots, 8$). The coordinates X_R^i are world sheet bosons and Ψ_R^i are world sheet fermions. The physical left moving degrees of freedom are denoted by X_L^i, $i = 1, \ldots, 8$, and X_L^I, $I = 1, \ldots, 16$, both world sheet bosons. Since the coordinates X_L^I are compactified on the $T_{\mathrm{E}_8\times\mathrm{E}'_8}$, they are called gauge degrees of freedom. Fig. (2.1) summarizes the notation.

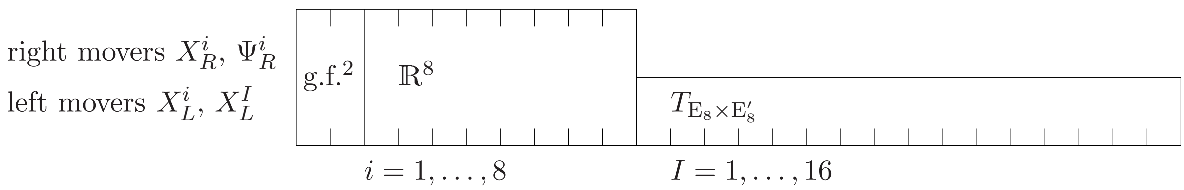


Figure 2.1: The coordinates of the heterotic string.

[1] Another possibility is the $\mathrm{Spin}(32)/\mathbb{Z}_2$ lattice, which leads to an SO(32) gauge group.
[2] gauge fixed due to light cone coordinates.

2.1.1 The Equations of Motion

The equations of motion are obtained from the world sheet action using the Hamilton principle. The result for the **bosonic coordinates** is (for $i = 1, \ldots, 8$ and $I = 1, \ldots, 16$)

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^i(\tau,\sigma) = 0 \tag{2.1}$$

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X_L^I(\tau,\sigma) = 0 \tag{2.2}$$

together with the boundary conditions (for $\Lambda \in T_{\mathrm{E}_8\times\mathrm{E}_8'}$)

$$X^i(\tau,\sigma+2\pi) = X^i(\tau,\sigma) \tag{2.3}$$

$$X_L^I(\tau,\sigma+2\pi) = X_L^I(\tau,\sigma) + 2\pi\Lambda^I\ . \tag{2.4}$$

Because one end of the string corresponds to $\sigma = 0$ and the other to $\sigma = 2\pi$, these equations state that both ends coincide. Therefore, the string is closed.

The general solution is given by a mode expansion. For $i = 1, \ldots, 8$ we split the solution into right and left moving parts ($X = X_R + X_L$, see app. (B.1)),

$$X_R^i(\tau,\sigma) = \frac{1}{2}x^i + \frac{1}{2}p^i(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^i}{n}e^{-in(\tau-\sigma)} \qquad \text{(right mover)} \tag{2.5}$$

$$X_L^i(\tau,\sigma) = \frac{1}{2}x^i + \frac{1}{2}p^i(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^i}{n}e^{-in(\tau+\sigma)} \qquad \text{(left mover)} \tag{2.6}$$

$$X_L^I(\tau,\sigma) = x_L^I + p^I(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^I}{n}e^{-in(\tau+\sigma)} \qquad \text{(gauge d.o.f.)} \tag{2.7}$$

with $(\alpha_n^i)^\dagger = \alpha_{-n}^i$, $(\tilde{\alpha}_n^i)^\dagger = \tilde{\alpha}_{-n}^i$ and $(\tilde{\alpha}_n^I)^\dagger = \tilde{\alpha}_{-n}^I$ so that X_R and X_L are real.

The equations of motion for the (right moving) **fermionic coordinates** are given by

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)\Psi_R^i(\tau,\sigma) = 0 \qquad \text{for } i = 1, \ldots, 8. \tag{2.8}$$

It turns out that Ψ_R^i can either have periodic or anti-periodic boundary conditions, called Ramond (R) and Neveu-Schwarz (NS) boundary conditions, respectively. They read

$$\Psi_R^i(\tau,\sigma+2\pi) = \pm\Psi_R^i(\tau,\sigma)\ . \tag{2.9}$$

The general solution of the equation of motion is given by the mode expansions

$$\Psi_R^i(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^i e^{-in(\tau-\sigma)} \qquad \text{R} \tag{2.10}$$

$$\Psi_R^i(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^i e^{-ir(\tau-\sigma)} \qquad \text{NS} \tag{2.11}$$

depending on the boundary condition.

2.1.2 The Quantized Right Movers

In general, a quantized right mover is given by the bosonic oscillators α_n^i and the fermionic oscillators d_n^i and b_r^i acting on the right moving ground state $|0\rangle_R$.

We restrict ourselves to massless right movers. They are given by acting with either the NS-oscillators $b_{-1/2}^i$ or with the R-oscillators d_0^i on the right moving ground state (for $i = 1, \ldots, 8$). Therefore, the corresponding states are denoted by NS or R, respectively. Furthermore, it turns out that the NS state transforms as an SO(8) vector and the R state as an SO(8) spinor. Thus we can write the NS and R state as representations of SO(8):

- The **SO(8) vector NS state** can be represented by a weight vector

$$q = \left(\underline{\pm 1, 0, 0, 0}\right) \qquad \text{NS}. \tag{2.12}$$

  The underscore denotes that all permutations are included.

- The **SO(8) spinor R state** can be written as

$$q = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \qquad \text{R} \tag{2.13}$$

  with an even number of plus signs. The first component of this spinor defines the four-dimensional chirality of the state,

$$\begin{aligned} -\frac{1}{2} &\;\Leftrightarrow\; \text{left-handed} \\ +\frac{1}{2} &\;\Leftrightarrow\; \text{right-handed.} \end{aligned}$$

Both the NS and the R states $|q\rangle_R$ fulfill the same equation for massless right movers,

$$\frac{m_R^2}{4} = \frac{1}{2}q^2 - \frac{1}{2} = 0. \tag{2.14}$$

2.1.3 The Quantized Left Movers

A quantized left mover is given by the oscillators $\tilde{\alpha}^i_{-n}$ and $\tilde{\alpha}^I_{-n}$ acting on the left moving ground state $|0\rangle_L$.

Since the sixteen gauge degrees of freedom X^I_L are compactified on a sixteen-torus $T_{\mathrm{E}_8\times\mathrm{E}_8'}$, the corresponding momenta p^I are quantized and correspond to the root vectors of the $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge group (see appendix A.1). Therefore, the momenta p^I can be seen as internal quantum numbers of the left moving ground state $|p\rangle_L$ that characterize the transformation properties of the ground state under the $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge group.

The massless left movers fulfill the equation

$$\frac{m_L^2}{4} = \frac{1}{2}p^2 + \tilde{N} - 1 = 0 \ , \tag{2.15}$$

where $p^2 = \sum_{I=1}^{16}(p^I)^2$, and $\tilde{N}$ is the number operator. It counts the number of oscillators $\tilde{\alpha}_{-n}$ acting on the ground state, i.e.

$$\begin{aligned} \tilde{N}\left(\tilde{\alpha}^i_{-n}|p\rangle_L\right) &= n\left(\tilde{\alpha}^i_{-n}|p\rangle_L\right) & (2.16)\\ \tilde{N}\left(\tilde{\alpha}^i_{-n}\tilde{\alpha}^j_{-m}|p\rangle_L\right) &= (n+m)\left(\tilde{\alpha}^i_{-n}\tilde{\alpha}^j_{-m}|p\rangle_L\right) \quad \text{for } i \neq j \ . & (2.17)\end{aligned}$$

This yields the following massless left movers

$$\begin{aligned} \tilde{\alpha}^i_{-1}|0\rangle_L &\qquad \mathrm{i} = 1, \ldots, 8 & (2.18)\\ \tilde{\alpha}^I_{-1}|0\rangle_L &\qquad \mathrm{I} = 1, \ldots, 16 & (2.19)\\ |p^I\rangle_L &\qquad \text{with } p^2 = 2 \ . & (2.20)\end{aligned}$$

The ten-dimensional Mass

The ten-dimensional mass-squared operator is given by

$$m^2 = m_R^2 + m_L^2 \tag{2.21}$$

and the level-matching condition, which follows from modular invariance, requires

$$m_R^2 = m_L^2 \ . \tag{2.22}$$

The discussion of the ten dimensional heterotic string is very brief and mainly has the purpose of introducing the notation. For more details see [2, 3, 4, 5].

2.1.4 The Spectrum of the Heterotic String

Now, we can tensor right and left movers together, to create all massless physical states:

$\vert q\rangle_R \otimes \tilde{\alpha}^i_{-1}\vert 0\rangle_L$	**supergravity multiplet** for ten-dimensional $\mathcal{N}=1$ supergravity ($i=1,\dots,8$)
$\vert q\rangle_R \otimes \tilde{\alpha}^I_{-1}\vert 0\rangle_L$	16 **uncharged gauge bosons** of $\mathrm{E}_8 \times \mathrm{E}_8'$ and their superpartners ($I=1,\dots,16$ correspond to the compactified dimensions)
$\vert q\rangle_R \otimes \vert p^I\rangle_L$	240 + 240 **charged gauge bosons** of $\mathrm{E}_8 \times \mathrm{E}_8'$ and their superpartners (internal quantized momenta p^I with condition $p^2=2$ give 240 root vectors for each E_8)

$|q\rangle_R$ is either an NS or R state, leading to a space-time boson or fermion. In general, due to $\mathcal{N}=1$ supergravity, two states with the same left mover, but one with an NS right mover and the other with an R right mover, are SUSY partners.

Restricted to massless states, the heterotic string is a ten-dimensional $\mathcal{N}=1$ supergravity theory with $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge group.

2.2 Compactification on a Six-Torus

To make contact with the physical world and its four space-time dimensions, one has to compactify six spatial dimensions. We choose to compactify the dimensions $i=3,\dots,8$. Thus, our four-dimensional Minkowski space is given by the two uncompactified dimensions $\mu=1,2$ plus the two coordinates that were gauge fixed due to the light cone coordinates. But doing so on a six-torus T^6, one gets four gravitini,

$$|\underbrace{\pm\frac{1}{2}}_{\text{chirality}},\underbrace{\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}}_{\text{internal}}\rangle_R \otimes \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \text{(even \# of plus signs)}, \tag{2.23}$$

as explained in the following. The first component of the right mover gives the chirality, because it corresponds to the two uncompactified dimensions. The three other components, which are correlated to the six compactified dimensions, can now be seen as internal quantum numbers. For these internal quantum numbers, there are four possibilities for each left- and right-handed state. Because one gravitino consists of both left- and right-handed parts, we have four gravitini. Thus, our theory has $\mathcal{N}=4$ supersymmetry in four dimensions, which is non-chiral.

The problem we have to solve in the following is to compactify six spatial dimensions and, at the same time, to project out three gravitini to get a theory with four-dimensional $\mathcal{N}=1$ supersymmetry. The compactification on an orbifold is promising. A discrete symmetry projects out three gravitini, leaving $\mathcal{N}=1$ supersymmetry unbroken.

2.3 Compactification on an Orbifold

An orbifold [6, 7] is defined as a six-torus T^6, where a discrete symmetry P has been moded out[3]. P is called the **point group**, its elements are called **twists**.

The following considerations restrict our choices for P: The holonomy group of the six-dimensional space is SO(6) which is locally isomorphic to SU(4). If we additionally require to maintain $\mathcal{N} = 1$ SUSY, P must be a subgroup of SU(3). Since we can only mode out a symmetry that is a symmetry of the torus T^6, the elements of P are required to be automorphisms of the T^6 lattice and therefore the point group is discrete. Here we restrict ourselves to abelian groups only. Thus, there are two classes of point groups:

- $\mathbb{Z}_N$: The cyclic group of order N consists of the elements
$$\mathbb{Z}_N = \left\{\vartheta^k \mid k = 0, \ldots, N-1\right\} \tag{2.24}$$
and ϑ can be seen as a rotation by $2\pi/N$.

- $\mathbb{Z}_N \times \mathbb{Z}_M$: Each factor is given by a twist ϑ_1 and ϑ_2 of order N and M, respectively. Thus, the elements of the point group are given by
$$\mathbb{Z}_N \times \mathbb{Z}_M = \left\{\vartheta_1^k \circ \vartheta_2^l \mid k = 0, \ldots, N-1 \text{ and } l = 0, \ldots, M-1\right\} \tag{2.25}$$

All admissible point groups have been classified in [6, 8]. In chapter 3 and 4, we will concentrate on the cases $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. In app. D we will briefly consider the $\mathbb{Z}_4$ case.

The six-dimensional space on which we compactify is defined by
$$T^6/P = \mathbb{R}^6/S \tag{2.26}$$
where we have introduced the **space group** S. S is defined to be the point group P plus the translations given by the lattice vectors e_α for $\alpha = 1, \ldots, 6$. By definition, the vectors e_α (with $|e_\alpha| = \sqrt{2}$) span the six-torus T^6. To summarize, the space group is defined as the set
$$S = \{(\theta, n_\alpha e_\alpha) \mid \theta \in P, n_\alpha \in \mathbb{Z}\}\ , \tag{2.27}$$
where the sum over α from 1 to 6 is implied. The action of an element $g = (\theta, n_\alpha e_\alpha)$ of S on the six-dimensional space $\mathbb{R}^6$ is given by
$$gx = \theta x + n_\alpha e_\alpha \tag{2.28}$$
for $x \in \mathbb{R}^6$. Then, two points $x, y \in \mathbb{R}^6$ are identified, if they differ by the action of a space group element g
$$x \simeq y \quad \text{if there is a } g \in S \text{ with } x = gy \tag{2.29}$$
and thereby the six-dimensional space $\mathbb{R}^6/S$ is defined.

[3] "to mode out P" means to identify points that are being mapped onto each other via P.

Due to modular invariance (see sec. (2.5)), the action of the space group S must be embedded into the gauge degrees of freedom, $S \hookrightarrow G$. G is called the **gauge twisting group**. In general, G is a subgroup of the automorphisms of the $\mathrm{E}_8 \times \mathrm{E}_8'$ Lie algebra. Here, we restrict ourselves to the case of inner automorphisms, which can be realized as a shift V in the $\mathrm{E}_8 \times \mathrm{E}_8'$ root lattice.

The embedding of the space group into the gauge degrees of freedom for both cases, $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_M$, is given by:

- $\mathbb{Z}_N$:
$$S \hookrightarrow G \qquad (\vartheta^k, n_\alpha e_\alpha) \mapsto (kV, n_\alpha A_\alpha) \tag{2.30}$$

- $\mathbb{Z}_N \times \mathbb{Z}_M$:
$$S \hookrightarrow G \qquad (\vartheta_1^k \circ \vartheta_2^l, n_\alpha e_\alpha) \mapsto (kV_1 + lV_2, n_\alpha A_\alpha). \tag{2.31}$$

The vectors A_α are the embeddings of the torus shifts e_α into the gauge degrees of freedom. They correspond to gauge transformations associated to non-contractible loops and are thus **Wilson lines** [9].

For the rest of this chapter, we focus on the case of the $\mathbb{Z}_N$ point group. It is easy to generalize to the $\mathbb{Z}_N \times \mathbb{Z}_M$ case. The action of an element $\tilde{g} = (kV, n_\alpha A_\alpha)$ of the gauge twisting group G on the gauge degrees of freedom is given by:

$$\tilde{g} X_L^I = X_L^I + 2\pi(kV^I + n_\alpha A_\alpha^I) \tag{2.32}$$

for $I = 1, \ldots, 16$. Points in the sixteen dimensions of the gauge degrees of freedom are identified, if they differ by the action of an element $\tilde{g}$.

In summary

$$\mathcal{O} = \mathbb{R}^6/S \otimes T_{\mathrm{E}_8 \times \mathrm{E}_8'}/G \tag{2.33}$$

defines a heterotic orbifold. The elements of the so called **orbifold group** $S \otimes G$ are given by

$$(\vartheta^k, n_\alpha e_\alpha; kV, n_\alpha A_\alpha) \in S \otimes G\ . \tag{2.34}$$

Its action on the coordinates of the heterotic string is illustrated in fig. (2.2), where the two physical degrees of freedom of the four-dimensional Minkowski space are denoted by $\mu = 1, 2$.

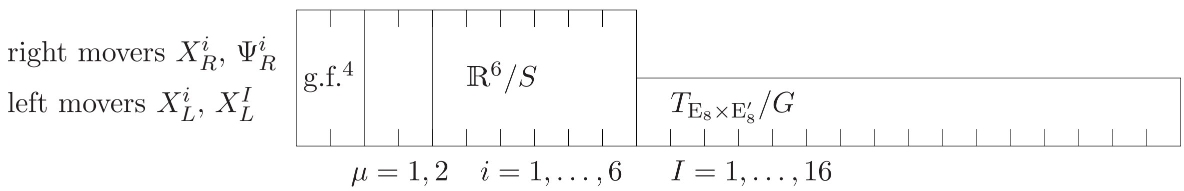


Figure 2.2: The coordinates of the heterotic string compactified on an orbifold.

[4]gauge fixed due to light cone coordinates.

2.4 Fixed Points

Having defined the space group S, we also have fixed the geometry of the six-dimensional space $\mathbb{R}^6/S$. The geometry of an orbifold is characterized by the existence of fixed points. Fixed points are defined as points $X_f \in T^6$ that are invariant under the action of a non-trivial element of S,

$$X_f^i = (\vartheta^k X_f)^i + n_\alpha e_\alpha^i \tag{2.35}$$

for $i = 1, \ldots, 6$ and $k = 1, \ldots, N-1$. It is important to note that due to this equation a fixed point corresponds to a space group element $(\vartheta^k, n_\alpha e_\alpha)$. We rewrite eqn. (2.35) to

$$\left[(\mathbb{1} - \vartheta^k) X_f\right]^i = n_\alpha e_\alpha^i . \tag{2.36}$$

Because the mapping $(\mathbb{1} - \vartheta^k)$ does not need to be invertible, it is not always possible to solve this equation explicitly with respect to X_f^i.

Two examples are presented in order to explain the geometry of an orbifold. The first example shows that, whenever possible, it is more convenient to write the six-torus T^6 in a complex basis. The second example shows the case where the mapping $(\mathbb{1} - \vartheta^k)$ is not invertible, and gives a geometrical interpretation for this case.

2.4.1 Example 1: $T^6/\mathbb{Z}_3$

In this example, we choose the six-torus to be the product of three two-tori

$$T^6 = T^2 \times T^2 \times T^2 \ . \tag{2.37}$$

Whenever this is possible, it is more convenient to write the twist as a rotation in $1+3$ complex planes. The first complex plane corresponds to our four-dimensional Minkowski space. We do not want to perform the orbifolding in our Minkowski space, thus there is no rotation in these dimensions. Each of the other three complex planes corresponds to a two-torus. The T^6 torus is chosen to be spanned by the root lattice of $\mathrm{SU}(3)^3$. In complex coordinates, these basis vectors are given by

$$e_1^a = \sqrt{2} \tag{2.38}$$
$$e_2^a = \sqrt{2}\ e^{2\pi i/3} \quad \text{for } a = 1, 2, 3. \tag{2.39}$$

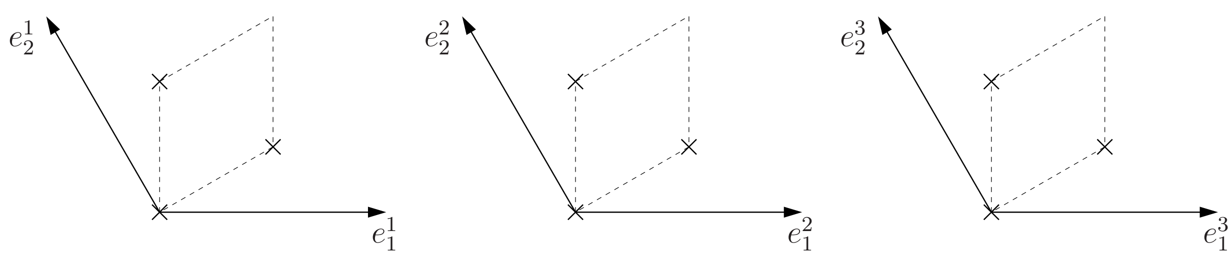


Figure 2.3: $\mathbb{Z}_3$ orbifold with $3 \times 3 \times 3 = 27$ fixed points on the six-torus T^6 for the $k = 1$ twisted sector. The three complex planes are mutually orthogonal.

The torus lattice is illustrated in fig. (2.3). The coordinates of the six compactified dimensions are written as

$$Z^1 = \left(X^1 + iX^2\right), \quad Z^2 = \left(X^3 + iX^4\right), \quad Z^3 = \left(X^5 + iX^6\right) \tag{2.40}$$

and Z^0 denotes the two uncompactified dimensions. The point group P of the $\mathbb{Z}_3$ orbifold consists of the three elements $\{\mathbb{1}, \vartheta, \vartheta^2\}$, where ϑ is a rotation by 120°. The twist acts on the complex coordinates by multiplication

$$\vartheta^k : Z^a \mapsto e^{2\pi i k v^a} Z^a \tag{2.41}$$

where $a = 0, \ldots, 3$ and $k = 0, 1, 2$. The **twist vector** v is given by

$$v = \left(0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) , \tag{2.42}$$

and the following correspondence holds:

$$\vartheta^k \Leftrightarrow kv \quad \text{for } k = 0, 1, 2 \tag{2.43}$$

The reason why v^3 is chosen to be $-2/3$ will become clear later in sec. (2.5). Obviously, the action of the twist vector v is an automorphism of the chosen six-torus. Hence, we have defined the space group S of the $\mathbb{Z}_3$ orbifold and we can solve the fixed point equations. To do this, we rewrite the fixed point equation (2.36) in complex coordinates. We use the fact that the three complex planes are mutually orthogonal and that the twist acts separately on each one of them. Thus,

$$\left(1 - e^{2\pi i k v^a}\right) Z^a_f = n^a_i e^a_i \tag{2.44}$$

for $a = 1, 2, 3$ without summation over a, and $n^a_i \in \mathbb{Z}$, arbitrary, with summation over $i = 1, 2$. We solve this equation for $k = 1$ and the solution is that in each complex plane of the six-torus T^6 there are three fixed points

$$\begin{array}{lll} Z^a_{f1} = 0 & & \text{for } n^a = (0,0) \\ Z^a_{f2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\, i & & \text{for } n^a = (1,0) \\ Z^a_{f3} = \sqrt{\frac{2}{3}}\, i & & \text{for } n^a = (1,1) \end{array}$$

Now, we have calculated the fixed points. But, of course, it is also possible to explain them graphically. Fig. (2.4) is separated into three parts, each explaining one fixed point:

(I) Clearly the origin is invariant under rotations, thus it is a fixed point. It corresponds to the space group element $(\vartheta, 0)$.

(II) Rotating the indicated point by 120° and shifting by e_1 gives the starting point. Thus we have a fixed point, which is denoted by the space group element (ϑ, e_1).

(III) Analogously, the point shown here corresponds to the space group element $(\vartheta, e_1{+}e_2)$.

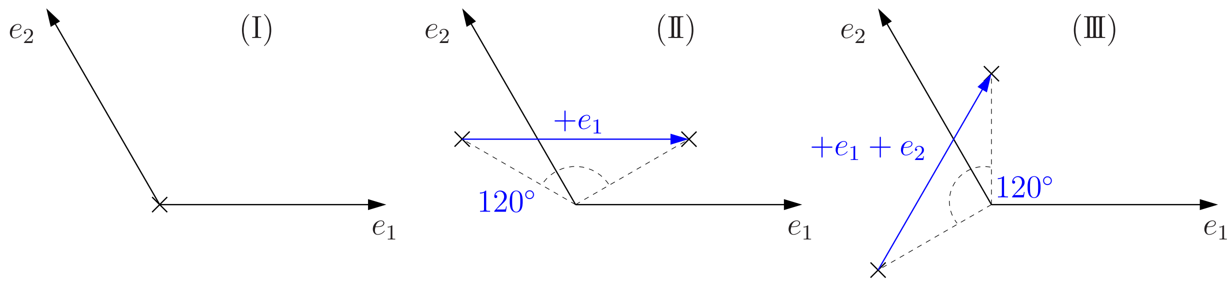


Figure 2.4: The tree fixed points of the $\mathbb{Z}_3$ orbifold in the first torus.

Thus, the first torus has three fixed points, which can be combined with three fixed points of the second torus and with three fixed points of the last torus. Therefore, the $k = 1$ twisted sector of the $\mathbb{Z}_3$ orbifold has $3 \times 3 \times 3 = 27$ fixed points, as illustrated in fig. (2.3).

A very important characteristic of any orbifold is that it is not a manifold. To illustrate this, fig. (2.5) shows the three fixed points in the first complex plane again. In fig. (2.6) these two dimensions have been folded together to a pillow-like object. Here, one can see that the fixed points are singular points and therefore an orbifold is not a manifold.

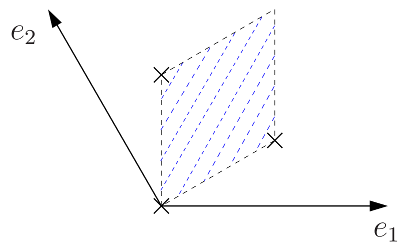


Figure 2.5: Here, the orbifold is not folded together. However, the dashed blue lines indicate that points which are connected by these lines, are identified on the orbifold.

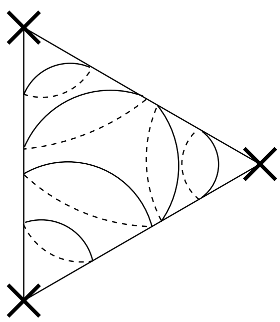


Figure 2.6: Now, the orbifold is folded together to a pillow-like object along the dashed blue lines in fig. (2.5). Evidently, the curvature is not defined at the three fixed points, thus they are singular points. (The size is scaled up.)

2.4.2 Example 2: $T^6/\mathbb{Z}_2$

Here, we choose the six-torus T^6 to be spanned by the orthogonal basis of $\mathbb{R}^6$: e_α for $\alpha = 1, \dots, 6$. In order to show another way of calculating the fixed points, we will not use complex notation here. Furthermore, we choose the point group to be $\mathbb{Z}_2$ with the twist

$$\vartheta = \begin{pmatrix} -1 & & & & & \\ & -1 & & & 0 & \\ & & -1 & & & \\ & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix} \tag{2.45}$$

which is an automorphism of the six-torus and fulfills $\vartheta^2 = \mathbb{1}$. Using our choices for the six-torus and the twist and, we define the space group S. Now, we can look for fixed points in the space $\mathbb{R}^6/S$ by solving eqn. (2.36),

$$\begin{pmatrix} 2 & & & & & \\ & 2 & & 0 & & \\ & & 2 & & & \\ & & & 2 & & \\ & 0 & & 0 & & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} X_f^1 \\ X_f^2 \\ X_f^3 \\ X_f^4 \\ X_f^5 \\ X_f^6 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 = 0 \\ n_6 = 0 \end{pmatrix} \tag{2.46}$$

for arbitrary $n_\alpha \in \mathbb{Z}$, $\alpha = 1, \dots, 4$. Notice that the matrix on the left hand side is not invertible. The solutions are

$$X_f = \left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}, \frac{n_4}{2}, X_f^5, X_f^6\right)^T \qquad X_f^5, X_f^6 \in \mathbb{R}, \text{ arbitrary.} \tag{2.47}$$

In this example where X_f^5 and X_f^6 are arbitrary we can see that we need to understand fixed points in a more general way: fixed points do not necessarily have to be zero-dimensional objects. This can be understood geometrically as fixed tori, which are mapped onto themselves under the action of S.

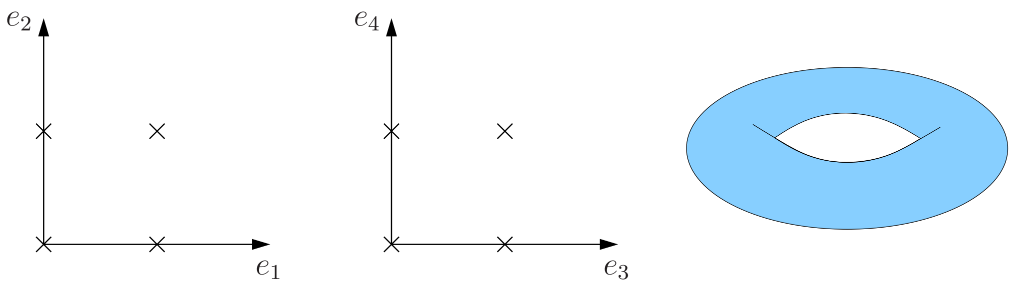


Figure 2.7: $\mathbb{Z}_2$ orbifold with $4 \times 4 = 16$ fixed tori in the space T^6. The two planes and the torus are mutually orthogonal.

It turns out that there are 16 fixed tori in the space T^6, corresponding to eqn. (2.47) with $n_i = 0, 1$ for $i = 1, \ldots, 4$, thus $2^4 = 16$. They are drawn in fig. (2.7).

Actually, the fixed points in this example could also have been calculated using the notation of the first example. The benefits of complex notation will become clear later, when we discuss the transformation properties of strings (in sec. (2.7) and in sec. (2.7.2)). Therefore, we will restrict ourselves to the case that the six-torus can be written as the product of three (complex) two-tori,

$$T^6 = T^2 \times T^2 \times T^2 \ . \tag{2.48}$$

Fixed points (and hyperplanes) play an important role for the strings living on an orbifold. In sec. (2.7) we will discuss a new kind of closed strings, called the twisted strings. A twisted string cannot propagate away from a fixed point, it is localized there.

2.5 Further Conditions on the Orbifold Group $S \otimes G$

There are further constraints on consistent orbifold groups $S \otimes G$. In the following the main results are highlighted by frames.

- The embedding $S \hookrightarrow G$ is a homomorphism

  N is the order of the twist: $\vartheta^N = \mathbb{1}$. Therefore, every space group element $(\vartheta, n_\alpha e_\alpha)$ must also satisfy

  $$(\vartheta, n_\alpha e_\alpha)^N X^i = X^i \text{ mod } \textit{torus shifts} \tag{2.49}$$

  for all $n_\alpha \in \mathbb{Z}$. Because the embedding is a homomorphism, an analogous equation holds in the gauge degrees of freedom for an element of the gauge twisting group $(V, n_\alpha A_\alpha)$,

  $$(V, n_\alpha A_\alpha)^N X^I = X^I \text{ mod } 2\pi\Lambda^I \tag{2.50}$$

  for all $n_\alpha \in \mathbb{Z}$ and with $\Lambda \in T_{\mathrm{E}_8\times\mathrm{E}_8'}$. From this, it is easy to show that the shift and the Wilson lines are constrained by

  $$\boxed{NV \in T_{\mathrm{E}_8\times\mathrm{E}_8'} \quad \text{and} \quad NA_\alpha \in T_{\mathrm{E}_8\times\mathrm{E}_8'} \ .} \tag{2.51}$$

- (Weak) modular invariance

  The modular invariance condition is a result of the calculation of the one-loop amplitude for vacuum to vacuum transitions, i.e. the one-loop partition function. The one-loop vacuum to vacuum transition of a string has the geometry of a torus. But the world sheet coordinates τ and σ of the string that sweeps out the torus are only defined up to the so called modular transformations. Since the one-loop amplitude has to be invariant under these transformations, the following condition must be imposed on all elements of the orbifold group

$$\boxed{N\left[(kV + n_\alpha A_\alpha)^2 - (kv)^2\right] = 0 \bmod 2 \quad \text{for } k = 0, \ldots, N-1 \ .} \tag{2.52}$$

This is equivalent to the equations

$$\begin{aligned} N\left(V^2 - v^2\right) &= 0 \bmod 2 && (2.53)\\ NV \cdot A_\alpha &= 0 \bmod 1 && (2.54)\\ NA_\alpha \cdot A_\beta &= 0 \bmod 1, \quad \alpha \neq \beta && (2.55)\\ NA_\alpha^2 &= 0 \bmod 2 \ . && (2.56) \end{aligned}$$

- (Strong) modular invariance

The weak modular invariance condition is enough to guarantee anomaly freedom of the orbifold model. But there may be problems in computing the transformation properties of the states in the next section (2.6). In order to avoid them it is better to bring the shift V and the Wilson lines A_α to a special form, in which they satisfy the so called strong modular invariance condition

$$\boxed{\left[(kV + n_\alpha A_\alpha)^2 - (kv)^2\right] = 0 \bmod 2 \quad \text{for } k = 0, \ldots, N-1 \ .} \tag{2.57}$$

Therefore, the shift and the Wilson lines first have to satisfy the weak modular invariance condition. Then, one adds $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice vectors to fulfill the strong condition. This does not change the physics. Again, the strong modular invariance condition is equivalent to the following conditions

$$\begin{aligned} V^2 - v^2 &= 0 \bmod 2 && (2.58)\\ V \cdot A_\alpha &= 0 \bmod 1 && (2.59)\\ A_\alpha \cdot A_\beta &= 0 \bmod 1, \quad \alpha \neq \beta && (2.60)\\ A_\alpha^2 &= 0 \bmod 2 \ . && (2.61) \end{aligned}$$

- $\mathcal{N} = 1$ SUSY in four dimensions

When the twist $\vartheta^k \Leftrightarrow kv$ acts on the SO(8) spinor of the gravitini, it projects out most of them. In order to retain $\mathcal{N} = 1$ supersymmetry, one has to demand the following condition on the twist vector v

$$\pm v^1 \pm v^2 \pm v^3 = 0 \tag{2.62}$$

for one combination of signs. We can always choose

$$\boxed{v^1 + v^2 + v^3 = 0 \ .} \tag{2.63}$$

Details are given in the next section (2.7.1), where we will derive the supergravity multiplet in the orbifold theory.

2.6 Constraints on States of the Orbifold

In this section we consider the action of orbifolding on states

$$|q\rangle_R \otimes |p\rangle_L \qquad \text{and} \qquad |q\rangle_R \otimes \tilde{\alpha}_{-1}|p\rangle_L \ . \tag{2.64}$$

In order to see how an element of the orbifold group acts on the left movers, we have to consider the generator of translation

$$e^{ip\cdot X}|0\rangle_L = |p\rangle_L \tag{2.65}$$

with $p \cdot X = \sum_{I=1}^{16} p^I X^I$. Then, the action of an element

$$h = (kv, n_\alpha e_\alpha; kV, n_\alpha A_\alpha) \in S \otimes G \qquad k = 0, \ldots, N-1 \tag{2.66}$$

of the orbifold group on the gauge degrees of freedom X^I gives a phase for the left movers, i.e.

$$X \overset{h}{\mapsto} X + 2\pi(kV + n_\alpha A_\alpha) \tag{2.67}$$

$$e^{ip\cdot X} \overset{h}{\mapsto} \underbrace{e^{2\pi i p\cdot(kV+n_\alpha A_\alpha)}}_{\text{phase}} e^{ip\cdot X} \ . \tag{2.68}$$

The right movers also transform under h,

$$|q\rangle_R \overset{h}{\mapsto} \underbrace{e^{2\pi i q\cdot(kv)}}_{\text{phase}} |q\rangle_R \ . \tag{2.69}$$

Furthermore, the oscillators $\tilde{\alpha}^\mu_{-1}$ and $\tilde{\alpha}^I_{-1}$ for the left movers transform trivially

$$\tilde{\alpha}^\mu_{-1} \overset{h}{\mapsto} \tilde{\alpha}^\mu_{-1} \quad \text{for } \mu = 1, 2 \tag{2.70}$$

$$\tilde{\alpha}^I_{-1} \overset{h}{\mapsto} \tilde{\alpha}^I_{-1} \quad \text{for } I = 1, \ldots, 16 \ , \tag{2.71}$$

because neither the twists nor the shifts act on them.

To get $S \otimes G$ invariant states we have to calculate the transformation properties with respect to all elements

$$h = (kv, n_\alpha e_\alpha; kV, n_\alpha A_\alpha) \in S \otimes G \qquad k = 0, \ldots, N-1 \ . \tag{2.72}$$

Only invariant states (with a trivial phase of 1) are consistent with the geometry of the underlying orbifold space. States that transform with a non-trivial phase are not consistent and must be projected out.

2.7 The Spectrum

The massless spectrum of the orbifold contains all massless closed strings which are consistent with the geometry of the orbifold. Since the geometry offers two different ways to close a string, the spectrum is divided into two corresponding parts:

- The first part is called the **untwisted sector**. The strings of the untwisted sector are closed even before twisting. Thus, in principle they are equal to the strings of the heterotic string, but they need to be consistent with the geometry of the underlying space.
- The second part of the spectrum is called the **twisted sectors**. Due to the geometry, new closed strings are possible with a new kind of boundary condition which leads to twisted strings localized at fixed points.

We will now discuss both parts of the spectrum in detail.

2.7.1 The Untwisted Sector

The massless untwisted string is closed even before compactification. It is illustrated in fig. (2.8). This kind of closed string is well known, it is the heterotic string (discussed in sec. (2.1)).

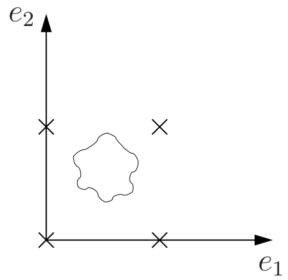


Figure 2.8: The massless untwisted string is closed in $\mathbb{R}^6$ and on the six-torus T^6.

But the new geometry of the underlying orbifold space imposes additional conditions: as described in the last section (2.6) we have to project onto $S \otimes G$ invariant states. In the following we do so for all massless states of the heterotic string, i.e. for the supergravity multiplet, the charged and uncharged $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge bosons and their SUSY partners.

The supergravity multiplet

The transformation of the supergravity multiplet is given by

$$|q\rangle_R \otimes \tilde{\alpha}^\mu_{-1}|0\rangle_L \overset{h}{\mapsto} \underbrace{e^{2\pi i(q\cdot(kv)+0\cdot(kV+n_\alpha A_\alpha))}}_{=1}|q\rangle_R \otimes \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1,2\ . \tag{2.73}$$

For the supergravity multiplet the vectors q must correspond to oscillators in the dimensions $\mu = 1,2$. Therefore, the NS right movers are given by

$$q = (\pm 1, 0, 0, 0) \qquad \text{NS} \tag{2.74}$$

which fulfill $q \cdot v = 0$, hence the right movers of the bosonic state are invariant. The correspondence between $\mu = 1,2$ and the R state is less apparent. But, in order to satisfy $q \cdot v = 0$ for the R states, we conclude

$$q = \left(\pm\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right) \qquad \text{R}\ , \tag{2.75}$$

compare to eqn. (2.63). Therefore, there are two R states, corresponding to both chiralities of one gravitino and the orbifold-theory has $\boldsymbol{\mathcal{N} = 1}$ **susy in four dimensions**.

It turns out that there are further states of the ten-dimensional $\mathcal{N} = 1$ supergravity multiplet which are invariant under the orbifolding. These states are of the form $|q\rangle_R \otimes \tilde{\alpha}^a_{-1}|0\rangle_L$ for $a = 1, \ldots, 3$. They are singlets and we will omit them in the following.

The gauge bosons and gauginos

Gauge bosons are given by massless vector fields with spin 1. Thus, in four dimensions, they have two degrees of freedom that correspond to two transverse polarizations. Therefore, a right mover of a gauge boson must consist of two NS states with oscillators in the $\mu = 1,2$ direction, i.e. $q = (\pm 1, 0, 0, 0)$.

Gauginos are described by spin-$\frac{1}{2}$ particles. In four dimensions, a spin-$\frac{1}{2}$ particle has two chiralities, i.e. it has a left- and a right-handed part. Therefore, a right mover of a gaugino must consist of two R states, again with oscillators in the $\mu = 1,2$ direction, i.e. $q = (\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}))$.

The uncharged gauge bosons (and their gaugino partners) transform as

$$|q\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L \overset{h}{\mapsto} \underbrace{e^{2\pi i(q\cdot(kv)+0\cdot(kV+n_\alpha A_\alpha))}}_{=1}|q\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L \qquad I = 1,\ldots,16\ . \tag{2.76}$$

The right movers of the uncharged gauge bosons (and of the gauginos) are given by

$$q = \begin{cases} (\pm 1, 0, 0, 0) & \text{NS} \\ (\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})) & \text{R}\ . \end{cases} \tag{2.77}$$

All 16 uncharged gauge bosons (and their gaugino partners) survive the orbifold projection on $S \otimes G$ invariant states. Because these gauge bosons correspond to the Cartan generators of the algebra, the **rank of the algebra is never reduced** with shift embedding[5].

The charged gauge bosons of the heterotic string transform as

$$|q\rangle_R \otimes |p^I\rangle_L \overset{h}{\mapsto} \underbrace{e^{2\pi i(q\cdot(kv)+p\cdot(kV+n_\alpha A_\alpha))}}_{\overset{!}{=}1} |q\rangle_R \otimes |p^I\rangle_L \ . \tag{2.78}$$

The right movers of the new charged gauge bosons (and gauginos) are given by

$$q = \begin{cases} (\pm 1, 0, 0, 0) & \text{NS} \\ (\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})) & \text{R} \ . \end{cases} \tag{2.79}$$

Thus, the right movers are invariant: $q \cdot v = 0$. Therefore, the invariant gauge bosons must fulfill the following conditions

$$p \cdot (kV) = 0 \bmod 1 \tag{2.80}$$
$$p \cdot (n_\alpha A_\alpha) = 0 \bmod 1 \tag{2.81}$$

for all combinations of k and n_α. Because not all former charged gauge bosons (and gauginos) are invariant, the gauge group is broken. The **unbroken gauge group** is represented by the invariant charged gauge bosons plus the 16 uncharged gauge bosons.

There are also other former gauge bosons (and gauginos) that survive the projection, but without fulfilling $q \cdot v = 0$. In order to be invariant they have to obey

$$2\pi i \left(q \cdot (kv) + p \cdot (kV + n_\alpha A_\alpha)\right) = 0 \bmod 1 \ . \tag{2.82}$$

These states are called **charged matter**. The former root vectors p of these charged matter states are now weights with respect to the unbroken gauge group, thus they form representations of the unbroken gauge group.

The spectrum of massless states for orbifold models contains not only the $S \otimes G$ invariant states of the heterotic string, but further closed strings. They arise due to the changed geometry of the orbifold space. We consider these states in the next section.

[5] However, it is possible to embed the action of the space group $(\theta, n_\alpha e_\alpha) \in S$ into the gauge degrees of freedom via a rotation $(\Theta, n_\alpha A_\alpha) \in G$,

$$G : X^I \mapsto \Theta X^I + 2\pi n_\alpha A^I_\alpha \ .$$

With this setup it is possible to reduce the rank of the algebra [10, 11].

2.7.2 The Twisted Sectors

In the twisted sectors, due to the non-trivial geometry of an orbifold, there are new possibilities for a string to close. Thus, we have new boundary conditions for the six bosonic coordinates. Each of these new conditions corresponds to a so called **constructing element** $g = (\vartheta^k, n_\alpha e_\alpha)$ of the space group S,

$$X^i(\tau, \sigma + 2\pi) = (gX)^i(\tau, \sigma) \tag{2.83}$$

$$= (\vartheta^k X(\tau, \sigma))^i + n_\alpha e^i_\alpha \qquad i = 1, \dots, 6 \; . \tag{2.84}$$

The endpoints of a string are given by evaluating this equation at $\sigma = 0$. Thus, the string closes only after performing a twist ϑ^k and torus shifts $n_\alpha e^i_\alpha$. Fig. (2.9) shows a twisted string on a $\mathbb{Z}_2$ orbifold.

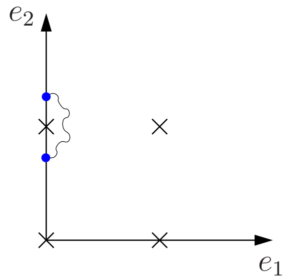


Figure 2.9: Since the blue circles are identified under the action of the constructing element $g = (\vartheta, e_2)$ of the space group, the twisted string is closed on the orbifold (ϑ is a rotation by 180°).

The spectrum of massless states for these new boundary conditions is divided into **twisted sectors**. Each twisted sector corresponds to one constructing twist $\vartheta^k \in P$ of eqn. (2.84) and is thus labeled by k.

Due to eqn. (2.84), the boundary condition for the "twist" $\mathbb{1} \in P$ corresponds to the boundary condition on a six-torus. Since we are only interested in massless states, which have no winding on the six-torus, we only consider the space group element $(\mathbb{1}, 0) \in S$. Obviously, this element corresponds to the untwisted sector.

Due to the changed boundary conditions of eqn. (2.84) the mode expansions for right and left movers, respectively, change. In app. (B.2) they are derived. Here, we restrict ourselves to the results.

We define complex coordinates

$$Z^1 = \left(X^1 + iX^2\right), \quad Z^2 = \left(X^3 + iX^4\right), \quad Z^3 = \left(X^5 + iX^6\right) \tag{2.85}$$

which can be summarized as

$$Z^a = X^{(2a-1)} + i\; X^{(2a)} \;, \tag{2.86}$$

for $a = 1, 2, 3$. Furthermore, we denote the complex conjugate of any complex quantity with a bared index $\bar{a}$. For example

$$Z^{\bar{a}} := \overline{Z^{a}} = X^{(2a-1)} - i\ X^{(2a)}\ . \tag{2.87}$$

Then the mode expansions for the twisted boundary condition eqn. (2.84) read

$$Z_R^a(\tau,\sigma) = \frac{1}{2}Z_f^a + \frac{i}{2}\sum_{m=0}\frac{\alpha^a_{m+\eta^a}}{m+\eta^a}e^{-i(m+\eta^a)(\tau-\sigma)} \tag{2.88}$$

$$Z_L^a(\tau,\sigma) = \frac{1}{2}Z_f^a + \frac{i}{2}\sum_{m=0}\frac{\tilde{\alpha}^a_{m-\eta^a}}{m-\eta^a}e^{-i(m-\eta^a)(\tau+\sigma)} \tag{2.89}$$

for $a = 1, 2, 3$ and $\eta^a = kv^a \bmod 1$ so that $0 < \eta^a < 1$. Note that there is no momentum p^i in the twisted directions and furthermore the centre-of-mass of the string is attached to a fixed point

$$Z_f^a = (\vartheta^k Z_f)^a + n_\alpha e_\alpha^a\ . \tag{2.90}$$

Therefore, the string is localized at a fixed point.

The quantized right movers in the twisted sectors

Due to the twist in the six bosonic coordinates X^i, the boundary conditions for the fermionic right moving coordinates also change (for $i = 1, \ldots, 6$),

$$\Psi_R^i(\tau,\sigma+2\pi) = \pm(\vartheta^k\Psi_R)^i(\tau,\sigma) \tag{2.91}$$

for R and NS state, respectively[6]. As a consequence the mode expansions change. Therefore, one can show that the corresponding SO(8) weight lattice (of both, the SO(8) vector and spinor) for the k-th twisted sector is shifted,

$$q^a_{\text{sh}} = q^a + kv^a \qquad a = 0, \ldots, 3\ . \tag{2.92}$$

The twisted right mover, denoted by

$$|q_{\text{sh}}\rangle_R = |q + kv\rangle_R\ , \tag{2.93}$$

fulfills the new mass-equation

$$\frac{m_R^2}{4} = \frac{1}{2}\left(q+kv\right)^2 - \frac{1}{2} + \delta c = 0 \tag{2.94}$$

and is thus massless. The quantity δc is a change in the zero point energy that appears because of the moded oscillators. It is given by

$$\delta c = \frac{1}{2}\sum_{a=0}^{3}\eta^a(1-\eta^a) \tag{2.95}$$

where $\eta^a = kv^a \bmod 1$ so that $0 \leq \eta^a < 1$. Notice that the zero point energy depends on the twisted sector k.

[6]This ensures invariance of the world sheet supercurrent.

The quantized left movers in the twisted sectors

From modular invariance, it follows that the space group has to be embedded into the gauge degrees of freedom. We restrict ourselves to shift embeddings, where the constructing element $g = (kv, n_\alpha e_\alpha)$ of the space group is embedded as $(kV, n_\alpha A_\alpha) \in G$. This element of the gauge twisting group corresponds to a new boundary condition for a closed string in the gauge degrees of freedom (compare to eqns. (2.4) and (2.32)),

$$X_L^I(\tau, \sigma + 2\pi) = X_L^I(\tau, \sigma) + 2\pi(kV^I + n_\alpha A_\alpha^I + \Lambda^I) \qquad I = 1, \ldots, 16 \; , \tag{2.96}$$

where $\Lambda \in T_{\mathrm{E}_8 \times \mathrm{E}_8'}$. The new (twisted) mode expansion is equal to eqn. (2.7), but with a shifted momentum,

$$p_{\mathrm{sh}}^I = p^I + kV^I + n_\alpha A_\alpha^I \qquad I = 1, \ldots, 16 \; . \tag{2.97}$$

Remark: In the bosonic formulation, the left and right moving momenta lie on an even, self-dual lattice $\Gamma_{(22;6)}$ of lorentzian signature $(22, 6)$ refs. [12]. This implies that a state with right moving momentum q_{sh} and left moving momentum p_{sh} transforms as

$$|q_{\mathrm{sh}}\rangle_R \otimes |p_{\mathrm{sh}}\rangle_L \overset{h}{\mapsto} e^{2\pi i(q_{\mathrm{sh}}\cdot\tilde{k}v - (p+kV+n_\alpha A_\alpha)\cdot(\tilde{k}V + \tilde{n}_\alpha A_\alpha))} \; |q_{\mathrm{sh}}\rangle_R \otimes |p_{\mathrm{sh}}\rangle_L \tag{2.98}$$

for $h = (\tilde{k}v, \tilde{n}_\alpha e_\alpha; \tilde{k}V, \tilde{n}_\alpha A_\alpha) \in S \otimes G$. For calculational convenience, we have decided to absorb the relative minus sign between the right and left moving momenta into the definition of p_{sh}, i.e.

$$-(p + kV + n_\alpha A_\alpha) = -p - kV - n_\alpha A_\alpha = \tilde{p} - kV - n_\alpha A_\alpha \; . \tag{2.99}$$

This coincides with the conventions of [13, 14]. Therefore, the twisted left mover is denoted by

$$|p - (kV + n_\alpha A_\alpha)\rangle_L \; . \tag{2.100}$$

It fulfills the mass formula for massless twisted left movers,

$$\frac{m_L^2}{4} = \frac{1}{2}\left(p - (kV + n_\alpha A_\alpha)\right)^2 + \tilde{N} - 1 + \delta c = 0 \; . \tag{2.101}$$

Since oscillators $\tilde{\alpha}^a_{m-\eta^a}$ are moded if they correspond to a complex dimension a in which the twist acts non-trivially (i.e. if $\eta^a \neq 0$), the number operator $\tilde{N}$ can be fractional. Furthermore, the change in the zero point energy δc is equal to the right mover case of eqn. (2.95).

Now, left and right movers must be tensored together to form $S \otimes G$ invariant states.

Transformation of the twisted left and right movers

We consider massless right and left movers, created with the constructing element $g = (kv, n_\alpha e_\alpha; kV, n_\alpha A_\alpha)$. Then, the transformation properties under an element

$$h = (\tilde{k}v, \tilde{n}_\alpha e_\alpha; \tilde{k}V, \tilde{n}_\alpha A_\alpha) \tag{2.102}$$

of the orbifold group $S \otimes G$ are given by

$$|q_{\rm sh}\rangle_R \quad \stackrel{h}{\mapsto} \quad e^{2\pi i q_{\rm sh}\cdot(\tilde{k}v)}\ |q_{\rm sh}\rangle_R \tag{2.103}$$
$$|p_{\rm sh}\rangle_L \quad \stackrel{h}{\mapsto} \quad e^{2\pi i p_{\rm sh}\cdot(\tilde{k}V+\tilde{n}_\alpha A_\alpha)}\ |p_{\rm sh}\rangle_L \tag{2.104}$$

where $q_{\rm sh} = q + kv$ and $p_{\rm sh} = p - (kV + n_\alpha A_\alpha)$. Furthermore, the moded oscillators of the left mover transform as

$$\vartheta^{\tilde{k}}\tilde{\alpha}^a_{m-\eta^a} \quad = \quad e^{2\pi i\tilde{\eta}^a}\ \tilde{\alpha}^a_{m-\eta^a} \qquad \simeq \mathbf{3} \text{ of SU(3)} \tag{2.105}$$
$$\vartheta^{\tilde{k}}\tilde{\alpha}^{\bar{a}}_{m-\eta^a} \quad = \quad e^{-2\pi i\tilde{\eta}^a}\ \tilde{\alpha}^{\bar{a}}_{m-\eta^a} \qquad \simeq \overline{\mathbf{3}} \text{ of SU(3)}\ , \tag{2.106}$$

where $\eta^a = kv^a \bmod 1$ corresponds to the constructing element g and $\tilde{\eta}^a = \tilde{k}v^a \bmod 1$ corresponds to h. See app. (B.2).

It is important to remark that the conditions for an invariant state in the twisted sectors are more involved than in the untwisted case. The details are given in the next section.

2.8 $S \otimes G$ Invariant States in the Twisted Sectors

In the twisted sectors it is more complicated to tensor left and right movers together to build states that are consistent with the geometry of the orbifold, i.e. that are $S \otimes G$ invariant. The problem and its solution are described in the following.

- We consider a given fixed point on the six-torus T^6 which corresponds to a **constructing element** $g = (\theta, n_\alpha e_\alpha; V, n_\alpha A_\alpha) \in S \otimes G$. The boundary condition for the six compactified coordinates reads

$$X(\tau,\sigma+2\pi) \quad = \quad gX(\tau,\sigma) \tag{2.107}$$
$$= \quad \theta X(\tau,\sigma) + n_\alpha e_\alpha\ . \tag{2.108}$$

  There is an analogous boundary condition for the gauge degrees of freedom, which is not relevant for our discussion. The set of all states that close under the constructing element g defines the **Hilbert space $\boldsymbol{H_g}$**.

- Let us now assume, we have a second element $h = (\tilde{\theta}, \tilde{n}_\alpha e_\alpha; \tilde{V}, \tilde{n}_\alpha A_\alpha) \in S \otimes G$ that commutes with g,

$$[h, g] = 0\ . \tag{2.109}$$

  A trivial example for a commuting element is g itself, but in general it can be any element of the orbifold group. Then, multiplying the twisted boundary condition of eqn. (2.107) by h yields

$$(hX)(\tau,\sigma+2\pi) \quad = \quad hgX(\tau,\sigma) \tag{2.110}$$
$$= \quad g(hX)(\tau,\sigma)\ . \tag{2.111}$$

Therefore, the states that correspond to the bosonic coordinates $(hX)(\tau,\sigma)$ also close under the constructing element g. Then, by definition of the Hilbert space, they are elements of H_g. Thus, h must act as the identity on the states of H_g, i.e. the states of H_g must transform as

$$|q_{\text{sh}}\rangle_R \otimes |p_{\text{sh}}\rangle_L \overset{h}{\mapsto} 1|q_{\text{sh}}\rangle_R \otimes |p_{\text{sh}}\rangle_L \tag{2.112}$$

where q_{sh} and p_{sh} correspond to the constructing element g. Furthermore, states of H_g, which transform non-trivially ($\zeta \neq 1$)

$$|q_{\text{sh}}\rangle_R \otimes |p_{\text{sh}}\rangle_L \overset{h}{\mapsto} \zeta|q_{\text{sh}}\rangle_R \otimes |p_{\text{sh}}\rangle_L \tag{2.113}$$

have to be removed from H_g. They are projected out. This procedure has to be applied with respect to all commuting elements. Hence, we introduce the notion of the **centralizer**

$$Z_g = \{h \in S \otimes G \text{ with } [h,g] = 0\} \tag{2.114}$$

as the set of all elements of the orbifold group that commute with the constructing element g. The elements of the centralizer must act as the identity on the Hilbert space H_g. In order to ensure this, non-invariant states are projected out.

- Finally, we consider an element $h = (\tilde{\theta}, \tilde{n}_\alpha e_\alpha; \tilde{V}, \tilde{n}_\alpha A_\alpha)$ of the orbifold group that does not commute with g,

$$[h,g] \neq 0 \ . \tag{2.115}$$

Again, we multiply the twisted boundary condition eqn. (2.107) by h. This time we get

$$\begin{aligned} (hX)(\tau,\sigma+2\pi) &= hgX(\tau,\sigma) & (2.116)\\ &= \left(hgh^{-1}\right)(hX)(\tau,\sigma) \ . & (2.117)\end{aligned}$$

However, this boundary condition corresponds to states of the Hilbert space $H_{hgh^{-1}}$. Thus, h maps a state of H_g to the corresponding one of $H_{hgh^{-1}}$. It is easy to see that h maps a state of $H_{hgh^{-1}}$ to $H_{h^2gh^{-2}}$ and so on,

$$H_g \xrightarrow{h} H_{hgh^{-1}} \xrightarrow{h} H_{h^2gh^{-2}} \xrightarrow{h} H_{h^3gh^{-3}} \xrightarrow{h} \ldots \ . \tag{2.118}$$

In order to get a state that is invariant under the action of h, one has to build a linear combination of the corresponding states of all these Hilbert spaces. The coefficients of this linear combination are given by the transformation properties with respect to the non-commuting element h.

If in addition the following holds

$$h^n = \mathbb{1} \quad n \in \mathbb{N} \ , \tag{2.119}$$

the sequence of eqn. (2.118) aborts and the sum in the linear combination is finite. Otherwise, the sum is infinite.

We summarize the general strategy used to obtain $S \otimes G$ invariant states:

How to Get $S \otimes G$ Invariant States

- Take all constructing elements $g \in S \otimes G$ and solve the corresponding equations for massless right and left movers.
- Then, for all constructing elements $g \in S \otimes G$ project onto their centralizers $Z_g = \{h \in S \otimes G \text{ with } [h,g] = 0\}$ and combine right and left movers to centralizer-invariant states of the Hilbert spaces H_g.
- Finally, build the linear combination of states of the Hilbert spaces H_g, $H_{hgh^{-1}}, \ldots, H_{h^n g h^{-n}}$, with $[h,g] \neq 0$ and $h^n = \mathbb{1}$ or $n = \infty$. Then, the sum is h-invariant. Repeat this step for all non-commuting $h \in S \otimes G$.

Notice that the constructing element of the untwisted sector is $g = (\mathbb{1}, 0; 0, 0)$. All elements of the orbifold group commute with this constructing one. Therefore, one has to project onto all elements of $S \otimes G$ in the untwisted sector (see sec. (2.7.1)).

However, this procedure has a disadvantage. Since the orbifold group $S \otimes G$ has an infinite number of elements, it is obvious that this procedure is not applicable. But, in sec. (3.2.2) we will see at a $\mathbb{Z}_3$ example that it is always possible to restrict the problem to a finite number of elements of $S \otimes G$.

2.9 The Role of Wilson Lines

If we start with a model without Wilson lines ($A_\alpha = 0$) and then switch on non-trivial Wilson lines A_α, it will effect the following:

- Wilson lines project out charged gauge bosons (and gauginos) and therefore they break the gauge group,
$$p \cdot A_\alpha = 0 \text{ mod } 1 \tag{2.120}$$
where the invariant p's are the roots of the unbroken gauge group.
- Depending on the fixed point, Wilson lines change the equation for massless left movers,
$$\frac{m_L^2}{4} = \frac{1}{2}\left(p - (kV + n_\alpha A_\alpha)\right)^2 + \tilde{N} - 1 + \delta c = 0 \ . \tag{2.121}$$
Furthermore, they project out twisted states due to the centralizer. Therefore, Wilson lines change the representations at different fixed points.

In sec. (4.4) this is demonstrated at a $\mathbb{Z}_2 \times \mathbb{Z}_2$ example with one non-trivial Wilson line.

Chapter 3

$\mathbb{Z}_3$ Orbifold

In this chapter we illustrate the general statements of the last chapter taking as an example the standard embedding for the $\mathbb{Z}_3$ orbifold without Wilson lines [15]. At the end, Wilson lines are briefly discussed.

3.1 Definition of the $\mathbb{Z}_3$ Orbifold

The point group P of the $\mathbb{Z}_3$ orbifold consists of the three elements $\{\mathbb{1}, \vartheta, \vartheta^2\}$, where ϑ is a rotation of 120°. The T^6 torus is chosen to be spanned by the root lattice of $SU(3)^3$ (e_1^a, e_2^a for $a = 1, 2, 3$ in fig. (3.1)). The action of P on the uncompactified space and on the six-torus is given in complex coordinates by the twist vector

$$v = \left(0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \text{ with } \vartheta^k \Leftrightarrow kv \ . \tag{3.1}$$

Obviously, this twist is an automorphism of the six-torus and it fulfills the condition for $\mathcal{N} = 1$ SUSY stated in eqn. (2.63),

$$v^1 + v^2 + v^3 = \frac{1}{3} + \frac{1}{3} - \frac{2}{3} = 0 \ . \tag{3.2}$$

Hence, we have now defined the space group S for a $\mathbb{Z}_3$ orbifold,

$$S = \{(kv, n_\alpha e_\alpha) \mid k = 0, 1, 2 \text{ and } n_\alpha \in \mathbb{Z}\} \ . \tag{3.3}$$

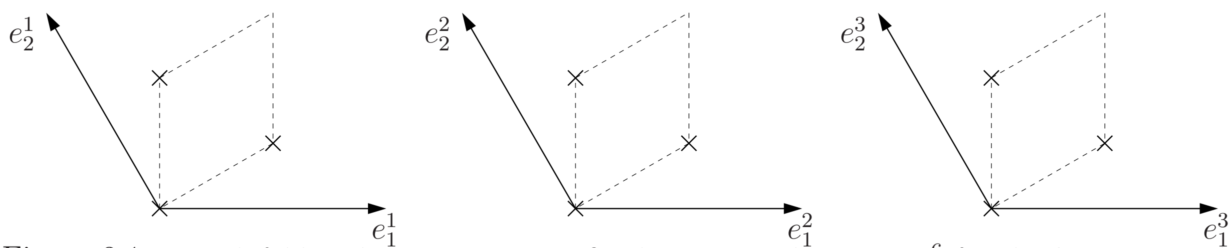


Figure 3.1: $\mathbb{Z}_3$ orbifold with $3 \times 3 \times 3 = 27$ fixed points in the space T^6 for the $k = 1$ twisted sector. The three complex planes are mutually orthogonal.

Having specified the space group, the geometry of this orbifold model is fixed. As explained in section (2.4.1), the $k = 1$ twisted sector of the $\mathbb{Z}_3$ orbifold has 27 fixed points (fig. 3.1).

3.2 The Standard Embedding

For the gauge twisting group G we choose shift embedding without Wilson lines ($A_\alpha = 0$). The shift for standard embedding is given by

$$V = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0^5\right)\left(0^8\right) \ , \tag{3.4}$$

where we introduce the following abbreviations

$$\left(\ldots, 0^5\right) \quad \text{for} \quad (\ldots, 0, 0, 0, 0, 0) \tag{3.5}$$

$$\left(\ldots, \left(\pm\frac{1}{2}\right)^3\right) \quad \text{for} \quad \left(\ldots, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \tag{3.6}$$

with the constraint of an even number of plus signs for the whole vector.
The shift has order three,

$$3V \in \mathrm{E}_8 \times \mathrm{E}_8' \text{ lattice.} \tag{3.7}$$

Thus, the gauge twisting group is given by

$$G = \{(kV, 0) \mid k = 0, 1, 2\} \ . \tag{3.8}$$

The orbifold group $S \otimes G$ with its shift V and its twist v trivially fulfills the strong modular invariance condition eqn. (2.57)

$$\left[(kV)^2 - (kv)^2\right] = 0 \bmod 2 \quad \text{for } k = 0, 1, 2 \ . \tag{3.9}$$

Therefore, anomaly freedom is guaranteed for the resulting spectrum of this orbifold model.

3.2.1 The Untwisted Sector

The untwisted sector is equal to the spectrum of the heterotic string, projected onto $S \otimes G$ invariant states. We concentrate on the particles of the $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge group, which breaks due to the orbifolding to the unbroken gauge group and to charged matter as described in sec. (2.7.1).

The transformations of right and left movers under a general element $h = (kv, n_\alpha e_\alpha; kV, 0)$ of the orbifold group are given by

$$|q\rangle_R \overset{h}{\mapsto} e^{2\pi i q\cdot(kv)}|q\rangle_R \tag{3.10}$$

$$|p\rangle_L \overset{h}{\mapsto} e^{2\pi i p\cdot(kV)}|p\rangle_L \ . \tag{3.11}$$

We see that it is enough to check invariance for $k = 1$, because this implies invariance for $k = 2$. The transformation properties of the right moving SO(8) vector/spinor and of the left moving $\mathrm{E}_8 \times \mathrm{E}_8'$ roots are listed in tab. (3.1) using the definition $\zeta := e^{2\pi i/3}$.

right movers $|q\rangle_R$

	q	$q\cdot v$	#
NS	$(\pm1,0,0,0)$	0	2
R	$\left(\pm\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)$	0	2

$\Rightarrow$ transforming as ζ^0

	q	$q\cdot v$	#
NS	$(0,\underline{1,0},0)$	$\frac{1}{3}$	2
	$(0,0,0,1)$	$-\frac{2}{3}$	1
R	$\left(\frac{1}{2},\underline{\frac{1}{2},-\frac{1}{2}},-\frac{1}{2}\right)$	$\frac{1}{3}$	2
	$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$-\frac{2}{3}$	1

$\Rightarrow$ transforming as ζ^1

	q	$q\cdot v$	#
NS	$(0,\underline{-1,0},0)$	$-\frac{1}{3}$	2
	$(0,0,0,-1)$	$\frac{2}{3}$	1
R	$\left(-\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2}},\frac{1}{2}\right)$	$-\frac{1}{3}$	2
	$\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\frac{2}{3}$	1

$\Rightarrow$ transforming as ζ^2

left movers $|p\rangle_L$

$p=(p_1)(0^8)$	$p\cdot V$	#
$(\pm(1,-1),0^6)$	0	2
$(\pm(1,0,-1),0^5)$	±1	2
$(0,\pm(1,-1),0^5)$	±1	2
$(0,0,0,\underline{\pm1,\pm1,0^3})$	0	40
$\left(\pm\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{2}\right)^5\right)$	0	32

$\Rightarrow$ 78 roots, transforming as ζ^0

$p=(0^8)(p_2)$	$p\cdot V$	#
$(\underline{\pm1,\pm1,0^6})$	0	112
$\left(\left(\pm\frac{1}{2}\right)^8\right)$	0	128

$\Rightarrow$ 240 roots, transforming as ζ^0

$p=(p_1)(0^8)$	$p\cdot V$	#
$(1,0,0,\underline{\pm1,0^4})$	$\frac{1}{3}$	10
$(0,1,0,\underline{\pm1,0^4})$	$\frac{1}{3}$	10
$(0,0,1,\underline{\pm1,0^4})$	$-\frac{2}{3}$	10
$(\underline{-1,0},-1,0^5)$	$\frac{1}{3}$	2
$(-1,-1,0,0^5)$	$-\frac{2}{3}$	1
$\left(\underline{-\frac{1}{2},\frac{1}{2}},-\frac{1}{2},\left(\pm\frac{1}{2}\right)^5\right)$	$\frac{1}{3}$	32
$\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\left(\pm\frac{1}{2}\right)^5\right)$	$\frac{1}{3}$	16

$\Rightarrow$ 81 roots, transforming as ζ^1

$p=(p_1)(0^8)$	$p\cdot V$	#
$(-1,0,0,\underline{\pm1,0^4})$	$-\frac{1}{3}$	10
$(0,-1,0,\underline{\pm1,0^4})$	$-\frac{1}{3}$	10
$(0,0,-1,\underline{\pm1,0^4})$	$\frac{2}{3}$	10
$(\underline{1,0},1,0^5)$	$-\frac{1}{3}$	2
$(1,1,0,0^5)$	$\frac{2}{3}$	1
$\left(\underline{-\frac{1}{2},\frac{1}{2}},\frac{1}{2},\left(\pm\frac{1}{2}\right)^5\right)$	$-\frac{1}{3}$	32
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\left(\pm\frac{1}{2}\right)^5\right)$	$\frac{2}{3}$	16

$\Rightarrow$ 81 roots, transforming as ζ^2

Table 3.1: Transformation properties of right and left movers in the untwisted sector. The symbol # denotes the number of right and left movers, respectively. The transformation phase is denoted by $\zeta=e^{2\pi i/3}$.

Tensoring right and left movers of tab. (3.1) together yields the following invariant states for the untwisted sector

$$\left.\begin{array}{l} |\pm 1,0,0,0\rangle_R \\ |\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle_R \end{array}\right\} \otimes \quad |p\rangle_L \qquad \text{with 78 roots, transforming as } \zeta^0 \tag{3.12}$$

$$\left.\begin{array}{l} |\pm 1,0,0,0\rangle_R \\ |\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle_R \end{array}\right\} \otimes \quad |p\rangle_L \qquad \text{with 240 roots, transforming as } \zeta^0 \tag{3.13}$$

and

$$\left.\begin{array}{l} |0,\underline{-1,0,0}\rangle_R \\ |-\frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\rangle_R \end{array}\right\} \otimes \quad |p\rangle_L \qquad \text{with 81 weights, transforming as } \zeta^1 \tag{3.14}$$

$$\left.\begin{array}{l} |0,\underline{1,0,0}\rangle_R \\ |\frac{1}{2},\underline{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\rangle_R \end{array}\right\} \otimes \quad |p\rangle_L \qquad \text{with 81 weights, transforming as } \zeta^2\ . \tag{3.15}$$

Now, we have to identify the charged gauge bosons (and gauginos) of the unbroken gauge group and the charged matter states. Using some general properties discussed in sec. (2.7.1) and combinatorics we find:

- Following the discussion of sec. (2.7.1), we conclude that the states of eqns. (3.12) and (3.13) correspond to the **charged gauge bosons** (and gauginos) of the unbroken gauge group.

  Since the shift $V = (\ldots)(0^8)$ acts trivially on the second part of the E_8' root lattice, it does not project out any state that only transforms in the second E_8'. Therefore, the 240 roots of eqn. (3.13) are obviously the invariant roots of the adjoint representation of the E_8'. For the other 78 roots of eqn. (3.12) we use combinatorics and guess the gauge group
  $$\mathrm{E}_6 \times \mathrm{SU}(3)\ , \tag{3.16}$$
  which has 72 plus 6 "charged" roots.

- Due to the non-trivial transformation of the right mover, we conclude that states of eqns. (3.14) and (3.15) correspond to **charged matter** and therefore, the roots of the broken gauge group become weights of representations under the unbroken gauge group. It is rather difficult to guess these representations, therefore we will use group theory to determine them.

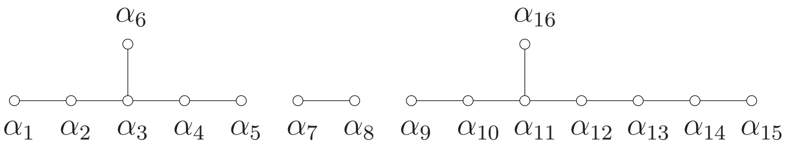
In the next section it is shown how to derive the unbroken gauge group and the matter representations in the untwisted sector.

Unbroken gauge group

In order to derive the gauge group, we have to find all simple roots of the 78+240 roots. We denote them by α_i. The simple roots depend on the basis that fixes the semi-ordering in the weight space [16]. Since this basis is arbitrary, the simple roots are only listed (together with the chosen basis) in the appendix (C.1). The simple roots are used to compute the Cartan matrix (taking into account that the roots have length 2)

$$A_{ij} = (\alpha_i, \alpha_j) \ . \tag{3.17}$$

Now, we can draw the Dynkin diagrams:



$$\mathrm{E}_6 \quad \times \quad \mathrm{SU}(3) \quad \times \quad \mathrm{E}_8' \ . \tag{3.18}$$

Charged matter

Next, we determine the representations, under which the charged matter states of eqns. (3.14, 3.15) transform. To do this, we use the simple roots to rewrite the weights as Dynkin labels (DL), both with respect to E_6 and with respect to SU(3) and finally with respect to E_8',

$$p_{\mathrm{DL(E_6)}} = (\alpha_1 \cdot p, \alpha_2 \cdot p, \alpha_3 \cdot p, \alpha_4 \cdot p, \alpha_5 \cdot p, \alpha_6 \cdot p) \tag{3.19}$$
$$p_{\mathrm{DL(SU(3))}} = (\alpha_7 \cdot p, \alpha_8 \cdot p) \tag{3.20}$$
$$p_{\mathrm{DL(E_8')}} = (\alpha_9 \cdot p, \ldots, \alpha_{16} \cdot p) \ . \tag{3.21}$$

Then, we compare the highest weights in Dynkin labels with ref. [17] and thus identify the representations. Since there are too many weights in the untwisted sector, not all steps are presented explicitly and we focus only on the results. In the twisted sector, where the representations are smaller, we will follow these steps in greater detail.

It turns out that the charged matter states of the untwisted sector transform under the gauge group $\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8'$ as:

$$3(\mathbf{27}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{L}} \quad \text{for states of eqn. (3.14)} \tag{3.22}$$
$$3(\overline{\mathbf{27}}, \mathbf{3}, \mathbf{1})_{\mathrm{R}} \quad \text{for states of eqn. (3.15)} \tag{3.23}$$

where the subscript denotes left- and right-handed states. The multiplicity 3 appears due to the three right movers for both the R and NS state (see tab. (3.1)). We decide to count left-chiral states as particles, thus the states $3(\mathbf{27}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{L}}$ are counted as particles and the states $3(\overline{\mathbf{27}}, \mathbf{3}, \mathbf{1})_{\mathrm{R}}$ give rise to their antiparticles.

3.2.2 The Twisted Sectors

The $\mathbb{Z}_3$ orbifold has two twisted sectors corresponding to $k = 1, 2$. We focus on the first twisted sector $k = 1$. However, the result of the $k = 2$ twisted sector is given at the end of this chapter. The $k = 1$ sector has 27 fixed points, and each corresponds one-to-one to a space group element. Since there are no non-trivial Wilson lines, these 27 space group elements are embedded as the same element of the gauge twisting group,

$$S \hookrightarrow G \qquad (v, n_\alpha e_\alpha) \mapsto (V, 0) \ , \tag{3.24}$$

and the 27 mass-equations for the 27 fixed points are identical. Thus, we only have to solve it once, e.g. with respect to the constructing element $g = (v, 0; V, 0)$. The resulting spectrum of the $k = 1$ twisted sector gets a multiplicity of 27 due to the number of fixed points.

The twisted right movers

The condition for massless right movers (eqn.(2.94)) reads

$$\frac{m_R^2}{4} = \frac{1}{2}(q+v)^2 - \frac{1}{6} = 0 \tag{3.25}$$

where the zero point energy is shifted by $\delta c = \frac{1}{3}$. Thus, we have to solve

$$(q+v)^2 = \frac{1}{3} \ . \tag{3.26}$$

In contrast to the difficult case of the left movers, it is easy to guess the solutions for massless twisted right movers. Along with their transformation properties they are listed in tab. (3.2). The transformation properties are calculated with respect to the constructing element g itself, leading to a transformation

$$\zeta^0 = e^{2\pi i q_{\mathrm{sh}} \cdot v} = 1 \tag{3.27}$$

for both twisted right movers.

In sec. (2.8) it was shown how to project onto $S \otimes G$ invariant states in principle. So why do we only check the invariance with respect to the constructing element. What about the other commuting and non-commuting elements of the orbifold group? The answer to this question is given in the following.

	q	$q_{\mathrm{sh}} = q + v$	$q_{\mathrm{sh}} \cdot v$
NS	$(0,0,0,1)$	$\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	0
R	$\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right)$	0

Table 3.2: The right movers in the $k = 1$ twisted sector.

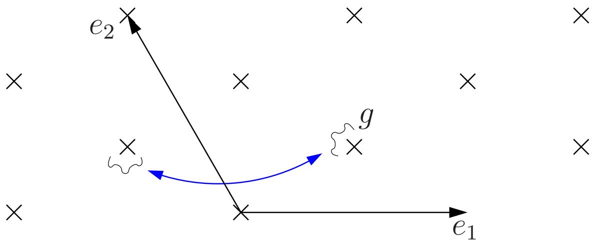


Figure 3.2: The twisted string corresponds to the constructing element $g = (\vartheta, e_1)$. The crosses mark the fixed points in $\mathbb{R}^6$.

Why do we check the invariance only with respect to the constructing element?

As an example we consider a constructing element of the $k = 1$ twisted sector

$$g = (\vartheta, e_1) \in S \; . \tag{3.28}$$

The constructing element g corresponds one-to-one to a fixed point. A string that closes under the action of g is localized there. The situation is illustrated in fig. (3.2). Following the discussion of sec. (2.8) we will first consider the centralizer

$$Z_g = \{(\mathbb{1}, 0)\, , (\vartheta, e_1)\, , \left(\vartheta^2, e_1 + e_2\right)\} \; . \tag{3.29}$$

The elements of the centralizer Z_g must act as the identity on the states of the Hilbert space H_g. Therefore, we have to project onto these elements. But the first element $(\mathbb{1}, 0)$ acts trivially. Furthermore, the transformation properties with respect to the third element $(\vartheta^2, e_1 + e_2)$ can be deduced from the second one, because there are no non-trivial Wilson lines ($A_\alpha = 0$). Thus, it is enough to project only onto (ϑ, e_1). This element is equal to the constructing one g.

But what about the non-commuting elements? In order to see why we can neglect them, we act with an arbitrary element, for example

$$h = (\mathbb{1}, e_1) \;\; , \tag{3.30}$$

on the boundary condition of the constructing element g. Since g and h do not commute

$$[h, g] = (\mathbb{1}, e_1 - e_2) \;\; , \tag{3.31}$$

h maps the string to a different fixed point in $\mathbb{R}^6$. This is shown in fig. (3.3).

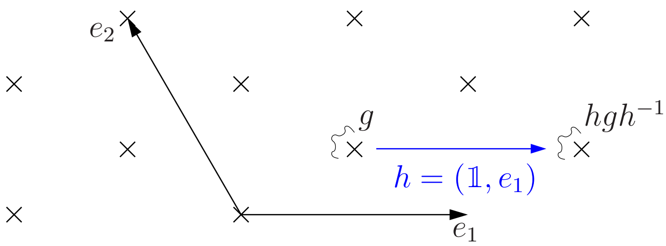


Figure 3.3: The twisted string at g is mapped to hgh^{-1} under the action of h.

The constructing element that corresponds to this shifted fixed point can be computed. Using the inverse $h^{-1} = (\mathbb{1}, -e_1)$ we find the constructing element

$$hgh^{-1} = (\vartheta, 2e_1 - e_2) \ . \tag{3.32}$$

The corresponding string is shown in fig. (3.4).

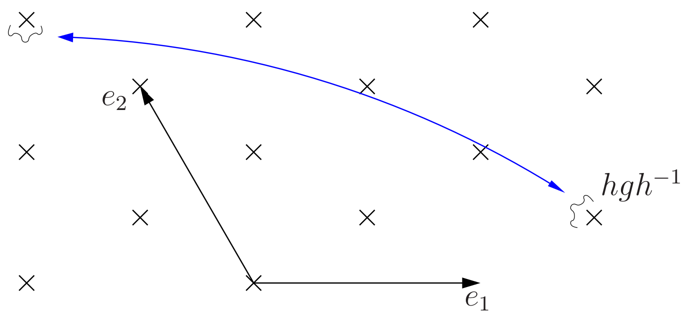


Figure 3.4: The twisted string corresponds to the constructing element $hgh^{-1} = (\vartheta, 2e_1 - e_2)$.

The fixed point of the space group element hgh^{-1} is apparently identified on the torus with the one of g. Since we defined fixed points on the six-torus T^6, we can neglect the action of h.

If h had mapped the fixed point of g to a different fixed point of the space T^6, the action of h would have been important. In this case h would identify two fixed points that were previously considered as inequivalent. In general, this only happens if the point group P has a non-trivial subgroup, for example the $\mathbb{Z}_4$ orbifold with a $\mathbb{Z}_2$ subgroup in app. (D). Furthermore, it is possible to define fixed points directly in the space $\mathbb{R}^6/S$. Then the action of non-commuting elements can always be disregarded.

So we can ignore the transformation properties with respect to the non-commuting element h in the case of a $\mathbb{Z}_3$ orbifold. Following these steps for other non-commuting elements, it can be shown that they can be ignored, too.

The twisted left movers

The condition for massless left movers (eqn.(2.101))

$$\frac{m_L^2}{4} = \frac{1}{2}(p - V)^2 + \tilde{N} - \frac{2}{3} = 0 \ , \tag{3.33}$$

or equivalently

$$(p - V)^2 = \frac{4}{3} - 2\tilde{N} \ , \tag{3.34}$$

can only be fulfilled for $\tilde{N} = 0, \frac{1}{3}$. The equation with $\tilde{N} = \frac{2}{3}$ cannot be solved for any p. According to the method described in the appendix (B.3), we will now solve the mass-equations for these two cases.

Case $\tilde{\mathbf{N}} = \mathbf{0}$. We have to solve the mass-equation

$$(p - V)^2 = \frac{4}{3} \tag{3.35}$$

with $V = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0^5\right)(0^8)$. First, we have to compute the bounds p_{up} and p_{low}. They are the edges of a 16 dimensional box in the $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice, in which the solutions p of the mass-equation can be found,

$$\underbrace{\left(p^1 - \frac{1}{3}\right)^2}_{\leq \frac{4}{3}} + \underbrace{\left(p^2 - \frac{1}{3}\right)^2}_{\leq \frac{4}{3}} + \underbrace{\left(p^3 + \frac{2}{3}\right)^2}_{\leq \frac{4}{3}} + \sum_{I=4}^{16} \underbrace{\left(p^I\right)^2}_{\leq \frac{4}{3}} = \frac{4}{3}\ . \tag{3.36}$$

Exemplarily, we compute the first component of the bounds (see eqn. (B.38))

$$\begin{aligned} p^1_{up} &= \lfloor \sqrt{\tfrac{4}{3}} + \tfrac{1}{3} \rfloor & &\approx \lfloor 1.49 \rfloor & &= 1.0 \\ p^1_{low} &= \lceil -\sqrt{\tfrac{4}{3}} + \tfrac{1}{3} \rceil & &\approx \lceil -0.82 \rceil & &= -0.5\ . \end{aligned}$$

Thus the bounds for the solutions of the mass-equation are given by

$$p_{up} = \left(1, 1, 0, 1^5\right)\left(1^8\right) \tag{3.37}$$

$$p_{low} = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, (-1)^5\right)\left(-1^8\right)\ . \tag{3.38}$$

Tab. (3.3) shows how all solutions can be obtained systematically by an iterative procedure. 27 weights of the $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice solve the mass-equation (3.35). They correspond to 27 left movers

$$|p - V\rangle_L\ . \tag{3.39}$$

Like in the right movers case, we determine the transformation properties of these left movers with respect to the constructing element $g = (v, 0; V, 0)$,

$$|p - V\rangle_L \overset{g}{\mapsto} e^{2\pi i (p-V)\cdot V} |p - V\rangle_L\ , \tag{3.40}$$

where all 27 weights transform as $\zeta^0 = e^{2\pi i (p-V)\cdot V} = 1$. Thus, the following states are invariant under the orbifolding

$$\left.\begin{array}{l} |0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\rangle_R \\ |\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\rangle_R \end{array}\right\} \otimes |p - V\rangle_L\ . \tag{3.41}$$

The states with R right movers are right-chiral fermions, thus we count them as antiparticles. The corresponding left-chiral particles arise in the $k = 2$ twisted sector.

Next, we determine the transformation under the $\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8'$ gauge group. Therefore, we need the simple roots we have identified in the untwisted sector. Thus we can convert the weights $p_{\mathrm{sh}} = p - V$ into Dynkin labels with respect to each part of the gauge group.

$$p_{\mathrm{sh,\ DL(E_6)}} = (\alpha_1 \cdot p_{\mathrm{sh}}, \ldots, \alpha_6 \cdot p_{\mathrm{sh}}) \tag{3.42}$$

$$p_{\mathrm{sh,\ DL(SU(3))}} = (\alpha_7 \cdot p_{\mathrm{sh}}, \alpha_8 \cdot p_{\mathrm{sh}}) = (0, 0) \tag{3.43}$$

$$p_{\mathrm{sh,\ DL(E_8')}} = (\alpha_9 \cdot p_{\mathrm{sh}}, \ldots, \alpha_{16} \cdot p_{\mathrm{sh}}) = \left(0^8\right) \tag{3.44}$$

The Dynkin labels with respect to the SU(3) and E_8' gauge groups are trivial: $(0, 0)$ and (0^8), respectively. Hence, these states are singlets under the corresponding gauge groups. But with respect to E_6 the Dynkin labels are non-trivial. They are listed in tab. (3.4). Comparing them to ref. [17] yields the result that the 27 states form the representation

$$(\overline{\mathbf{27}}, \mathbf{1}, \mathbf{1})_{\mathrm{R}} \tag{3.45}$$

under the gauge group.

p^1		p^2	$m(2)$	p^3	$m(3)$	$p^4 \ldots p^8$	$m(8)$	$p^9 \ldots p^{16}$	$m(16)$	#
$-\frac{1}{2}$	half-int.	$-\frac{1}{2}$	$\frac{25}{18}\dagger$							
	$\cdots$	$+\frac{1}{2}$	$\frac{13}{18}$	$-\frac{3}{2}$	$\frac{17}{12}\dagger$					
			$\cdots$	$-\frac{1}{2}$	$\frac{3}{4}$	$\left(\pm\frac{1}{2}\right)^5$	$2\dagger$			
0	int.	0	$\frac{2}{9}$	-1	$\frac{1}{3}$	$\underline{\pm 1, 0^4}$	$\frac{4}{3}$	0^8	$\frac{4}{3}$	10
			$\cdots$	0	$\frac{2}{3}$	$\cdots$		$\cdots$	$\dagger$	
	$\cdots$	1	$\frac{5}{9}$	-1	$\frac{2}{3}$	$\cdots$		$\cdots$	$\dagger$	
			$\cdots$	0	1	$\underline{\pm 1, 0^4}$	$2\dagger$			
$+\frac{1}{2}$	half-int.	$-\frac{1}{2}$	$\frac{13}{18}$	$-\frac{3}{2}$	$\frac{17}{12}\dagger$					
			$\cdots$	$-\frac{1}{2}$	$\frac{3}{4}$	$\left(\pm\frac{1}{2}\right)^5$	$2\dagger$			
	$\cdots$	$+\frac{1}{2}$	$\frac{1}{18}$	$-\frac{3}{2}$	$\frac{3}{4}$	$\left(\pm\frac{1}{2}\right)^5$	$2\dagger$			
			$\cdots$	$-\frac{1}{2}$	$\frac{1}{12}$	$\left(\pm\frac{1}{2}\right)^5$	$\frac{4}{3}$	0^8	$\frac{4}{3}$	16
1	int.	0	$\frac{5}{9}$	-1	$\frac{2}{3}$	$\cdots$		$\cdots$	$\dagger$	
			$\cdots$	0	1	$\underline{\pm 1, 0^4}$	$2\dagger$			
	$\cdots$	1	$\frac{8}{9}$	-1	1	$\underline{\pm 1, 0^4}$	$2\dagger$			
			$\cdots$	0	$\frac{4}{3}$	0^5	$\frac{4}{3}$	0^8	$\frac{4}{3}$	1

Table 3.3: If the result of $m(J) = \sum_{I=1}^{J}(p^I - V^I)^2$ is greater than $\frac{4}{3}$, this p is apparently no solution of the twisted mass-equation (denoted by †). Thus, we have to change p^J or a previous p^K (with $K < J$). This depends on the bound p_{up}. The dots $\cdots$ in the first columns denote that the previous components p^I are equal to the ones of the previous line.

$p=(p_1)(0^8)$	$p_{\mathrm{sh}}=p-V$	$p_{\mathrm{sh,\ DL(E_6)}}$
$(0,0,-1,-1,0,0,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-1,0,0,0,0\right)$	$(0,1,-1,0,0,1)$
$(0,0,-1,0,-1,0,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,-1,0,0,0\right)$	$(1,-1,0,0,0,1)$
$(0,0,-1,0,0,-1,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,-1,0,0\right)$	$(1,0,0,-1,1,0)$
$(0,0,-1,0,0,0,-1,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,-1,0\right)$	$(1,0,-1,1,0,0)$
$(0,0,-1,0,0,0,0,-1)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,0,-1\right)$	$(0,0,-1,0,1,1)$
$(0,0,-1,0,0,0,0,1)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,0,1\right)$	$(1,0,0,0,-1,0)$
$(0,0,-1,0,0,0,1,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,1,0\right)$	$(0,0,0,-1,0,1)$
$(0,0,-1,0,0,1,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,1,0,0\right)$	$(0,0,-1,1,-1,1)$
$(0,0,-1,0,1,0,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,1,0,0,0\right)$	$(0,1,-1,0,0,0)$
$(0,0,-1,1,0,0,0,0)$	$\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},1,0,0,0,0\right)$	$(1,-1,0,0,0,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\mathbf{(0,0,0,0,1,0)}$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,0,1,-1,0,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,0,1,-1,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,0,0,0,1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,1,0,0,0,-1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(-1,1,0,-1,1,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(-1,1,-1,1,0,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(-1,1,0,0,-1,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(1,-1,1,0,0,-1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,-1,1,-1,1,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,-1,0,1,0,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,-1,1,0,-1,0)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,0,0,1,-1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,0,1,-1,0,-1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,0,1,-1,-1)$
$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,0,0,0,0)$
$(1,1,0,0,0,0,0,0)$	$\left(\frac{2}{3},\frac{2}{3},\frac{2}{3},0,0,0,0,0\right)$	$(-1,0,1,0,0,-1)$

Table 3.4: The 27 roots which fulfill the mass-equation. In the first column they are unshifted and in the second they are shifted by V. In the last column the shifted roots are written in Dynkin Labels w.r.t. the E_6. $\mathbf{(0,0,0,0,1,0)}$ is the highest weight of the $\overline{\mathbf{27}}$.

Case $\tilde{\mathbf{N}} = \frac{1}{3}$. Due to the moded oscillator the resulting states are of the form

$$\left.\begin{array}{l} |0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\rangle_R \\ |\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\rangle_R \end{array}\right\} \otimes \tilde{\alpha}^a_{-\frac{1}{3}} |p - V\rangle_L \tag{3.46}$$

where the vector p of the $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice is a solution of the equation

$$(p - V)^2 = \frac{2}{3} \tag{3.47}$$

for massless left movers. The bounds for the solutions of this mass-equation are given by

$$p_{up} = \left(1, 1, 1, \left(\frac{1}{2}\right)^5\right)\left(\left(\frac{1}{2}\right)^8\right) \tag{3.48}$$

$$p_{low} = \left(0, 0, 0, \left(-\frac{1}{2}\right)^5\right)\left(\left(-\frac{1}{2}\right)^8\right) \tag{3.49}$$

and the weights together with the Dynkin labels are listed in tab. (3.5). The weights transform as

$$\zeta = e^{2\pi i p_{\mathrm{sh}} \cdot V} = e^{2\pi i/3} \tag{3.50}$$

under the action of the constructing element g. It follows that the $S \otimes G$ invariant states with $\tilde{N} = \frac{1}{3}$ are

$$|q + v\rangle_R \otimes \tilde{\alpha}^{\bar{a}}_{-\frac{1}{3}} |p - V\rangle_L \overset{g}{\mapsto} \underbrace{\zeta\, \zeta^2}_{=1}\ |q + v\rangle_R \otimes \tilde{\alpha}^{\bar{a}}_{-\frac{1}{3}} |p - V\rangle_L \tag{3.51}$$

for $\bar{a} = \bar{1}, \bar{2}, \bar{3}$ (complex index). The moded oscillators $\tilde{\alpha}^{\bar{a}}_{-1/3}$ of the left mover corresponds to the $\overline{\mathbf{3}}$ of SU(3), therefore they transform with ζ^2 (see eqn. (2.106) with $m = 0$, and $\eta^a = 1/3$ in all three complex planes $a = 1, 2, 3$). The invariant states form the following representation under the unbroken gauge group

$$3(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{R}} \tag{3.52}$$

where the factor 3 comes from the three (complex) moded oscillators of the left movers.

$p = (p_1)(0^8)$	$p_{\mathrm{sh}} = p - V$	$p_{\mathrm{sh,\ DL(SU(3))}}$	$p_{\mathrm{sh}} \cdot V$
(0^8)	$\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, 0^5\right)$	$(\mathbf{0}, \mathbf{1})$	$-\frac{2}{3}$
$(1, 0, -1, 0^5)$	$\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0^5\right)$	$(1, -1)$	$\frac{1}{3}$
$(0, 1, -1, 0^5)$	$\left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, 0^5\right)$	$(-1, 0)$	$\frac{1}{3}$

Table 3.5: The left movers in the $k = 1$ twisted sector for $\tilde{N} = \frac{1}{3}$. The highest weight is $(\mathbf{0}, \mathbf{1})$. It corresponds to the representation $\overline{\mathbf{3}}$ of SU(3).

The Spectrum of the first twisted sector Taking the multiplicity of 27 fixed points into account, the spectrum of the $k = 1$ twisted sector reads:

$$27 \quad (\overline{\mathbf{27}}, \mathbf{1}, \mathbf{1})_{\mathrm{R}} \tag{3.53}$$
$$81 \quad (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{R}} \tag{3.54}$$

Since these states are all right-chiral, they are counted as antiparticles. The corresponding particles can be found in the $k = 2$ twisted sector, which is not discussed here in detail.

3.2.3 The Spectrum

To summarize, the standard embedding of the $\mathbb{Z}_3$ orbifold has $\mathcal{N} = 1$ SUSY in four dimensions with the gauge group

$$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8' \tag{3.55}$$

and the following matter representations:

sector	**matter content**
in the untwisted sector:	$3 \quad (\mathbf{27}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{L}}$
	$3 \quad (\overline{\mathbf{27}}, \mathbf{3}, \mathbf{1})_{\mathrm{R}}$
in the $k = 1$ twisted sector:	$27 \quad (\overline{\mathbf{27}}, \mathbf{1}, \mathbf{1})_{\mathrm{R}}$
	$81 \quad (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})_{\mathrm{R}}$
in the $k = 2$ twisted sector:	$27 \quad (\mathbf{27}, \mathbf{1}, \mathbf{1})_{\mathrm{L}}$
	$81 \quad (\mathbf{1}, \mathbf{3}, \mathbf{1})_{\mathrm{L}}$

A chiral supermultiplet consists of a left-handed particle and its right-handed antiparticle.

3.2.4 Heterotic Brane Picture

Since the states of the untwisted sector are not attached to a fixed point, they are in addition to our four-dimensional space free to move in the six compact dimensions of the orbifold. Following the notation of [18], they can be seen as **bulk fields**.

The states of the twisted sectors are localized at fixed points. Since the fixed points of the $\mathbb{Z}_3$ orbifold are zero-dimensional objects, the states of the twisted sectors are completely fixed in the six compact dimensions. Of course, these states can propagate freely in (3+1)-dimensional Minkowski space, now called **3-branes**. Thus each fixed point in the compact dimensions corresponds to a 3-brane from the ten-dimensional point-of-view. It is easy to see that all 3-branes are parallel. The situation is illustrated in fig. (3.5).

Note that a state on such a 3-brane is only fixed with respect to its centre-of-mass. So it is still possible that strings of different branes interact. The interaction strength depends on the overlap of the states of different branes. Therefore, it decreases with the distance of the branes.

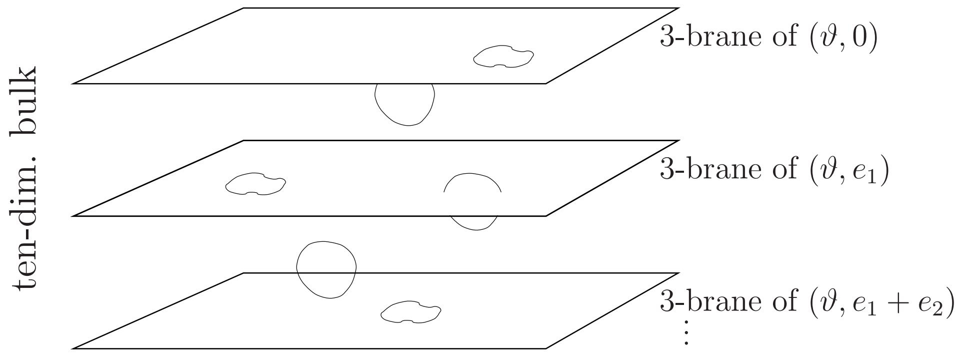


Figure 3.5: Brane picture for the $\mathbb{Z}_3$ orbifold: 27 parallel 3-branes in the ten-dimensional bulk. Each 3-brane corresponds to a fixed point and thus to a constructing element of the space group. Their distances in the six extra dimensions are approximately of order M_{Planck}^{-1} and correspond to the compactification radii.

3.3 Wilson Lines

Since the standard embedding leads to a spectrum with 36 generations of E_6, Wilson lines are needed to break the gauge group and to reduce the number of generations. Some examples can be found in [9, 19, 20, 21, 22]. However, we restrict ourselves to the following consideration about Wilson lines in $\mathbb{Z}_3$ orbifolds.

In general, a torus shift e_i^a (in the complex plane that corresponds to $a = 1, 2, 3$) is embedded as a Wilson line A_i^a. Thus, six Wilson lines are possible. However, some torus-shifts are identified on the $\mathbb{Z}_3$ orbifold. For example the torus-shifts in the first complex plane e_1^1 and e_2^1 correspond to the same direction on the $\mathbb{Z}_3$ orbifold,

$$e_2^1 = \vartheta e_1^1 \quad \Rightarrow \quad e_2^1 \simeq e_1^1 \ . \tag{3.56}$$

Compare to fig. (3.6). Therefore, the corresponding Wilson lines must be equal and only three Wilson lines can be chosen independently,

$$A_1^1 = A_2^1 \tag{3.57}$$
$$A_1^2 = A_2^2 \tag{3.58}$$
$$A_1^3 = A_2^3 \ . \tag{3.59}$$

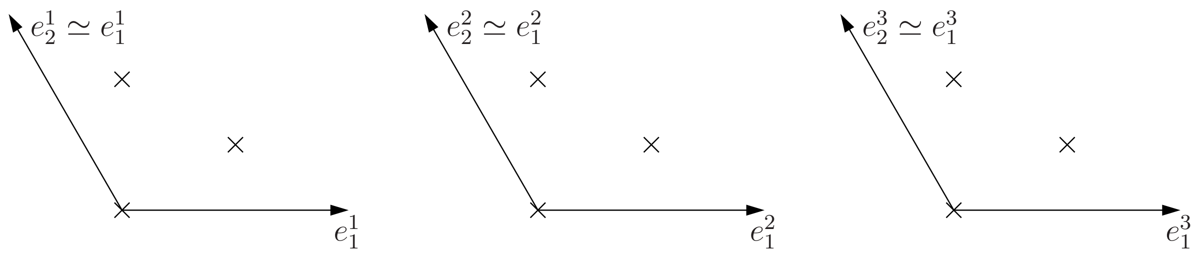


Figure 3.6: For the $\mathbb{Z}_3$ orbifold only three of six Wilson lines are independent.

Chapter 4

$\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

In this chapter we focus on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We will see that the geometry is more involved than in the $\mathbb{Z}_3$ case, but therefore the structure is richer. The chapter starts with an explanation of the geometry. Afterwards, a model with standard embedding is presented. In order to demonstrate the action of Wilson lines on the spectrum, a model with one Wilson line is analyzed in detail.

4.1 Definition of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

First, we start with the definition of the point group, and consequently of the space group. To do this, we choose a six-torus. There are many torus-lattices that have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Here, we choose the following one

$$T^6 = T^2 \times T^2 \times T^2 \ , \tag{4.1}$$

where we denote the basis of the torus lattice by e_i for $i = 1, \ldots, 6$. Furthermore, we choose these basis vectors to be orthogonal. The action of the point group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathbb{1},\ \vartheta_1,\ \vartheta_2,\ \vartheta_1 \circ \vartheta_2\} \tag{4.2}$$

can be written as rotations in four complex planes. The twist vectors are

$$\mathbb{1} \quad \Leftrightarrow \quad 0v_1 + 0v_2 = \left(0, 0, 0, 0\right) \tag{4.3}$$

$$\vartheta_1 \quad \Leftrightarrow \quad 1v_1 + 0v_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0\right) \tag{4.4}$$

$$\vartheta_2 \quad \Leftrightarrow \quad 0v_1 + 1v_2 = \left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) \tag{4.5}$$

$$\vartheta_1 \circ \vartheta_2 \quad \Leftrightarrow \quad 1v_1 + 1v_2 = \left(0, \frac{1}{2}, 0, -\frac{1}{2}\right) \ . \tag{4.6}$$

Obviously, these twists are automorphisms of the six-torus $T^2 \times T^2 \times T^2$, thus they are admissible and we define the space group

$$S = \{(kv_1 + lv_2, n_\alpha e_\alpha) \mid k, l = 0, 1, n_\alpha \in \mathbb{Z}\} \ . \tag{4.7}$$

We label the twisted sectors by $T_{(k,l)}$ with $k, l = 0, 1$. The case $k = l = 0$ corresponds to the untwisted sector.

The supergravity multiplet

The general discussion is given in sec. (2.7.1). In order to compute the number of supersymmetries, we count the number of gravitini. Since the left mover of the supergravity multiplet is invariant under the orbifold projection, each twist breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ in four dimensions, i.e. the two gravitini

$$\left| \pm \left(\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \frac{1}{2}\right) \right\rangle_R \otimes \ \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1, 2 \tag{4.8}$$

$$\left| \pm \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \right\rangle_R \otimes \ \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1, 2 \tag{4.9}$$

are invariant under the twist $\vartheta_1 \Leftrightarrow v_1 = (0, \frac{1}{2}, -\frac{1}{2}, 0)$. Furthermore, the two gravitini

$$\left| \pm \left(\frac{1}{2}, \frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right) \right\rangle_R \otimes \ \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1, 2 \tag{4.10}$$

$$\left| \pm \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \right\rangle_R \otimes \ \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1, 2 \tag{4.11}$$

are invariant under the second twist $\vartheta_2 \Leftrightarrow v_2 = (0, 0, \frac{1}{2}, -\frac{1}{2})$. Consequently, the combination of all twists leaves only one gravitino invariant,

$$\left| \pm \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\rangle_R \otimes \tilde{\alpha}^\mu_{-1}|0\rangle_L \qquad \mu = 1, 2 \ , \tag{4.12}$$

and therefore four-dimensional $\mathcal{N} = 1$ supersymmetry is unbroken.

4.2 The Geometry of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

In sec. (2.4.2) we discussed the number of fixed points in the example of a $\mathbb{Z}_2$ orbifold and it turned out that there are 16 fixed tori.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold has three twisted sectors. They correspond to the twist vectors $kv_1 + lv_2$ with ($k = 1$ and $l = 0$), ($k = 0$ and $l = 1$), ($k = 1$ and $l = 1$), respectively. Each twist is analogous to the one of the $\mathbb{Z}_2$ example (compare to fig. (4.1)). Even so, we cannot say that the fixed points are fixed tori, because at the end, the space group with all its twists is moded out. Thus, there is no torus anymore and we will denote the fixed points by fixed hypersurfaces. Therefore, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold has $16 + 16 + 16 = 48$ fixed hypersurfaces.

Note that the six orthogonal directions e_α of the six-torus T^6 are inequivalent on the orbifold. Hence we can define six inequivalent non-contractible loops on the orbifold, consequently six Wilson lines A_α. But first we consider the case of no Wilson lines ($A_\alpha = 0$). The action of one Wilson line is explained in sec. (4.4).

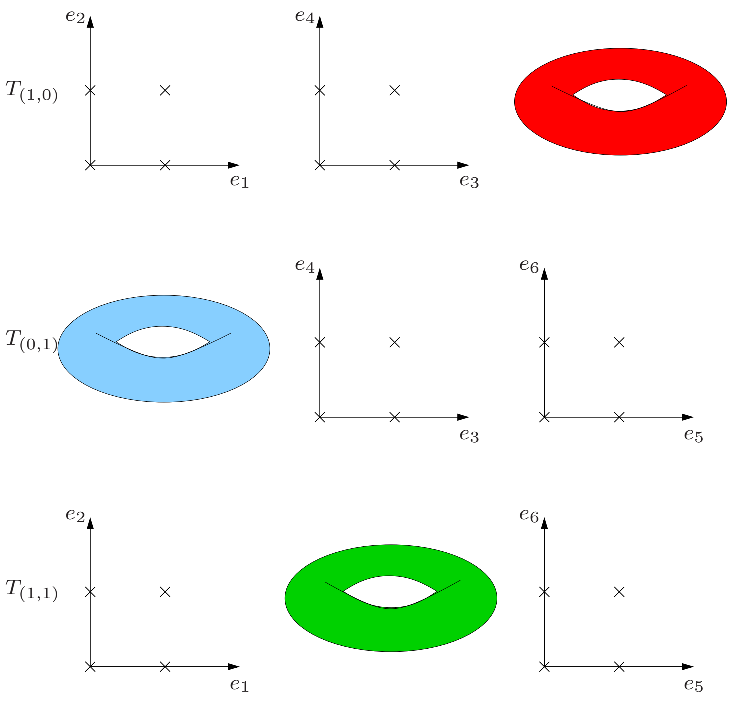


Figure 4.1: The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold has 48 fixed hypersurfaces. Only before moding out the other twists they look like fixed tori. Each row shows the 16 parallel fixed hypersurfaces of the particular twisted sector $T_{(k,l)}$. Fixed hypersurfaces of different sectors are mutually orthogonal.

4.3 The Standard Embedding

First, we study the standard embedding without non-trivial Wilson lines, i.e. $A_\alpha = 0$ for $\alpha = 1, \ldots, 6$. The result is in agreement with ref. [8]. The shifts are given by

$$v_1 \quad \Leftrightarrow \quad V_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0^6\right)\left(0^8\right) \tag{4.13}$$

$$v_2 \quad \Leftrightarrow \quad V_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0^5\right)\left(0^8\right) \tag{4.14}$$

and they satisfy

$$2V_i \in T_{\mathrm{E}_8 \times \mathrm{E}_8'} \text{ for } i = 1, 2\ . \tag{4.15}$$

Now, the gauge twisting group G and thus the orbifold group $S \otimes G$ are defined. The orbifold group of the standard embedding trivially fulfills the (strong) modular invariance condition eqn. (2.57), i.e.

$$\left[(kV_1 + lV_2)^2 - (kv_1 + lv_2)^2\right] = 0 \bmod 2 \quad \text{for } k, l = 0, 1\ . \tag{4.16}$$

4.3.1 The Untwisted Sector

The general discussion of the untwisted sector of an orbifold is given in sec. (2.7.1). Here we restrict ourselves only to the $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge sector. The transformation properties of right and left movers with respect to a general element

$$h = (kv_1 + lv_2, n_\alpha e_\alpha; kV_1 + lV_2, 0) \tag{4.17}$$

of the orbifold group are given by

$$|q\rangle_R \quad \overset{h}{\mapsto} \quad e^{2\pi i q\cdot(kv_1+lv_2)}|q\rangle_R \tag{4.18}$$

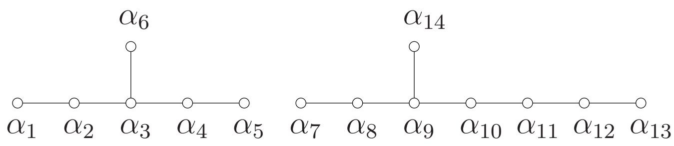
$$|p\rangle_L \quad \overset{h}{\mapsto} \quad e^{2\pi i p\cdot(kV_1+lV_2)}|p\rangle_L\ . \tag{4.19}$$

The $\mathrm{E}_8 \times \mathrm{E}_8'$ gauge bosons (and their gaugino partners) of the heterotic string are projected onto $S \otimes G$ invariant states. Thus, they become gauge bosons of the unbroken gauge group and charged matter. The transformation properties of all right and left movers of the untwisted sector are listed in tab. (C.3) in the appendix. It turns out that the 240 roots of the second E_8' are invariant. The other invariant roots of the unbroken gauge group are reproduced from tab. (C.3) and listed here in tab.(4.1).

$p = (p_1)(0^8)$	$p \cdot V_1$	$p \cdot V_2$	$p \cdot (V_1 + V_2)$	#
$\left(0^3, \underline{\pm 1, \pm 1, 0, 0, 0}\right)$	0	0	0	40
$\left(\pm\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\pm\frac{1}{2}\right)^5\right)$	0	0	0	32

Table 4.1: The 72 invariant roots of the unbroken E_6 gauge group.

Next, all simple roots are computed. They are listed in tab. (C.5), and the basis that fixes the semi-ordering is given in tab.(C.4). It turns out that there are only 14 simple roots corresponding to a algebra of rank 14. However, the rank 16 of the unbroken gauge group is not reduced, so we get additional U(1) factors. The Dynkin diagram of the unbroken gauge group reads



$$\mathrm{E}_6 \qquad \times \qquad \mathrm{E}_8' \ . \tag{4.20}$$

The complete gauge group is given by

$$\mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8' \ . \tag{4.21}$$

Furthermore, we have to tensor right and left movers of tab. (C.3) together to form $S \otimes G$ invariant charged matter states. Then the representations are determined by rewriting the weights in Dynkin labels and by finding the highest weights. Since the untwisted sector has to many matter states, we restrict ourselves to present the results here. However, in the first twisted sector we will follow these steps in greater detail.

The charged matter states in the untwisted sector are

$$\begin{array}{ll} 3(\overline{\mathbf{27}},\mathbf{1})_{\mathrm{L}} & 3(\mathbf{27},\mathbf{1})_{\mathrm{R}} \\ 3(\mathbf{27},\mathbf{1})_{\mathrm{L}} & 3(\overline{\mathbf{27}},\mathbf{1})_{\mathrm{R}} \\ 6(\mathbf{1},\mathbf{1})_{\mathrm{L}} & 6(\mathbf{1},\mathbf{1})_{\mathrm{R}} \ . \end{array}$$

Notice that no root of the former E_8 gauge group is projected out,

$$\underbrace{72}_{\text{gauge bosons}} + \underbrace{3 \times 27 + 3 \times 27 + 6 \times 1}_{\text{charged matter}} = 240 \ . \tag{4.22}$$

This will change in the presence of non-trivial Wilson lines (i.e. $A_\alpha \neq 0$).

4.3.2 The Twisted Sectors

The general discussion of the twisted sectors can be found in sec.(2.7.2). The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold has three twisted sectors. Each sector has 16 fixed hypersurfaces and is labeled by $T_{(k,l)}$ with ($k = 1$ and $l = 0$), ($k = 0$ and $l = 1$), ($k = 1$ and $l = 1$), respectively.

These 16 fixed hypersurfaces correspond to 16 different space group elements. Since there are no Wilson lines, the embeddings of the space group elements of the same twisted sector into the gauge degrees of freedom are identical,

$$S \hookrightarrow G \qquad (kv_1 + lv_2, n_\alpha e_\alpha) \mapsto (kV_1 + lV_2, 0) \ . \tag{4.23}$$

Thus, for each twisted sector $T_{(k,l)}$ we only have to solve the mass-equations for the twisted right and left movers once,

$$\frac{m_R^2}{4} = \frac{1}{2}\left(q + (kv_1 + lv_2)\right)^2 - \frac{1}{4} = 0 \tag{4.24}$$

$$\frac{m_L^2}{4} = \frac{1}{2}\left(p - (kV_1 + lV_2)\right)^2 + \tilde{N} - \frac{3}{4} = 0 \tag{4.25}$$

where the shift in the zero point energy $\delta c = \frac{1}{4}$ is the same for all three twisted sectors.

The $T_{(1,0)}$ twisted sector

Here, we restrict ourselves to the main results. However, calculational details are given in the appendix C.2.

Since there are no Wilson lines, the centralizer for all three twisted sectors only imposes projection conditions with respect to ϑ_1 and ϑ_2. Then, invariance with respect to ϑ_1 and ϑ_2 automatically implies invariance with respect to $\vartheta_1 \circ \vartheta_2$.

The twisted right movers are given in tab. (C.8). For the twisted left movers it turns out that there are 56 weights that solve the mass-equation for $\tilde{N} = 0$. Sorted with respect to their transformation properties, they split into $28 + 28$ weights, as listed in the tables (C.6) and (C.7). Tensored together with the right movers, they form the following representations of the gauge group:

$$\begin{array}{ll} 1\,(\mathbf{27},\mathbf{1})_{\mathrm{L}} & 1\,\left(\overline{\mathbf{27}},\mathbf{1}\right)_{\mathrm{R}} \\ 1\,(\mathbf{1},\mathbf{1})_{\mathrm{L}} & 1\,(\mathbf{1},\mathbf{1})_{\mathrm{R}}\ . \end{array}$$

Furthermore, the mass-equation for $\tilde{N} = \frac{1}{2}$ yields two solutions for the left movers, they are listed in tab. (C.9). The eight invariant states with $\tilde{N} = \frac{1}{2}$ are given in tab. (C.11), where we can check their invariance using the transformation properties of the moded oscillators listed in tab. (C.10). It turns out that these states are singlets:

$$4\,(\mathbf{1},\mathbf{1})_{\mathrm{L}} \qquad 4\,(\mathbf{1},\mathbf{1})_{\mathrm{R}}\ .$$

To summarize, taking the multiplicity of 16 fixed hypersurface into account, the $T_{(1,0)}$ contains the following $S \otimes G$ invariant states:

$$\begin{array}{ll} 16\,(\mathbf{27},\mathbf{1})_{\mathrm{L}} & 16\,\left(\overline{\mathbf{27}},\mathbf{1}\right)_{\mathrm{R}} \\ 80\,(\mathbf{1},\mathbf{1})_{\mathrm{L}} & 80\,(\mathbf{1},\mathbf{1})_{\mathrm{R}}\ . \end{array}$$

The $T_{(0,1)}$ and $T_{(1,1)}$ twisted sector can be computed analogously, leading to the same spectrum.

4.3.3 CPT Partners

Each twisted sector corresponds to an orbifold $T^4/\mathbb{Z}^2$, which gives $\mathcal{N}=2$ SUSY in four dimensions, and the states transform in hypermultiplets. The second twist breaks to $\mathcal{N}=1$ SUSY and projects out half of the chiralities, giving half-hypermultiplets. One left-chiral half-hypermultiplet and its CPT partner are combined to give one chiral supermultiplet of $\mathcal{N}=1$, $d=4$ SUSY.

4.3.4 The Spectrum

To summarize, the standard embedding of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold without Wilson lines has $\mathcal{N}=1$ SUSY in four dimensions with the gauge group

$$\mathrm{E}_6 \times \mathrm{U}(1)^2 \times \mathrm{E}_8' \tag{4.26}$$

and the following matter representations in chiral supermultiplets:

sector	matter content
in the untwisted sector:	3 $(\overline{\mathbf{27}}, \mathbf{1})$
	3 $(\mathbf{27}, \mathbf{1})$
	6 $(\mathbf{1}, \mathbf{1})$
in the $T_{(1,0)}$ twisted sector:	16 $(\mathbf{27}, \mathbf{1})$
	80 $(\mathbf{1}, \mathbf{1})$
in the $T_{(0,1)}$ twisted sector:	16 $(\mathbf{27}, \mathbf{1})$
	80 $(\mathbf{1}, \mathbf{1})$
in the $T_{(1,1)}$ twisted sector:	16 $(\mathbf{27}, \mathbf{1})$
	80 $(\mathbf{1}, \mathbf{1})$

As described in sec. (4.3.3), the CPT partners have been combined to give chiral supermultiplets.

4.3.5 Heterotic Brane Picture

The states of the untwisted sector are free to move in all ten dimensions. Therefore, they are called **bulk states**. Since the states of the twisted sectors are localized on fixed hyperplanes, they can only propagate in a six-dimensional space. Consequently, the states of the twisted states can be seen as living on **5-branes**. For every twisted sector the 16 fixed hyperplanes lie parallel in ten-dimensional space, thus the 16 5-branes of one sector are parallel. Furthermore, 5-branes of different sectors are orthogonal. The brane picture is depicted in fig. (4.2).

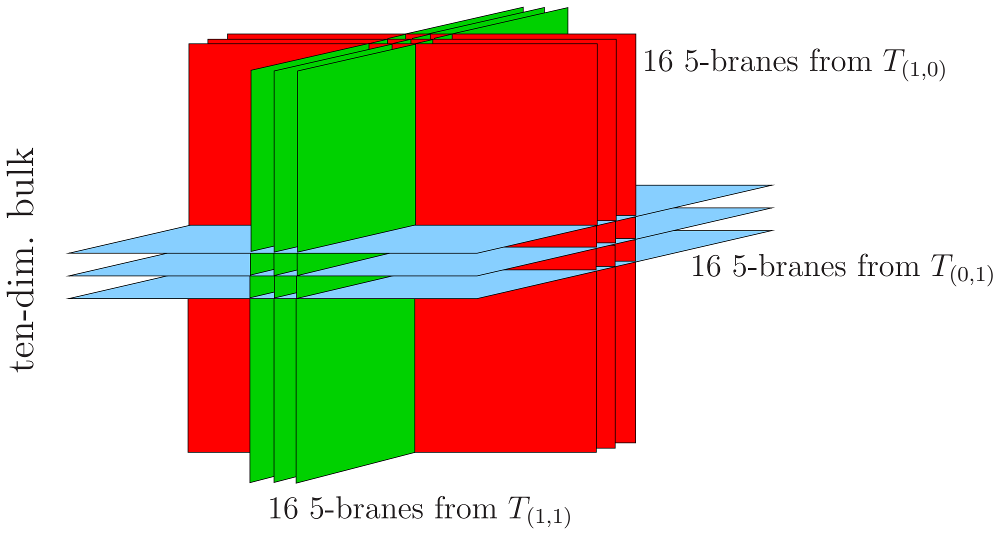


Figure 4.2: Brane picture for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold: for each twisted sector $T_{(k,l)}$ there are 16 parallel 5-branes in the ten-dimensional bulk. The branes of the three twisted sectors are mutually orthogonal. The distances of parallel branes in the six extra dimensions are approximately of order M_{Planck}^{-1} and correspond to the compactification radii.

4.4 Model with one Wilson Line

In this section, we consider an example with one Wilson line in order to see its effect on the spectrum. Of course, this model can be solved by straightforward calculations. However, the qualitative aspects of Wilson lines are pointed out here.

We take the space group of eqn. (4.7) and choose standard embedding with one Wilson line

$$A_1 = \left(0^7, 1\right)\left(1, 0^7\right) \ . \tag{4.27}$$

The Wilson line is of order 2,

$$2A_1 \in T_{\mathrm{E}_8 \times \mathrm{E}_8'} \ . \tag{4.28}$$

Now, we have to check whether the (strong) modular invariance conditions are fulfilled. In the case of non-trivial Wilson lines, it is easier to consider the following, equivalent version of the (strong) modular invariance condition (eqn. (2.58)),

$$\begin{aligned} V^2 - v^2 &= 0 \bmod 2 && (4.29)\\ V \cdot A_\alpha &= 0 \bmod 1 && (4.30)\\ A_\alpha \cdot A_\beta &= 0 \bmod 1, \quad \alpha \neq \beta && (4.31)\\ A_\alpha^2 &= 0 \bmod 2 \ . && (4.32) \end{aligned}$$

It can be easily seen that the shifts and the Wilson lines satisfy these conditions.

4.4.1 The Untwisted Sector

The massless states of the untwisted sector are identical to the massless states of the heterotic string, projected onto $S \otimes G$ invariant states. This projection guarantees that the surviving states are consistent with the geometry of the orbifold.

The transformation of right and left movers under a general element

$$h = (kv_1 + lv_2, n_\alpha e_\alpha; kV_1 + lV_2, n_1 A_1) \tag{4.33}$$

of the orbifold group is given by (sec. (2.7.1))

$$|q\rangle_R \overset{h}{\mapsto} e^{2\pi i q\cdot(kv_1+lv_2)}|q\rangle_R \tag{4.34}$$

$$|p\rangle_L \overset{h}{\mapsto} e^{2\pi i p\cdot(kV_1+lv_2+n_1A_1)}|p\rangle_L \tag{4.35}$$

for any combination of $k, l = 0, 1$ and $n_1 = 0, 1$. These transformation properties look almost the same as in the case of no Wilson lines in section (4.3.1). Therefore, we can compute the untwisted sector by imposing the condition

$$p \cdot A_1 = 0 \text{ mod } 1 \tag{4.36}$$

on the states of the untwisted sector of sec. (4.3.1). First, we consider the gauge bosons, where we focus on the first part, the E_6. The eighth entry of

$$A_1 = \left(0^7, 1\right)\left(1, 0^7\right) \tag{4.37}$$

projects out all spinorial roots of the E_6, leaving 40 roots invariant. The result is presented in tab. (4.2).

$p = (p_1)(0^8)$	$p \cdot A_1$	#invariant	remark
$\left(0^3, \underline{\pm1, \pm1, 0, 0, 0}\right)$	0	40	adjoint of SO(10)
$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \left(\pm\frac{1}{2}\right)^5\right)$	$\pm\frac{1}{2}$	0	$\overline{\mathbf{16}}$ of SO(10)
$\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \left(\pm\frac{1}{2}\right)^5\right)$	$\pm\frac{1}{2}$	0	**16** of SO(10)

Table 4.2: From the 72 charged roots of the E_6 gauge bosons only 40 are invariant. Thus the unbroken gauge group is SO(10) (which has 40 charged roots).

Calculating the simple roots, it turns out that the gauge group breaks as follows

$$\underbrace{E_6 \times U(1)^2 \times E_8'}_{\text{without Wilson lines}} \implies \underbrace{S0(10) \times U(1)^3 \times SO(16)'}_{\text{one Wilson line } A_1=(0^7,1)(1,0^7)} . \tag{4.38}$$

This can also be seen from the following considerations. The adjoint representation of the E_6 contains the adjoint of SO(10) plus the spinor and the antispinor of SO(10) and an additional U(1),

$$\underbrace{\mathbf{78}}_{\text{adjoint}} \text{ of } E_6 = \underbrace{\mathbf{45}}_{\text{adjoint}} \oplus \underbrace{\mathbf{16} \oplus \overline{\mathbf{16}}}_{\text{projected out}} \oplus \underbrace{\mathbf{1}}_{\text{U(1)}} \text{ of SO(10)} . \tag{4.39}$$

and we see that the spinor is projected out and the remaining symmetry is SO(10). Furthermore, the projection $p \cdot A_1 = 0 \bmod 1$ acts on the charged matter states of the untwisted sector of the model without Wilson lines, listed in the table in sec. (4.3.4). This yields

$$\mathbf{27} \text{ of } E_6 = \underbrace{\mathbf{16}}_{\text{projected out}} \oplus \mathbf{10} \oplus \mathbf{1} \text{ of SO(10)} . \tag{4.40}$$

So, from the three **27** of E_6 three **10** and three singlets of SO(10) survive the projection with A_1. The $\overline{\mathbf{27}}$ decomposes analogously. Additionally the six singlets of E_6 are invariant. Therefore, the invariant charged matter states of the untwisted sector are:

$$6 \quad (\mathbf{10}, \mathbf{1}) \tag{4.41}$$
$$12 \quad (\mathbf{1}, \mathbf{1}) . \tag{4.42}$$

4.4.2 The Twisted Sectors

First, we consider the $\boldsymbol{T_{(1,0)}}$ **twisted sector**. Since we have one non-trivial Wilson line A_1, not all elements of the space group S will be embedded as the same element of the gauge twisting group G (compare to the case of no Wilson line, discussed in sec. (4.3.2)). But we can divide the space group into two classes. The elements of the first class contain the torus shift e_1, the ones of the second class do not. Now, we consider their embeddings into the gauge degrees of freedom,

$$(v_1, n_\alpha e_\alpha) \in S \text{ with } n_1 = 0 \quad \mapsto \quad (V_1, 0) \in G \tag{4.43}$$
$$(v_1, n_\alpha e_\alpha) \in S \text{ with } n_1 = 1 \quad \mapsto \quad (V_1, A_1) \in G . \tag{4.44}$$

As usual, we restrict ourselves to space group elements that correspond to fixed points in the six-torus T^6. It turns out that there are eight elements of the first class and eight of the second class. They are illustrated in fig. (4.3).

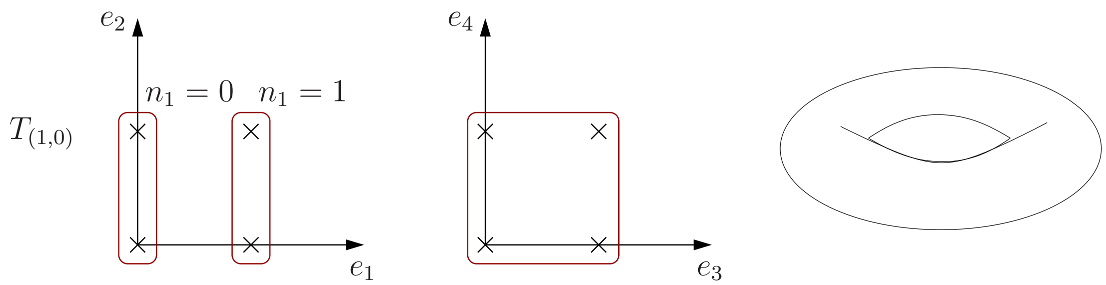


Figure 4.3: The Wilson line in e_1 direction breaks the multiplicity of fixed points that are equivalent with respect to the resulting spectrum as indicated by the dark red boxes.

An element of the gauge twisting group corresponds to a mass-equation for the twisted left movers. We consider the mass-equations for two elements, one of the first and one of the second class, respectively. They read

$$\frac{m_L^2}{4} = \frac{1}{2}(p - V_1)^2 + \tilde{N} - \frac{3}{4} \tag{4.45}$$

$$\frac{m_L^2}{4} = \frac{1}{2}(p - (V_1 + A_1))^2 + \tilde{N} - \frac{3}{4} \; . \tag{4.46}$$

In general, the resulting states from these mass-equations are different. Here, the second mass-equation is more restrictive than the first one. Thus there are less solutions than in the case of no Wilson lines. The resulting massless spectrum from the eight elements of the space group with $n_1 = 1$ is

$$8 \quad (\mathbf{1}, \mathbf{16}) \tag{4.47}$$

compared to the spectrum from the other eight elements with $n_1 = 0$

$$8 \quad (\overline{\mathbf{16}}, \mathbf{1}) \tag{4.48}$$

$$8 \quad (\mathbf{10}, \mathbf{1}) \tag{4.49}$$

$$48 \quad (\mathbf{1}, \mathbf{1}) \; , \tag{4.50}$$

In the $\boldsymbol{T_{(0,1)}}$ **twisted sector** the centralizer introduces a qualitatively new projection condition. The fixed points of this sector have $n_1 = 0$. Therefore, the mass-equations and their solutions for the twisted right and left movers are identical to the case of no Wilson line. In sec. (4.3.2) we found out that the resulting massless spectrum contains a **27** of E_6. Now, the gauge group is broken and the 27 roots form the following representations under the unbroken gauge group

$$\mathbf{27} \text{ of } E_6 = \mathbf{16} \oplus \mathbf{10} \oplus \mathbf{1} \text{ of } SO(10) \; . \tag{4.51}$$

Next, we have to tensor right and left movers together to create $S \otimes G$ invariant states (compare to sec. (2.8)). But the centralizer Z_g of any constructing element g of the $T_{(0,1)}$ sector contains the element

$$(\mathbb{1}, e_1; 0, A_1) \in Z_g \; . \tag{4.52}$$

Therefore, we have to impose the projection condition

$$p \cdot A_1 = 0 \bmod 1 \; , \tag{4.53}$$

which again projects out the spinor of SO(10), i.e. the **16**. Thus the matter content of the $T_{(0,1)}$ twisted sector is

$$16 \quad (\mathbf{10}, \mathbf{1}) \tag{4.54}$$

$$96 \quad (\mathbf{1}, \mathbf{1}) \; . \tag{4.55}$$

Finally, the $\boldsymbol{T_{(1,1)}}$ **twisted sector** is analogous to the first sector $T_{(1,0)}$ and has the same matter content. The complete spectrum is summarized in the next section.

4.4.3 The Spectrum

To summarize, the standard embedding of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with one Wilson line

$$A_1 = \left(0^7, 1\right)\left(1, 0^7\right) \tag{4.56}$$

has $\mathcal{N} = 1$ SUSY in four dimensions. The gauge group is

$$\mathrm{S0}(10) \times \mathrm{U}(1)^3 \times \mathrm{SO}(16)' \,, \tag{4.57}$$

and the matter representations of the massless spectrum are listed in tab. (4.3).

The Wilson line A_1 corresponds to the torus-shift e_1. Therefore, the mass-equations of states that have a constructing element containing e_1 change. This is the case in the $T_{(1,0)}$ and in the $T_{(1,1)}$ twisted sectors. There the multiplicity of 16 breaks into $8+8$, as indicated by the dark red boxes in fig. (4.3).

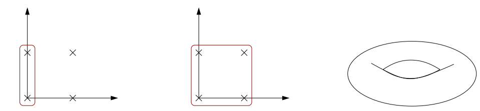
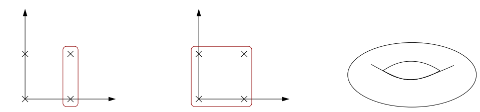
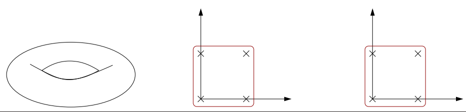
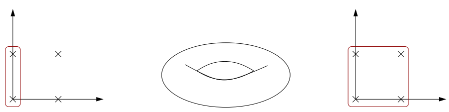
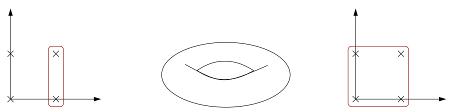
sector	**matter content**	**localization**
in the untwisted sector:	6 $(\mathbf{10}, \mathbf{1})$	*bulk*
	12 $(\mathbf{1}, \mathbf{1})$	
in the $T_{(1,0)}$ twisted sector:	8 $(\overline{\mathbf{16}}, \mathbf{1})$	
	8 $(\mathbf{10}, \mathbf{1})$	
	48 $(\mathbf{1}, \mathbf{1})$	
	8 $(\mathbf{1}, \mathbf{16})$	
in the $T_{(0,1)}$ twisted sector:	16 $(\mathbf{10}, \mathbf{1})$	
	96 $(\mathbf{1}, \mathbf{1})$	
in the $T_{(1,1)}$ twisted sector:	8 $(\overline{\mathbf{16}}, \mathbf{1})$	
	8 $(\mathbf{10}, \mathbf{1})$	
	48 $(\mathbf{1}, \mathbf{1})$	
	8 $(\mathbf{1}, \mathbf{16})$	

Table 4.3: Massless spectrum of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with one Wilson line in the direction of e_1, omitting the gauge bosons, the gauginos and the supergravity multiplet. The dark red boxes indicate the multiplicity of the states.

Chapter 5

$\mathbb{Z}_2 \times \mathbb{Z}_2$ Toy Model with Three Generations

In this section it is shown that three generation models can be obtained in the framework of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds with Wilson lines. One SO(10) toy model [18] is presented in detail. Next, some qualitative features of the toy model are investigated, like the localization of the three generations and of the Higgs bosons, and the connection to 6D field theory orbifolds, using a zoom-in picture.
Further models with three generations and Standard Model or GUT gauge groups are listed in app. (E).

5.1 Definition of the Model

We choose the space group of eqn. (4.7) with standard embedding. Furthermore, we choose the following six Wilson lines

$$\begin{aligned}
A_1 &= \left(0^7, 1\right) \left(1, 0^7\right) & (5.1)\\
A_2 &= \left(0^8\right) \left(0, 1, \frac{1}{2}^4, 0^2\right) & (5.2)\\
A_3 &= \left(0^8\right) \left(0^2, \frac{1}{2}, -\frac{1}{2}^2, \frac{1}{2}, -1, 0\right) & (5.3)\\
A_4 &= \left(0^8\right) \left(1, 0^3, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^2\right) & (5.4)\\
A_5 &= \left(0^8\right) \left(0, \frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, 0\right) & (5.5)\\
A_6 &= \left(0^8\right) \left(1, 0^3, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^2\right) \; . & (5.6)
\end{aligned}$$

From the last section we know that A_1 breaks the gauge group to SO(10) times some hidden sector gauge group. Since the other Wilson lines A_α for $\alpha = 2, \ldots, 6$ only affect the hidden sector gauge group, we see that this model has a SO(10) gauge group. But before we can analyze the model in detail, we have to check that the Wilson lines fulfill the conditions of (strong) modular invariance

$$V \cdot A_\alpha = 0 \bmod 1 \tag{5.7}$$
$$A_\alpha \cdot A_\beta = 0 \bmod 1, \quad \alpha \neq \beta \tag{5.8}$$
$$A_\alpha^2 = 0 \bmod 2\ . \tag{5.9}$$

These conditions have been verified using the computer program described in the following.

5.2 Computer-Aided Orbifolding

Having defined a consistent set of Wilson lines and shifts, we can now compute the spectrum of massless states. But due to the existence of these six Wilson lines the degeneracy of the fixed points is completely lifted, i.e. for each constructing element separately we have to solve the mass-equations and project onto the centralizer. Since there are 48 different constructing elements, we see that this calculation is rather long and tedious. Therefore, the rules for orbifold construction have been implemented into a computer program[1]. As input parameter for this program one has to specify

- the point group $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$
- the twists v or v_1, v_2
- the torus-lattice T^6
- all fixed points $(\vartheta^k, n_\alpha e_\alpha)$ or $(\vartheta^k \circ \vartheta^l, n_\alpha e_\alpha)$
- the shifts V or V_1, V_2
- and the Wilson lines A_α.

The program automatically checks the consistency of these input parameters (e.g. the strong modular invariance conditions) and after a short operating time (of about 5 seconds) the program lists the complete massless spectrum. The program has been successfully tested by verifying many orbifold models from literature, e.g. $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$-II.

Using this computer program the following spectrum of the SO(10) toy model has been computed.

[1] The computer programs have been developed in collaboration with Akın Wingerter.

sector	**matter content**
in the untwisted sector:	6 $(\mathbf{10},\mathbf{1})$
	12 $(\mathbf{1},\mathbf{1})$
in the $T_{(1,0)}$ twisted sector:	2 $(\mathbf{10},\mathbf{1})$
	2 $(\overline{\mathbf{16}},\mathbf{1})$
	60 $(\mathbf{1},\mathbf{1})$
in the $T_{(0,1)}$ twisted sector:	2 $(\mathbf{10},\mathbf{1})$
	0 $(\overline{\mathbf{16}},\mathbf{1})$
	60 $(\mathbf{1},\mathbf{1})$
in the $T_{(1,1)}$ twisted sector:	1 $(\mathbf{10},\mathbf{1})$
	1 $(\overline{\mathbf{16}},\mathbf{1})$
	54 $(\mathbf{1},\mathbf{1})$

Table 5.1: Massless spectrum of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with $2+0+1=3$ generations of SO(10).

5.3 The Spectrum

The shifts of the standard embedding and the six Wilson lines break the gauge group to

$$\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{U}(1)'^8 \ . \tag{5.10}$$

The spectrum of massless states is given in tab. (5.1). The $\overline{\mathbf{16}}$ of SO(10) contains all particles of one Standard model generation plus the right-handed neutrino. We illustrate this by the following decomposition of the **16**:

$$\begin{array}{lcccc}
\mathrm{SO}(10) & & \mathbf{16} & & \\
 & & \downarrow & & \\
\mathrm{SU}(5) & & \mathbf{10} \oplus \mathbf{1} \oplus \overline{\mathbf{5}} & & \\
 & \swarrow & \searrow & & \searrow \\
\mathrm{SM} & (\mathbf{3},\mathbf{2}) \oplus (\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) & & (\mathbf{1},\mathbf{1}) & (\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \\
 & \begin{pmatrix} u \\ d \end{pmatrix}_L \quad \overline{u}_L \quad e^+_L & & \overline{\nu}_L & \overline{d}_L \quad \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L
\end{array}$$

where the intermediate SU(5) breaking is not necessary. Thus, this model has

$$2+0+1=3 \tag{5.11}$$

generations of an SO(10) GUT theory [2]. The notation in eqn. (5.11) indicates the origin of the three generations: Two from the $T_{(1,0)}$ twisted sector, none from $T_{(0,1)}$, and finally

[2]This is the first three generation model in the framework of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds[18]. This seems to be in apparent contradiction to the results obtained in ref. [23]. This discrepancy can be explained by the fact that in ref. [23] Wilson lines had not been included.

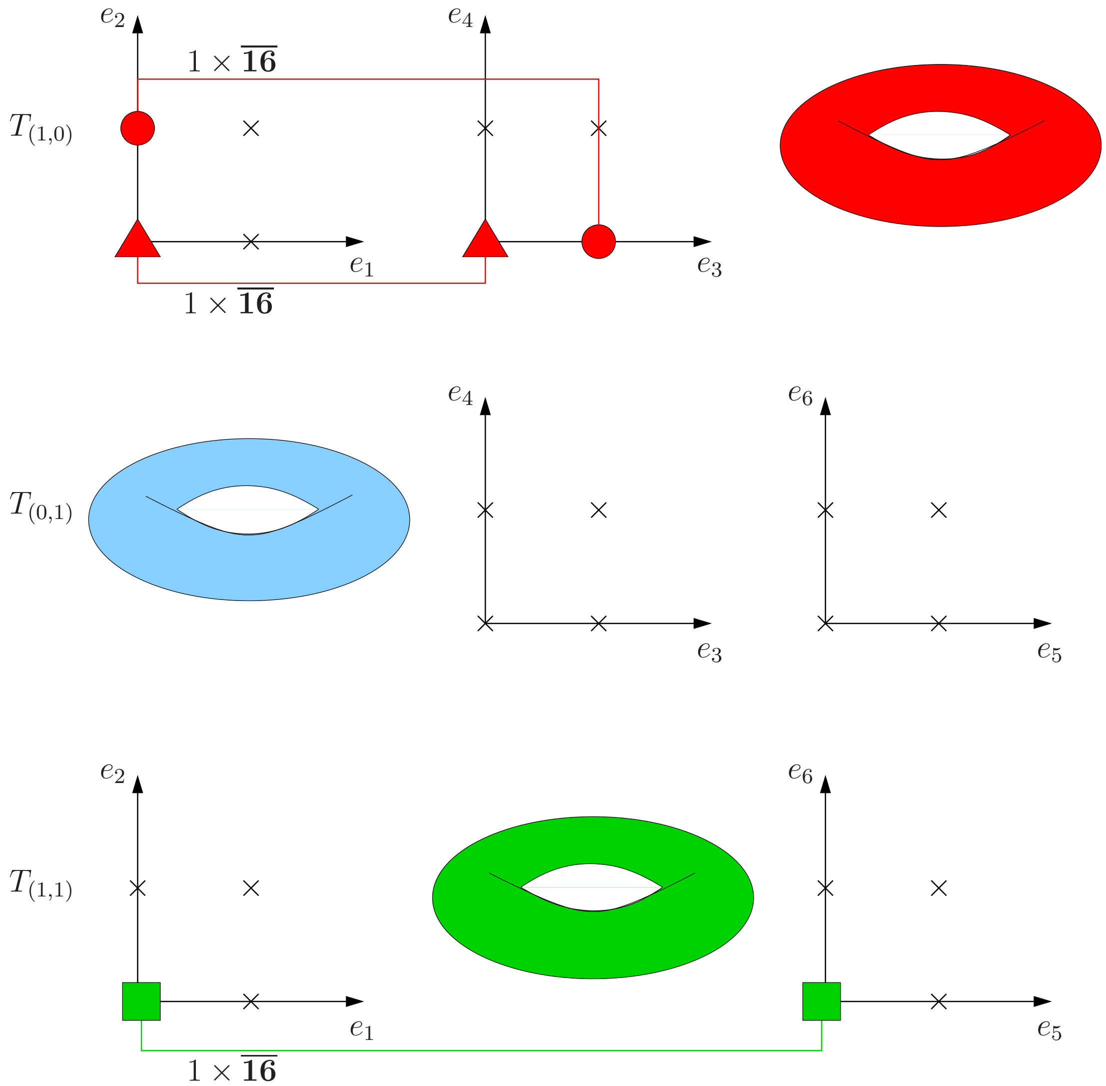


Figure 5.1: Localization of the three generations with SO(10) gauge group.

one generation from $T_{(1,1)}$, as illustrated in fig. (5.1).

The Standard Model Higgs boson is contained in the $\mathbf{10}$ of SO(10). The decomposition of the $\mathbf{10}$ to the $(\mathbf{1}, \mathbf{2})$ representation of the Higgs doublet is given by:

$$\begin{array}{lc}
\mathrm{SO}(10) & \mathbf{10} \\
 & \downarrow \\
\mathrm{SU}(5) & \mathbf{5} \oplus \overline{\mathbf{5}} \\
 & \swarrow \qquad \searrow \\
\mathrm{SM} & (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \qquad (\overline{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})
\end{array}$$

Therefore, this model has eleven candidates for a Standard Model Higgs boson. The additional color-triplets in the decomposition do not correspond to any observed particles.

Therefore, these states must either decouple from the observable particle spectrum by getting a large mass, or they have to be projected out (e.g. by a Wilson line). This problem is known as the doublet-triplet splitting problem.

Next, we take a closer look at the localization of the three generations (the $\overline{\mathbf{16}}$s) and of the Higgs bosons (the $\mathbf{10}$s), as listed in tab. (5.2).

sector	# of $\mathbf{10}$s	# of $\overline{\mathbf{16}}$s
untwisted	6	0
$(\theta_1, 0)$	1	1
$(\theta_1, e_2 + e_3)$	1	1
$(\theta_2, 0)$	1	0
$(\theta_2, e_4 + e_6)$	1	0
$(\theta_3, 0)$	1	1
sum	11	3

Table 5.2: Localization of the three generations and of the Higgs bosons.

If a Standard Model particle (contained in the $\overline{\mathbf{16}}$) is "closer" to the Higgs boson then their interaction is stronger and thus the particle is heavier. Therefore, the localization of the three generations with respect to the Higgs boson is important for the Yukawa couplings. For a qualitative investigation, we use the following zoom-in picture.

5.4 Zoom-In Picture

In fig. (5.1) the twisted sectors are drawn separately. Next, we merge all three twisted sectors and draw them in one figure (illustrated in fig. (5.2)). For a detailed discussion of

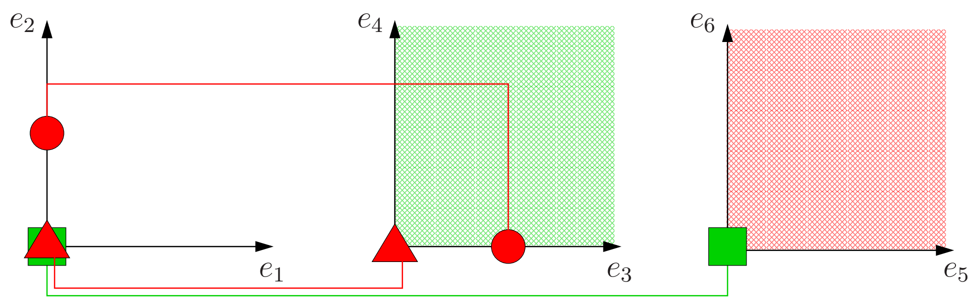


Figure 5.2: All three twisted sector of fig. (5.1) with the three generations of SO(10) drawn in one picture.

the localization we now zoom-in into each torus separately. Doing so in the **first torus**, we see that all three generations are localized at fixed points. One generation is located at

$e_2/2$ and the other two at the origin. Furthermore, each generation has a **10** living at the same fixed point (compare tab. (5.2)). Additionally, there are eight candidates for a Higgs boson (i.e. **10**s) living in the bulk of the first torus. The situation is depicted in fig. (5.3).

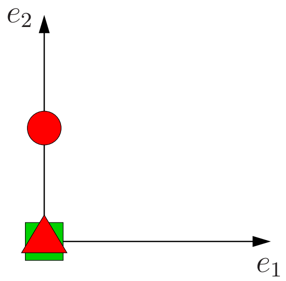


Figure 5.3: First torus.

Next, we zoom into the **second torus**. From this point of view, two generations are fixed (one at the origin and the other at $e_1/2$) and the third is free to move in the bulk, compare fig. (5.4). The Higgs bosons are localized as follows: two at the origin, one at the point $e_3/2$, one at $e_4/2$ and seven in the bulk.

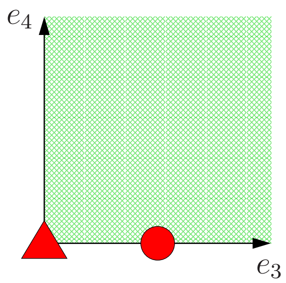


Figure 5.4: Second torus.

Finally, we zoom into the **third torus**. Here, one generation is localized at the origin and two live in the bulk (fig. (5.5)). Furthermore, there are two **10**s at the origin, one at $e_6/2$ and eight candidates for a Higgs boson live in the bulk.

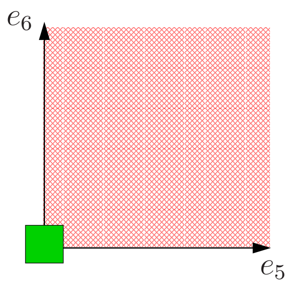


Figure 5.5: Third torus.

5.5 6d Intermediate Picture

In this section we extend the zoom-in picture from the last section to the gauge group. As an example we use the SO(10) toy model. Therefore, we first have to specify how the zooming works. We will see that the gauge group will be larger in the 6D picture. Furthermore, it is shown that the 6D gauge group breaks at the fixed point.

In order to get the six-dimensional intermediate picture, we start with a $T^6/\mathbb{Z}_2$ orbifold. We choose the $\mathbb{Z}_2$ twist vector to be

$$v_2 = \left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) \ . \tag{5.12}$$

This twist rotates in the second and third torus, and leaves the first one untouched, as illustrated in fig. (5.6).

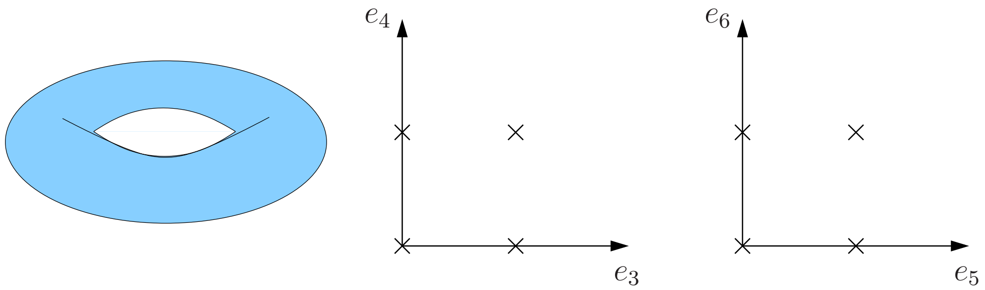


Figure 5.6: The twist vector v_2 rotates only in the second and third torus.

Furthermore, we choose the standard embedding of this twist, i.e.

$$V_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0^5\right) \left(0^8\right) \tag{5.13}$$

and we switch on the four Wilson lines that correspond to the second and third torus (i.e. A_3 - A_6) from the SO(10) toy model (listed in sec. (5.1)).

This yields a six-dimensional model with the gauge group

$$\mathrm{E}_7 \times \mathrm{SU}(2) \tag{5.14}$$

times a hidden sector gauge group, which we omit in this discussion.

Next, we shrink the size of the second and third torus, thus the first torus is much larger than the other ones. The gauge bosons and the matter content from the 16 fixed points are free to move in the first torus, they are bulk fields there.

Additionally, we put the three generations of the SO(10) toy model to the points, where they were before (compare to fig. (5.3)). The situation is depicted in fig. (5.7). This setup

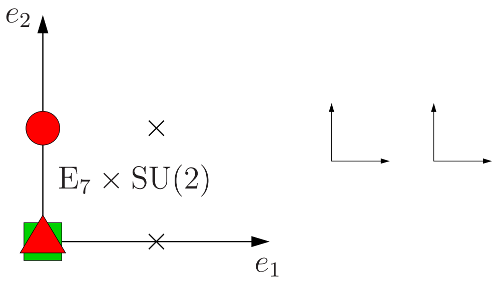


Figure 5.7: The starting point for the 6D intermediate picture: Two extra dimensions are much bigger than the other four. The gauge bosons are living in the bulk of the first torus and the three generations of the SO(10) toy model are put into the first torus by hand.

can be compared with the starting point for a 6D field theory orbifold GUT [24], derived from a heterotic orbifold.

If we now want to go to four dimensions, we have to perform the second twist, i.e. in this case

$$v_1 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0\right) \quad . \tag{5.15}$$

Consequently, also the shift V_1 and the Wilson lines A_1 and A_2 act. In the field theory language their action corresponds to parities, which must be assigned by hand. Here they can be derived.

From the last section, we know that due to the second shift and the Wilson lines the gauge group breaks to SO(10). However, here we want to analyze the local physics at the fixed points [25], i.e. the local action of the shift V_1 and the two Wilson lines. In the following table, it is listed which local shift acts at each fixed point on the gauge group:

fixed point	local shift
origin	V_1
$e_1/2$	$V_1 + A_1$
$e_2/2$	$V_1 + A_2$
$(e_1 + e_2)/2$	$V_1 + A_1 + A_2$

Unfortunately, the local shifts break at all four fixed points the gauge symmetry form $\mathrm{E}_7 \times \mathrm{SU}(2)$ to $\mathrm{E}_6 \times \mathrm{U}(1)^2$, as illustrated in fig. (5.8). However, an E_6 from the left-hand side is different to one of the right-hand side, i.e. they lie differently in the E_7 root lattice. The intersection of these E_6's gives the SO(10) gauge group.

As shown in fig. (5.8) the three generations (i.e. the three $\overline{\mathbf{16}}$ of SO(10)) live at fixed points where a larger gauge group is realized. Therefore, they have to be in complete

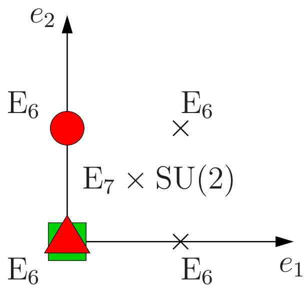


Figure 5.8: 6D intermediate picture with the gauge groups in the bulk and at the fixed points and the three generations.

representations of E$_6$, i.e.

$$\underbrace{\overline{\mathbf{16}} \oplus \mathbf{10} \oplus \mathbf{1}}_{\text{of 4D SO(10)}} = \underbrace{\overline{\mathbf{27}}}_{\text{of 6D E}_6} \tag{5.16}$$

In this manner, it is possible that a GUT is realized at high energies, when the extra dimensions are resolved. From a phenomenological point of view a Standard Model gauge group in four dimensions with a SO(10) realized in a higher dimension would be of interest.

5.6 Conclusions

One main result of my thesis is the construction of several models with three generations in the context of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The question whether or not it is possible to construct three generation models in this context could be clarified.

Special emphasis is given on the geometrical interpretation of the results. Keeping the geometry in mind, improves our understanding of many properties of the four-dimensional spectrum and is helpful for constructing realistic models. A very appealing feature is that the grand unified group may be present in the extra dimensions, whose fingerprints may be observable in our space-time.

Wilson lines have proven to be an efficient tool for controlling the number of generations and, at the same time, allowing for a further breakdown of the gauge symmetry. The qualitative features of general $\mathbb{Z}_N \times \mathbb{Z}_M$ models have been studied using a specific example, suggesting that the exploration of this class of models may be promising.

Appendix A

Some Aspects of Group Theory

A.1 The Exceptional Group E_8

Some important facts about the E_8 are:

- $\dim(\mathrm{adj}(\mathrm{E}_8)) = 248$
- $\mathrm{rank}(\mathrm{E}_8) = 8$

Therefore, the E_8 is generated by eight Cartan generators H_i and 240 raising and lowering operators $E_{\pm\alpha}$, where the α's denote the roots. The roots are given by

$$\left(\underline{\pm 1, \pm 1, 0^6}\right) \qquad \text{(and all permutations)} \tag{A.1}$$

$$\left(\left(\pm\tfrac{1}{2}\right)^8\right) \qquad \text{(even \# of plus signs)}. \tag{A.2}$$

Furthermore, the E_8 root lattice

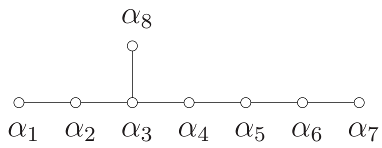
$$T_{\mathrm{E}_8} = \left\{ \Big(n_1, \ldots, n_8\Big), \Big(n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}\Big) \;\Big|\; n_i \in \mathbb{Z}, \sum n_i \text{ even} \right\} \tag{A.3}$$

is even and self-dual. Consequently, the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice is also even and self-dual.

After one has found the (basis dependent) simple roots, denoted by α_i for $i = 1, \ldots, 8$, one can compute the (basis independent) Cartan matrix

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad \Rightarrow \quad A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{A.4}$$

Finally, the Dynkin diagram is given by:



A.2 The Vector of SO(6)

The vector $X \in \mathbb{R}^6$ transforms under a rotation $\vartheta \in \mathrm{SO}(6)$ as:

$$X \overset{\vartheta}{\mapsto} \vartheta X \ . \tag{A.5}$$

Therefore, the vector X forms a six-dimensional representation under the SO(6), denoted by **6**. Since SO(6) is locally isomorphic to SU(3), the question arises how the **6** transforms under the SU(3).

Therefore, we consider a SU(3) matrix U in its general form:

$$U = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{A.6}$$

with complex entries $a_{ij} = \mathrm{Re}\ a_{ij} + i\ \mathrm{Im}\ a_{ij}$. This matrix can be embedded into SO(6) by the following replacement:

$$i \implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.7}$$

with the property

$$i^2 = -1 \implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\mathbb{1} \ . \tag{A.8}$$

Therefore, the unitary 3×3 matrix becomes an orthogonal 6×6 matrix.

Now we are going to show that the vector of SO(6) decomposes into $\mathbf{3} \oplus \overline{\mathbf{3}}$ of SU(3). We restrict ourselves to the simple case where $\vartheta \in \mathrm{SO}(6)$ is of block diagonal form. Its action on a vector is given by:

$$X \overset{\vartheta}{\mapsto} \vartheta X = \begin{pmatrix} \cos\theta & -\sin\theta & & & & 0 \\ \sin\theta & \cos\theta & & & & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & & & & \cos\theta & -\sin\theta \\ 0 & & & & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^6 \end{pmatrix} \tag{A.9}$$

where $\theta = 2\pi\eta$. Hence, the rotation ϑ acts in three orthogonal planes separately. Using the embedding of SU(3) into SO(6) stated in eqn. (A.7) we can rewrite ϑ as a unitary matrix U. Now, X transforms under ϑ as a **3**:

$$X \overset{\vartheta}{\mapsto} UX = \begin{pmatrix} e^{i\theta} & & 0 \\ & e^{i\theta} & \\ 0 & & e^{i\theta} \end{pmatrix} \begin{pmatrix} X^1 + iX^2 \\ X^3 + iX^4 \\ X^5 + iX^6 \end{pmatrix} \tag{A.10}$$

and X^* transforms under ϑ as a $\overline{\mathbf{3}}$:

$$X^* \overset{\vartheta}{\mapsto} U^* X^* = \begin{pmatrix} e^{-i\theta} & & 0 \\ & e^{-i\theta} & \\ 0 & & e^{-i\theta} \end{pmatrix} \begin{pmatrix} X^1 - iX^2 \\ X^3 - iX^3 \\ X^5 - iX^6 \end{pmatrix} \tag{A.11}$$

Therefore, the $\mathbf{6}$ of SO(6) decomposes into $\mathbf{3} \oplus \overline{\mathbf{3}}$ of SU(3).

Appendix B

Explicit Calculations

B.1 The Mode Expansion for the Untwisted Sector

In this section we are looking for the general solution of the equation of motion,

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^i(\tau,\sigma) = 0 \qquad i = 1,\ldots,8 \ , \tag{B.1}$$

in the untwisted sector, i.e. we have to solve this differential equation. Additionally, the solution must satisfy the boundary condition

$$X^i(\tau,\sigma+2\pi) = X^i(\tau,\sigma) \ . \tag{B.2}$$

The general solution is given by a mode expansion

$$X^i(\tau,\sigma) = x^i + p^i\tau + \frac{i}{2}\sum_{n\neq 0}\left(\frac{\alpha_n^i}{n}e^{-in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^i}{n}e^{-in(\tau+\sigma)}\right) \tag{B.3}$$

which can be split into right and left moving parts,

$$\begin{aligned} X^i(\tau,\sigma) &= X_R^i(\tau-\sigma) + X_L^i(\tau+\sigma) && \text{(B.4)}\\ X_R^i(\tau-\sigma) &= \frac{1}{2}x^i + \frac{1}{2}p^i(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^i}{n}e^{-in(\tau-\sigma)} \qquad \text{(right mover)} && \text{(B.5)}\\ X_L^i(\tau+\sigma) &= \frac{1}{2}x^i + \frac{1}{2}p^i(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^i}{n}e^{-in(\tau+\sigma)} \qquad \text{(left mover)} && \text{(B.6)} \end{aligned}$$

The operator X^i must be hermitian, i.e.

$$\left(X^i(\tau,\sigma)\right)^\dagger = X^i(\tau,\sigma) \ . \tag{B.7}$$

Therefore, the conditions

$$(\alpha_n^i)^\dagger = \alpha_{-n}^i \qquad \text{and} \qquad (\tilde{\alpha}_n^i)^\dagger = \tilde{\alpha}_{-n}^i \tag{B.8}$$

follow, and the oscillators α_n^i and $\tilde{\alpha}_n^i$ are complex. In the following we focus on the right mover and we decompose the right moving oscillator

$$\alpha_n^i = a_n^i - i\ b_n^i \tag{B.9}$$

with $a_n^i, b_n^i \in \mathbb{R}$ for $i = 1, \ldots, 6$. From eqn. (B.8) we deduce

$$\alpha_{-n}^i = a_n^i + i\ b_n^i\ . \tag{B.10}$$

Using eqns. (B.9) and (B.10) we rewrite the mode expansion of eqn. (B.5)

$$X_R^i(\tau-\sigma) = \frac{1}{2}x^i + \frac{1}{2}p^i(\tau-\sigma) + \sum_{n>0}\frac{1}{n}\big[a_n^i \sin\left(n(\tau-\sigma)\right) + b_n^i \cos\left(n(\tau-\sigma)\right)\big]\ . \tag{B.11}$$

We will use the real form of the mode expansion when we derive the twisted mode expansion in sec. (B.2).

B.2 The Mode Expansion for the Twisted Sector

In analogy to the untwisted case of eqn. (B.11) we take the ansatz

$$\begin{aligned} X^i(\tau,\sigma) \quad = \quad & x^i + p^i\tau + \sum_{n>0}\frac{1}{n}\big[\underbrace{a_n^i \sin\left(n(\tau-\sigma)\right) + b_n^i \cos\left(n(\tau-\sigma)\right)}_{\text{right mover}} \\ & + \underbrace{\tilde{c}_n^i \sin\left(n(\tau+\sigma)\right) + \tilde{d}_n^i \cos\left(n(\tau+\sigma)\right)}_{\text{left mover}}\big]\ , \end{aligned} \tag{B.12}$$

with $\tilde{\alpha}_n^i = \tilde{c}_n^i - i\ \tilde{d}_n^i$ for $i = 1, \ldots, 6$. Furthermore, we have to impose the twisted boundary condition

$$X^i(\tau,\sigma+2\pi) = \left(\vartheta^k X(\tau,\sigma)\right)^i + n_\alpha e_\alpha^i\ . \tag{B.13}$$

First we take $\vartheta^k \in \mathrm{SO}(6)$, where k denotes the twisted sector. Later we will specialize to $\vartheta^k \in \mathrm{SU}(3)$. Restricted to the right movers, the twisted boundary condition yields the following:

- The momentum p^i must fulfill
$$(\vartheta^k p)^i = p^i \tag{B.14}$$
for $i = 1, \ldots, 6$. Hence, components of the momentum p^i in which the twist ϑ^k acts non-trivially must vanish
$$p^i = 0\ . \tag{B.15}$$

- The centre-of-mass is fixed, i.e. $x^i = X_f^i$ and coincides with a fixed point
$$(\vartheta^k X_f)^i + n_\alpha e_\alpha^i = X_f^i\ . \tag{B.16}$$

- The real oscillators a_n^i and b_n^i are no longer independent

$$\vartheta^k a_n = a_n \cos(2\pi n) + b_n \sin(2\pi n) \quad \text{(B.17)}$$
$$\vartheta^k b_n = b_n \cos(2\pi n) - a_n \sin(2\pi n) \quad \text{(B.18)}$$

From eqn. (B.17) it follows that if $n \in \mathbb{Z}$, then ϑ must be the identity. However, if we restrict ourselves to $\vartheta \in \mathrm{SU}(3)$ (compare to app. (A.2)), the twist ϑ is of block diagonal form

$$\vartheta^k = \begin{pmatrix} \vartheta_1^k & & 0 \\ & \vartheta_2^k & \\ 0 & & \vartheta_3^k \end{pmatrix} , \quad \text{(B.19)}$$

for the k-th twisted sector. The 2×2 matrices ϑ_a for $a = 1, 2, 3$ are given by

$$\vartheta_a^k = \begin{pmatrix} \cos(2\pi\eta^a) & -\sin(2\pi\eta^a) \\ \sin(2\pi\eta^a) & \cos(2\pi\eta^a) \end{pmatrix} \quad \text{(B.20)}$$

and the three rotation angles η^a correspond to the twist vector v

$$\eta^a = kv^a \bmod 1 \text{ so that } 0 \leq \eta^a < 1 . \quad \text{(B.21)}$$

By choosing the twist ϑ block diagonal, we restrict ourselves to the case of the six-torus

$$T^6 = T^2 \times T^2 \times T^2 \quad \text{(B.22)}$$

and the twist ϑ_a acts in each two-torus separately. As an example we solve the equations (B.17) and (B.18) in the first complex plane ($a = 1$)

$$a_n^1 \cos(2\pi\eta^1) - a_n^2 \sin(2\pi\eta^2) = a_n^1 \cos(2\pi n) + b_n^1 \sin(2\pi n) \quad \text{(B.23)}$$
$$a_n^1 \sin(2\pi\eta^1) + a_n^2 \cos(2\pi\eta^2) = a_n^2 \cos(2\pi n) + b_n^2 \sin(2\pi n) \quad \text{(B.24)}$$

and

$$b_n^1 \cos(2\pi\eta^1) - b_n^2 \sin(2\pi\eta^2) = b_n^1 \cos(2\pi n) - a_n^1 \sin(2\pi n) \quad \text{(B.25)}$$
$$b_n^1 \sin(2\pi\eta^1) + b_n^2 \cos(2\pi\eta^2) = b_n^2 \cos(2\pi n) - a_n^2 \sin(2\pi n) . \quad \text{(B.26)}$$

It follows that the oscillators must fulfill

$$-a_n^2 = b_n^1 \quad \text{(B.27)}$$
$$a_n^1 = b_n^2 . \quad \text{(B.28)}$$

Furthermore n must be fractional

$$n = m + \eta^1 \qquad m \in \mathbb{Z} . \quad \text{(B.29)}$$

Therefore, the twisted mode expansion of eqn. (B.12) for the first two components reads

$$\begin{aligned}\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}(\tau,\sigma) &= \begin{pmatrix} X^1_f \\ X^2_f \end{pmatrix} + \sum_{m=0} \frac{1}{m+\eta^1} \times \\ &\left[\begin{pmatrix} a^1_{m+\eta^1} \\ a^2_{m+\eta^1} \end{pmatrix} \sin\left((m+\eta^1)(\tau-\sigma)\right) + \begin{pmatrix} -a^2_{m+\eta^1} \\ a^1_{m+\eta^1} \end{pmatrix} \cos\left((m+\eta^1)(\tau-\sigma)\right)\right],\end{aligned} \tag{B.30}$$

where we have omitted the left mover. We define complex coordinates and complex oscillators

$$Z^a := X^{(2a-1)} + i\, X^{(2a)} \tag{B.31}$$

$$\alpha^a_{m+\eta^a} := 2\left(a^{(2a-1)}_{m+\eta^a} + i\, a^{(2a)}_{m+\eta^a}\right), \tag{B.32}$$

for $a = 1, 2, 3$. The factor of 2 for the oscillators is convention. Rewriting the twisted mode expansion of eqn. (B.30) yields

$$Z^a(\tau,\sigma) = Z^a_f + \frac{i}{2}\sum_{m=0} \frac{\alpha^a_{m+\eta^a}}{m+\eta^a} e^{-i(m+\eta^a)(\tau-\sigma)} + \text{ left mover.} \tag{B.33}$$

Furthermore, it turns out that the (left moving) oscillators transform under the twist ϑ^k as follows

$$\tilde{\alpha}^a_{m-\eta^a} \overset{\vartheta^k}{\mapsto} e^{2\pi i\eta^a}\, \tilde{\alpha}^a_{m-\eta^a} \qquad \text{transform as } \mathbf{3} \tag{B.34}$$

$$\tilde{\alpha}^{\bar{a}}_{m-\eta^a} \overset{\vartheta^k}{\mapsto} e^{-2\pi i\eta^a}\, \tilde{\alpha}^{\bar{a}}_{m-\eta^a} \qquad \text{transform as } \overline{\mathbf{3}}\ . \tag{B.35}$$

This can also be seen in app. (A.2). There, it is shown that the vector of SO(6) decomposes into $\mathbf{3} \oplus \overline{\mathbf{3}}$ of SU(3).
Here, the real oscillator $a^i_{m+\eta^a}$ transforms as a vector of SO(6). By restricting the twist ϑ to be an element of SU(3), the vector decomposes into $\mathbf{3} \oplus \overline{\mathbf{3}}$.

B.3 How to Solve the Mass-Equation

Finding all $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice vectors p that solve the mass-equation (2.101) is quite a difficult challenge, therefore one way to solve this problem is presented in detail.

- Write down the **mass-equation** (2.101),

$$(p - V)^2 = e \ , \tag{B.36}$$

  using the shorthand V for $kV + n_\alpha A_\alpha$ and $e := 2(1 - \delta c - \tilde{N})$. The left hand side of this equation is positive, therefore e must be positive. This restricts the values of $\tilde{N}$.

- Derive **bounds** for the components of p.

  Denote the bounds by p_{up} and p_{low} for upper and lower bound, respectively. The components of the bounds are either half integral or integral. They do not have to be $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice vectors, because they only specify a 16 dimensional box in the $\mathrm{E}_8 \times \mathrm{E}_8'$ lattice in which the solutions of the mass-equation can be found.

  The bounds are given by

$$\underbrace{(p^1 - V^1)^2}_{\leq e} + \underbrace{(p^2 - V^2)^2}_{\leq e} + \underbrace{(p^3 - V^3)^2}_{\leq e} + \ldots + \underbrace{(p^{16} - V^{16})^2}_{\leq e} = e \ . \tag{B.37}$$

  Hence the bounds can be computed,

$$p^I_{\mathrm{up}} = \lfloor \sqrt{e} + V^I \rfloor \qquad p^I_{\mathrm{low}} = \lceil -\sqrt{e} + V^I \rceil \ , \tag{B.38}$$

  where the floor- and ceil-brackets shall be defined slightly different here. The return value of the floor-brackets $\lfloor$argument$\rfloor$ is an integer or a half integer equal to the greatest integer or half integer less than or equal to the argument. The return value of the ceil-brackets $\lceil$argument$\rceil$ is an integer or half integer equal to the smallest integer or half integer greater than or equal to the argument. We give examples:

$$\lfloor 1.49 \rfloor = 1.0 \qquad \lceil -0.82 \rceil = -0.5 \ .$$

- Finally, the components of the solutions p of the mass-equation (B.36) are computed by the following method.

  The idea is to run through all possible values between p^I_{low} and p^I_{up} for all components p^I iteratively. Every time we choose a new component p^I, we have to make sure that some restrictions are fulfilled. If one of the restrictions fails, we can skip this “solution”.

  The first restriction is: By fixing the first (or the ninth) component of p to be integer or half-integer, the first eight (or the second eight) components of p are fixed to belong to the integer lattice or to the half-integer lattice, respectively. Hence, if in addition the sum over both, the first eight and the second eight components is even, this p is a

$E_8 \times E_8'$ lattice vector.

The second restriction is: Every time we choose a component p^J, the contribution to the mass can be computed up to the J-th component by the equation

$$m(J) = \sum_{I=1}^{J} (p^I - V^I)^2 \ . \tag{B.39}$$

If the mass is greater than e, we can skip this "solution". If it is less than e, we continue with the next component.

An explicit example for this method is given in sec. (3.2.2) for the twisted sector of the $\mathbb{Z}_3$ orbifold.

Appendix C

Tables

C.1 Details of the $\mathbb{Z}_3$ Orbifold

$p = (p_1)\,(0^8)$	$p = (0^8)\,(p_2)$
$(-1, 0, 1, 0, 0, 0, 0, 0)$	$(-1, -1, 0, 0, 0, 0, 0, 0)$
$(-1, 1, 0, 0, 0, 0, 0, 0)$	$(-1, 1, 0, 0, 0, 0, 0, 0)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(-1, 0, -1, 0, 0, 0, 0, 0)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(-1, 0, 0, -1, 0, 0, 0, 0)$
$(0, 0, 0, 0, 0, -1, 0, 1)$	$(-1, 0, 0, 0, -1, 0, 0, 0)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(-1, 0, 0, 0, 0, -1, 0, 0)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(-1, 0, 0, 0, 0, 0, -1, 0)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(-1, 0, 0, 0, 0, 0, 0, -1)$

Table C.1: $\mathbb{Z}_3$ standard embedding. The used choice for the 16 basis vectors is listed. A choice of basis fixes the semi-ordering in the weight space [16].

simple roots of E_6	
α_1	$= (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})(0^8)$
α_2	$= (0, 0, 0, -1, 1, 0, 0, 0)(0^8)$
α_3	$= (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})(0^8)$
α_4	$= (0, 0, 0, 0, 0, 1, -1, 0)(0^8)$
α_5	$= (0, 0, 0, 0, 0, -1, 0, -1)(0^8)$
α_6	$= (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})(0^8)$
simple roots of SU(3)	
α_7	$= (-1, 1, 0, 0, 0, 0, 0, 0)(0^8)$
α_8	$= (0, -1, 1, 0, 0, 0, 0, 0)(0^8)$
simple roots of E_8'	
α_9	$= (0^8)(-1, 0, 0, 0, 0, 0, 0, -1)$
α_{10}	$= (0^8)(0, 0, 0, 0, 0, 0, -1, 1)$
α_{11}	$= (0^8)(0, 0, 0, 0, 0, -1, 1, 0)$
α_{12}	$= (0^8)(0, 0, 0, 0, -1, 1, 0, 0)$
α_{13}	$= (0^8)(0, 0, 0, -1, 1, 0, 0, 0)$
α_{14}	$= (0^8)(0, 0, -1, 1, 0, 0, 0, 0)$
α_{15}	$= (0^8)(0, -1, 1, 0, 0, 0, 0, 0)$
α_{16}	$= (0^8)(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

Table C.2: $\mathbb{Z}_3$ standard embedding. The simple roots in Cartan Weyl basis are listed. They are positive and simple with respect to the basis of tab. (C.1).

C.2 Details of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

right movers $|q\rangle_R$

	q	$q \cdot v_1$	$q \cdot v_2$
NS	$(\pm1,0,0,0)$	0	0
R	$\left(\pm\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)$	0	0

$\Rightarrow$ transforming as ζ^0 and γ^0

	q	$q \cdot v_1$	$q \cdot v_2$
NS	$(0,\pm1,0,0)$	$\frac{1}{2}$	0
R	$\left(\pm\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)$	$\frac{1}{2}$	0

$\Rightarrow$ transforming as ζ^2 and γ^0

	q	$q \cdot v_1$	$q \cdot v_2$
NS	$(0,0,\pm1,0)$	$\frac{1}{2}$	$\frac{1}{2}$
R	$\left(\pm\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\right)$	$\frac{1}{2}$	$\frac{1}{2}$

$\Rightarrow$ transforming as ζ^2 and γ^2

	q	$q \cdot v_1$	$q \cdot v_2$
NS	$(0,0,0,\pm1)$	0	$\frac{1}{2}$
R	$\left(\pm\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)\right)$	0	$\frac{1}{2}$

$\Rightarrow$ transforming as ζ^0 and γ^2

left movers $|p\rangle_L$

$p=(p_1)(0^8)$	$p \cdot V_1$	$p \cdot V_2$
$\left(0,0,0,\underline{\pm1,\pm1,0^3}\right)$	0	0
$\left(\pm\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{2}\right)^5\right)$	0	0

$\Rightarrow$ 72 roots, transforming as ζ^0 and γ^0

$p=(0^8)(p_2)$	$p \cdot V_1$	$p \cdot V_2$
$\left(\underline{\pm1,\pm1,0^6}\right)$	0	0
$\left(\left(\pm\frac{1}{2}\right)^8\right)$	0	0

$\Rightarrow$ 240 roots, transforming as ζ^0 and γ^0

$p=(p_1)(0^8)$	$p \cdot V_1$	$p \cdot V_2$
$\left(0,0,\pm1,\underline{\pm1,0^4}\right)$	0	$\frac{1}{2}$
$\left(\pm\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{2}\right)^5\right)$	0	$\frac{1}{2}$
$(\pm1,\pm1,0^6)$	0	$\frac{1}{2}$

$\Rightarrow$ 56 roots, transforming as ζ^0 and γ^2

$p=(p_1)(0^8)$	$p \cdot V_1$	$p \cdot V_2$
$\left(\pm1,0,0,\underline{\pm1,0^4}\right)$	$\frac{1}{2}$	0
$\left(\pm\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\pm\frac{1}{2}\right)^5\right)$	$\frac{1}{2}$	0
$(0,\pm1,\pm1,0^5)$	$\frac{1}{2}$	0

$\Rightarrow$ 56 roots, transforming as ζ^2 and γ^0

$p=(p_1)(0^8)$	$p \cdot V_1$	$p \cdot V_2$
$\left(0,\pm1,0,\underline{\pm1,0^4}\right)$	$\frac{1}{2}$	$\frac{1}{2}$
$\left(\pm\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),\left(\pm\frac{1}{2}\right)^5\right)$	$\frac{1}{2}$	$\frac{1}{2}$
$(\pm1,0,\pm1,0^5)$	$\frac{1}{2}$	$\frac{1}{2}$

$\Rightarrow$ 56 roots, transforming as ζ^2 and γ^2

Table C.3: Transformation properties of right and left movers in the untwisted sector. The transformation phases are denoted by $\zeta = i$ and $\gamma = i$ and they correspond to the twists v_1 and v_2, respectively.

$p = (p_1)\,(0^8)$	$p = (0^8)\,(p_2)$
$(0,0,0,-1,-1,0,0,0)$	$(-1,-1,0,0,0,0,0,0)$
$(0,0,0,-1,1,0,0,0)$	$(-1,1,0,0,0,0,0,0)$
$(0,0,0,-1,0,-1,0,0)$	$(-1,0,-1,0,0,0,0,0)$
$(0,0,0,-1,0,0,-1,0)$	$(-1,0,0,-1,0,0,0,0)$
$(0,0,0,-1,0,0,0,-1)$	$(-1,0,0,0,-1,0,0,0)$
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(-1,0,0,0,0,-1,0,0)$
	$(-1,0,0,0,0,0,-1,0)$
	$(-1,0,0,0,0,0,0,-1)$

Table C.4: $\mathbb{Z}_2 \times \mathbb{Z}_2$ standard embedding. The used choice for the 14 basis vectors is listed. A choice of basis fixes the semi-ordering in the weight space [16].

simple roots of E_6
$\alpha_1 = (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})(0^8)$
$\alpha_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0^8)$
$\alpha_3 = (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0^8)$
$\alpha_4 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})(0^8)$
$\alpha_5 = (0,0,0,0,1,1,0,0)(0^8)$
$\alpha_6 = (0,0,0,0,0,0,-1,1)(0^8)$
simple roots of E_8'
$\alpha_7 = (0^8)(-1,0,0,0,0,0,0,-1)$
$\alpha_8 = (0^8)(0,0,0,0,0,0,-1,1)$
$\alpha_9 = (0^8)(0,0,0,0,0,-1,1,0)$
$\alpha_{10} = (0^8)(0,0,0,0,-1,1,0,0)$
$\alpha_{11} = (0^8)(0,0,0,-1,1,0,0,0)$
$\alpha_{12} = (0^8)(0,0,-1,1,0,0,0,0)$
$\alpha_{13} = (0^8)(0,-1,1,0,0,0,0,0)$
$\alpha_{14} = (0^8)(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$

Table C.5: $\mathbb{Z}_2 \times \mathbb{Z}_2$ standard embedding. The simple roots in Cartan Weyl basis are listed. They are positive and simple with respect to the basis of tab. (C.4).

$p = (p_1)(0^8)$	$p_{\mathrm{sh}} = p - V$	$p_{\mathrm{sh,\ DL(E_6)}}$
$(0,-1,-1,0,0,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},-1,0,0,0,0,0\right)$	$(1,-1,1,-1,0,0)$
$(0,-1,1,0,0,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},1,0,0,0,0,0\right)$	$\mathbf{(0,0,0,0,0,0)}$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,0,0,-1,1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,1,0,-1,-1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,1,0,-1,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\mathbf{(1,0,0,0,0,0)}$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,1,-1,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,-1,1,0,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,0,-1,1,1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,1,-1,1,-1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,0,0,-1,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(1,-1,0,1,-1,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(1,0,-1,0,0,1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(1,0,0,0,0,-1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,-1,0,0,0,1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,-1,1,0,0,-1)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,0,-1,1,0)$
$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(1,-1,0,0,1,0)$
$(1,0,0,-1,0,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,-1,0,0,0,0\right)$	$(-1,1,0,0,0,0)$
$(1,0,0,0,-1,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,-1,0,0,0\right)$	$(0,1,-1,1,-1,0)$
$(1,0,0,0,0,-1,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,-1,0,0\right)$	$(-1,0,0,1,-1,0)$
$(1,0,0,0,0,0,-1,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,0,-1,0\right)$	$(-1,1,-1,0,0,1)$
$(1,0,0,0,0,0,0,-1)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,0,0,-1\right)$	$(-1,1,0,0,0,-1)$
$(1,0,0,0,0,0,0,1)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,0,0,1\right)$	$(0,0,-1,1,0,1)$
$(1,0,0,0,0,0,1,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,0,1,0\right)$	$(0,0,0,1,0,-1)$
$(1,0,0,0,0,1,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,0,1,0,0\right)$	$(0,1,-1,0,1,0)$
$(1,0,0,0,1,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,0,1,0,0,0\right)$	$(-1,0,0,0,1,0)$
$(1,0,0,1,0,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},0,1,0,0,0,0\right)$	$(0,0,-1,1,0,0)$

Table C.6: $\mathbb{Z}_2 \times \mathbb{Z}_2$ standard embedding. There are 56 weights in the $T_{(1,0)}$ twisted sector, 28 with transformation phases ζ^0 and γ^1. We find the highest weights $\mathbf{(0,0,0,0,0,0)}$ of a singlet and $\mathbf{(1,0,0,0,0,0)}$ of a **27** of E_6.

$p=(p_1)(0^8)$	$p_{\text{sh}}=p-V$	$p_{\text{sh, DL(E}_6)}$
$(0,-1,0,-1,0,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,-1,0,0,0,0\right)$	$(0,0,1,-1,0,0)$
$(0,-1,0,0,-1,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,-1,0,0,0\right)$	$(1,0,0,0,-1,0)$
$(0,-1,0,0,0,-1,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,-1,0,0\right)$	$(0,-1,1,0,-1,0)$
$(0,-1,0,0,0,0,-1,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,0,-1,0\right)$	$(0,0,0,-1,0,1)$
$(0,-1,0,0,0,0,0,-1)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,-1\right)$	$(0,0,1,-1,0,-1)$
$(0,-1,0,0,0,0,0,1)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,1\right)$	$(1,-1,0,0,0,1)$
$(0,-1,0,0,0,0,1,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,0,1,0\right)$	$(1,-1,1,0,0,-1)$
$(0,-1,0,0,0,1,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,0,1,0,0\right)$	$(1,0,0,-1,1,0)$
$(0,-1,0,0,1,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,0,1,0,0,0\right)$	$(0,-1,1,-1,1,0)$
$(0,-1,0,1,0,0,0,0)$	$\left(-\frac{1}{2},-\frac{1}{2},0,1,0,0,0,0\right)$	$(1,-1,0,0,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(-1,1,0,0,-1,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,0,0,1,-1,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,1,-1,0,0,1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,1,0,0,0,-1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(-1,0,0,0,0,1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,1,0,0,-1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(-1,1,0,-1,1,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\mathbf{(0,0,0,0,1,0)}$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,-1,1,-1,1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,0,1,-1,-1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(0,1,-1,0,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(1,0,-1,1,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,0,0,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(0,-1,0,1,0,0)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$	$(0,0,-1,0,1,1)$
$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$\left(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$	$(0,0,0,0,1,-1)$
$(1,0,-1,0,0,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},-1,0,0,0,0,0\right)$	$\mathbf{(0,0,0,0,0,0)}$
$(1,0,1,0,0,0,0,0)$	$\left(\frac{1}{2},\frac{1}{2},1,0,0,0,0,0\right)$	$(-1,1,-1,1,0,0)$

Table C.7: $\mathbb{Z}_2\times\mathbb{Z}_2$ standard embedding. There are 56 weights in the $T_{(1,0)}$ twisted sector, 28 with transformation phases ζ^0 and γ^3. We find the highest weights $\mathbf{(0,0,0,0,0,0)}$ of a singlet and $\mathbf{(0,0,0,0,1,0)}$ of a $\overline{\mathbf{27}}$ of E_6.

	q	$q_{\text{sh}} = q + v_1$	$q_{\text{sh}} \cdot v_1$	$q_{\text{sh}} \cdot v_2$
NS	$(0,\ \ 0, 1, 0)$	$\left(0,\ \ \frac{1}{2},\ \ \frac{1}{2}, 0\right)$	0	$\frac{1}{4}$
	$(0, -1, 0, 0)$	$\left(0, -\frac{1}{2}, -\frac{1}{2}, 0\right)$	0	$-\frac{1}{4}$
R	$\left(\ \ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(\ \ \frac{1}{2}, 0, 0, -\frac{1}{2}\right)$	0	$\frac{1}{4}$
	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2},\ \ \frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0, 0,\ \ \frac{1}{2}\right)$	0	$-\frac{1}{4}$

Table C.8: $\mathbb{Z}_2 \times \mathbb{Z}_2$ standard embedding. The twisted right movers in the $T_{(1,0)}$ twisted sector. All have a transformation phase ζ^0. Two right movers have γ^1 and the other two have γ^3.

p	$p_{\text{sh}} = p - V_1$	$p_{\text{sh}} \cdot V_1$	$p_{\text{sh}} \cdot V_2$	$p_{\text{sh, DL(E}_6)}$
$(0,\ \ 0, 0, 0, 0, 0, 0, 0, 0)$	$\left(-\frac{1}{2},\ \ \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\right)$	$-\frac{1}{2}$	$\frac{1}{4}$	(0^6)
$(1, -1, 0, 0, 0, 0, 0, 0, 0)$	$\left(\ \ \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\right)$	$\frac{1}{2}$	$-\frac{1}{4}$	(0^6)

Table C.9: $\mathbb{Z}_2 \times \mathbb{Z}_2$ standard embedding. The twisted left movers in the $T_{(1,0)}$ twisted sector for $\tilde{N} = \frac{1}{2}$. The first weight transforms with ζ^2 and γ^1, the second one with ζ^2 and γ^3. In Dynkin labels we see that they are singlets under the gauge group.

	ϑ_1	ϑ_2	$\vartheta_1 \circ \vartheta_2$
$\tilde{\alpha}^1_{-\frac{1}{2}}$	-1	1	-1
$\tilde{\alpha}^{\bar{1}}_{-\frac{1}{2}}$	-1	1	-1
$\tilde{\alpha}^2_{-\frac{1}{2}}$	-1	-1	1
$\tilde{\alpha}^{\bar{2}}_{-\frac{1}{2}}$	-1	-1	1
$\tilde{\alpha}^3_{-\frac{1}{2}}$	1	-1	-1
$\tilde{\alpha}^{\bar{3}}_{-\frac{1}{2}}$	1	-1	-1

Table C.10: Transformation phases of the moded oscillators with respect to the twists ϑ_1, ϑ_2 and $\vartheta_1 \circ \vartheta_2$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Compare to the equations (2.105) and (2.106) with $m = 0$ and $\eta^a = \frac{1}{2}$.

$$\left.\begin{array}{r}|\frac{1}{2},0,0,-\frac{1}{2}\rangle_R\\|0,\frac{1}{2},\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{1}_{-\frac{1}{2}}|\frac{1}{2},-\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|\frac{1}{2},0,0,-\frac{1}{2}\rangle_R\\|0,\frac{1}{2},\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{\bar{1}}_{-\frac{1}{2}}|\frac{1}{2},-\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|-\frac{1}{2},0,0,\frac{1}{2}\rangle_R\\|0,-\frac{1}{2},-\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{1}_{-\frac{1}{2}}|-\frac{1}{2},\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|-\frac{1}{2},0,0,\frac{1}{2}\rangle_R\\|0,-\frac{1}{2},-\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{\bar{1}}_{-\frac{1}{2}}|-\frac{1}{2},\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|\frac{1}{2},0,0,-\frac{1}{2}\rangle_R\\|0,\frac{1}{2},\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{2}_{-\frac{1}{2}}|-\frac{1}{2},\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|\frac{1}{2},0,0,-\frac{1}{2}\rangle_R\\|0,\frac{1}{2},\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{\bar{2}}_{-\frac{1}{2}}|-\frac{1}{2},\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|-\frac{1}{2},0,0,\frac{1}{2}\rangle_R\\|0,-\frac{1}{2},-\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{2}_{-\frac{1}{2}}|\frac{1}{2},-\frac{1}{2},0^6\rangle_L$$

$$\left.\begin{array}{r}|-\frac{1}{2},0,0,\frac{1}{2}\rangle_R\\|0,-\frac{1}{2},-\frac{1}{2},0\rangle_R\end{array}\right\}\otimes\ \tilde{\alpha}^{\bar{2}}_{-\frac{1}{2}}|\frac{1}{2},-\frac{1}{2},0^6\rangle_L$$

Table C.11: $\mathbb{Z}_2\times\mathbb{Z}_2$ standard embedding. Invariant twisted states in the $T_{(1,0)}$ twisted sector with moded oscillators ($\tilde{N}=\frac{1}{2}$). The left movers' ket $|p_1\rangle_L$ gives the first half of the weight, i.e. $p_{\rm sh}=(p_1)(0^8)$.

Appendix D

The Geometry of the $\mathbb{Z}_4$ Orbifold

The point group P of the $\mathbb{Z}_4$ orbifold consists of four elements,

$$P = \{\mathbb{1}, \vartheta, \vartheta^2, \vartheta^3\} \ , \tag{D.1}$$

where ϑ is a rotation of 90°. The six-torus is chosen to be $T^6 = T^2 \times T^2 \times T^2$ and in addition orthogonal. Thus, we can write the twist as a rotation in complex planes

$$v = \left(0, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \text{ with } \vartheta^k \Leftrightarrow kv \tag{D.2}$$

for $k = 0, \ldots, 3$. This choice of v is obviously an automorphism of the six-torus. Furthermore, the twist fulfills the condition for $\mathcal{N} = 1$ SUSY stated in eqn. (2.63) $v_1 + v_2 + v_3 = -\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 0$. As we have fulfilled all conditions of sec. (2.3) and (2.5) for a consistent space group

$$S = \{(kv, n_\alpha e_\alpha) \mid k = 0, 1, 2, 3 \text{ and } n_\alpha \in \mathbb{Z}\} \ , \tag{D.3}$$

we have fixed the geometry of the $\mathbb{Z}_4$ orbifold (up to the size). We analyze the structure of the fixed points sector by sector:

***k* = 1 twisted sector** The $k = 1$ sector has $4 \times 2 \times 2 = 16$ fixed points in the six-torus T^6, as illustrated in fig. (D.1).

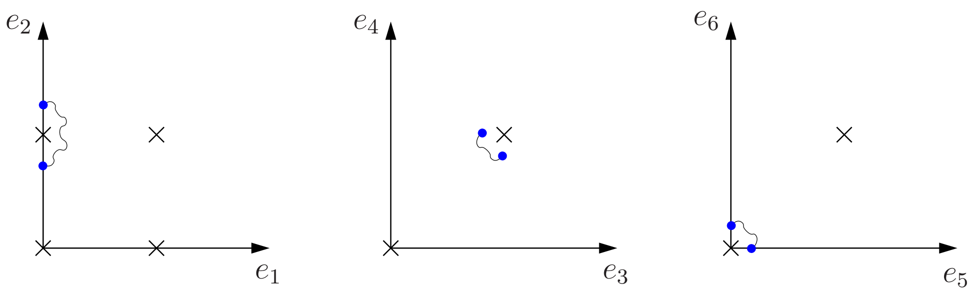


Figure D.1: $\mathbb{Z}_4$ orbifold: $4 \times 2 \times 2 = 16$ fixed points in the six-torus T^6 for the $k = 1$ twisted sector. As an example, a string is drawn. The blue circles are identified under the action of the constructing element $(\vartheta, e_2 + e_3)$. Thus the string is closed.

Since the six-dimensional space is defined as $\mathbb{R}^6/S$, we have to consider the action of the other space group elements on the 16 fixed points of this sector. It turns out that each fixed point is mapped to itself (modulo torus-shifts). Therefore, the number of fixed points in the space $\mathbb{R}^6/S$ equals the number of fixed points in T^6 (and the method of sec. (2.8) is simplified). This sector is analog to the $\mathbb{Z}_3$ case of sec. (3.2.2).

***k* = 2 twisted sector** In the $k = 2$ sector the situation is slightly more difficult. The twist

$$2v = \left(0, -1, \frac{1}{2}, \frac{1}{2}\right) \quad . \tag{D.4}$$

looks like a $\mathbb{Z}_2$ twist. In sec. (2.4.2) it was shown that the $\mathbb{Z}_2$ orbifold has 16 fixed tori in the space of the six-torus T^6. They are plotted in fig. (D.2).

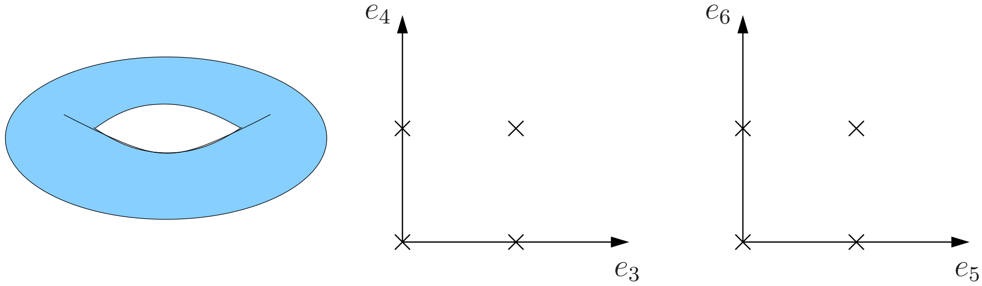


Figure D.2: $\mathbb{Z}_4$ orbifold: $4 \times 4 = 16$ fixed tori in the six-torus T^6 for the $k = 2$ twisted sector. If we consider the space $\mathbb{R}^6/S$, the number of fixed tori reduces to $4 + 6 = 10$.

Again, the six-dimensional space is defined as $\mathbb{R}^6/S$ and we have to check the action of the other space group elements on the 16 fixed tori. As an example we take a constructing element

$$g = (\vartheta^2, e_3) \tag{D.5}$$

of the $k = 2$ twisted sector. The string that closes with respect to g is localized at a fixed point, see fig. (D.3).

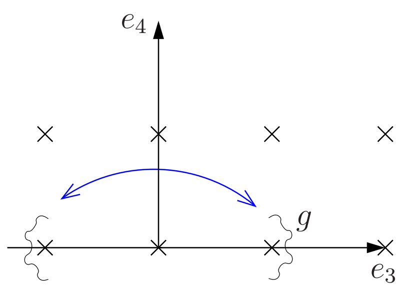


Figure D.3: $\mathbb{Z}_4$ orbifold: The action of the constructing element $g = (\vartheta^2, e_3)$ on a string in the $k = 2$ twisted sector, focusing on the plane of e_3 and e_4.

Following the considerations of sec. (2.8), we take a space group element

$$h = (\vartheta, 0) \tag{D.6}$$

that does not commute with the constructing one,

$$[h, g] = (\mathbb{1}, e_4 - e_3) \ . \tag{D.7}$$

The action of this element h on the constructing one is illustrated in fig. (D.4).

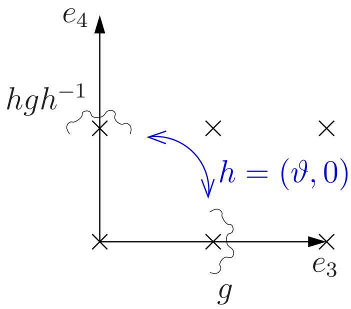


Figure D.4: $\mathbb{Z}_4$ orbifold: The non-commuting element $h = (\vartheta, 0)$ maps the fixed torus g to hgh^{-1}. Therefore, the fixed tori that correspond to g and hgh^{-1} are actually identified in the space $\mathbb{R}^6/S$.

h maps states from the Hilbert space H_g to $H_{hgh^{-1}}$, where hgh^{-1} is given by

$$hgh^{-1} = (\vartheta^2, e_4) \ . \tag{D.8}$$

Now we have two possibilities to take this into account:

- Either we have to build linear combinations of a state of H_g with the corresponding one of $H_{hgh^{-1}}$ for all non-commuting elements h in order to get $S \times G$ invariant states,
- or we define fixed points in the space $\mathbb{R}^6/S$ and ignore the action of non-commuting elements of $S \otimes G$. We can do so by identifying fixed points in the T^6-space that are mapped onto each other by some elements of the space group S. Obviously, this is easier than to build a linear combination of an infinite number of states.

We follow the second way. Unfortunately it is not possible to illustrate the space $\mathbb{R}^6/S$ and its fixed hypersurfaces. Therefore, we list the constructing elements that correspond to the fixed hypersurfaces. It turns out that there are $4 + 6 = 10$ fixed hypersurfaces in the space $\mathbb{R}^6/S$. The following four constructing elements correspond to fixed hypersurfaces that are not mapped to different ones in T^6:

$$\begin{array}{ll} (\vartheta^2, 0) & (\vartheta^2, e_5 + e_6) \\ (\vartheta^2, e_3 + e_4) & (\vartheta^2, e_3 + e_4 + e_5 + e_6) \end{array} \tag{D.9}$$

in contrast to the following six:

$$\begin{array}{lll} (\vartheta^2, e_6) & (\vartheta^2, e_4) & (\vartheta^2, e_4 + e_6) \\ (\vartheta^2, e_4 + e_5) & (\vartheta^2, e_4 + e_5 + e_6) & (\vartheta^2, e_3 + e_4 + e_6) \ . \end{array} \tag{D.10}$$

It is important to notice that the centralizer for these two sets of space group elements are different, imposing different projection conditions. Consequently, the resulting spectra at these fixed hypersurfaces are different.

$k = 3$ twisted sector Since the twist

$$3v = \left(0, -\frac{3}{2}, \frac{3}{4}, \frac{3}{4}\right) \tag{D.11}$$

is a true $\mathbb{Z}_4$ twist, this sector can be studied analogously to the $k = 1$ twisted sector and the $k = 3$ has 16 fixed points, both in T^6 and in $\mathbb{R}^6/S$.

For further discussion of the $\mathbb{Z}_4$ orbifold see e.g. refs. [26, 27].

Appendix E

$\mathbb{Z}_2 \times \mathbb{Z}_2$ Models with three Generations

In this part we present nine $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models with three generations. Their gauge groups are: two models with Standard Model like gauge groups, one with Pati-Salam gauge group, one with SU(5) and five with SO(10) gauge groups.

For each model the Wilson lines A_i $i = 1...6$ and the unbroken gauge group are listed. The complete[1] massless spectrum is presented in a table:

untwisted sector:	
3	$(\mathbf{3}, \mathbf{1}; \mathbf{1})$
number	*(representation)*

twisted sectors:				
$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1	2		3	$(\mathbf{3}, \mathbf{2}; \mathbf{1})$
9		6	15	$(\overline{\mathbf{3}}, \mathbf{1}; \mathbf{1})$
number in this sector	*number in this sector*	*number in this sector*	*total number*	*(representation)*
6	6		12	hidden matter: $(\mathbf{1}, \mathbf{1}; \mathbf{2})$ *hidden matter transforms only under the hidden gauge group*

After this, it is briefly explained how to count the generations in each model. At the end the localization of the three generations is shown in a picture.

[1] Gauge bosons and the supergravity multiplet are omitted.

E.1 Three Generations of $\mathbf{SU}(3)\times\mathbf{SU}(2)\times\mathbf{U}(1)^n$ - Model 1

- Standard embedding with Wilson lines:

$$
\begin{aligned}
A_1 &= \left(\tfrac{1}{2}^3, 0^4, \tfrac{1}{2}\right)\left(\tfrac{1}{2}^4, 0^4\right) & A_2 &= \left(0^6, \tfrac{1}{2}^2\right)\left(-\tfrac{1}{2}, 0^3, \tfrac{1}{2}, 1, 0^2\right) \\
A_3 &= \left(0^5, \tfrac{1}{2}^2, 0\right)\left(0^4, -\tfrac{1}{2}, 1, \tfrac{1}{2}, 0\right) & A_4 &= \left(0^8\right)\left(1, \tfrac{3}{2}, -\tfrac{3}{2}, 1, 0, \tfrac{3}{2}, -1, \tfrac{1}{2}\right) \\
A_5 &= \left(0^5, \tfrac{1}{2}^2, 0\right)\left(0^4, -\tfrac{1}{2}, 1, \tfrac{1}{2}, 0\right) & A_6 &= \left(0^8\right)\left(0^8\right)
\end{aligned}
$$

- Gauge group: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^5 \times \mathrm{SU}(4)' \times \mathrm{SU}(2)'^2 \times \mathrm{U}(1)'^3$

- Massless spectrum:

untwisted sector:	
3	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
3	$(\overline{\mathbf{3}}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
6	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1		2	3	$(\overline{\mathbf{3}}, \mathbf{2}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
8		4	12	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
2	4	2	8	$(\overline{\mathbf{3}}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
9	12	4	25	$(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
26	44	18	88	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
1			1	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1})$
1			1	$(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2}, \mathbf{1})$
				hidden matter:
4		2	6	$(\mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{1}, \mathbf{1})$
2		4	6	$(\mathbf{1}, \mathbf{1}; \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1})$
9		2	11	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1})$
6		6	12	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2})$

- One generation contains: $(\overline{\mathbf{3}}, \mathbf{2})$, $2(\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{2})$ and $2(\mathbf{1}, \mathbf{1})$. Thus, this model has three left-handed quark doublets, a net number of four $(12 - 8 = 4)$ right-handed quark singlets and many lepton-like particles.

- Localization of the three generations:

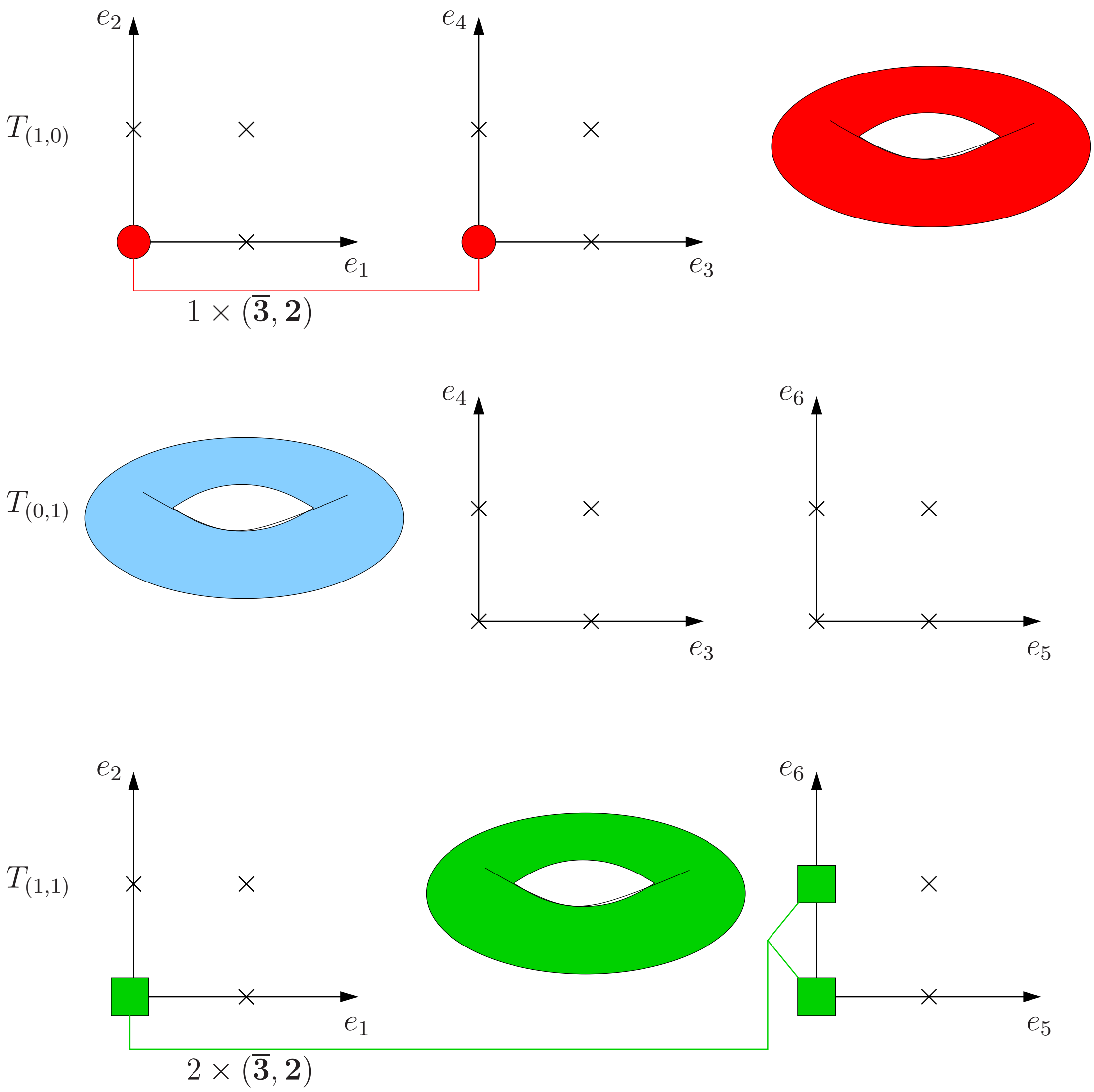


Figure E.1: Localization of the left-handed quarks.

E.2 Three Generations of SU(3)×SU(2)×U(1)^n - Model 2

- Standard embedding with Wilson lines:

$$A_1 = \left(0^5, \tfrac{1}{2}^2, 1\right)\left(0, \tfrac{1}{2}^2, 0^5\right) \qquad A_2 = \left(\tfrac{1}{4}^6, -\tfrac{1}{4}^2\right)\left(\tfrac{1}{4}^6, \tfrac{5}{4}^2\right)$$
$$A_3 = (0^7, 1)\,(1, 0^7) \qquad A_4 = (0^8)\left(1, 0^3, -\tfrac{1}{2}^3, \tfrac{1}{2}\right)$$
$$A_5 = (0^8)\left(1, -\tfrac{1}{2}, \tfrac{1}{2}, 0^3, -\tfrac{1}{2}^2\right) \qquad A_6 = (0^8)\,(0^8)$$

- Gauge group: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^5 \times \mathrm{SU}(2)'^4 \times \mathrm{U}(1)'^4$
- Massless spectrum:

untwisted sector:	
3	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
3	$(\overline{\mathbf{3}}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
6	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1	2		3	$(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1	2	6	9	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
9		6	15	$(\overline{\mathbf{3}}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
9		20	29	$(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
26	22	60	108	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
				hidden matter:
6	6		12	$(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
6	6		12	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
6	6		12	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
6	6		12	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
1	2		3	$(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
1	2		3	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$
	2		2	$(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
	2		2	$(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
	2		2	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$
	2		2	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$

- One generation contains: $(\mathbf{3}, \mathbf{2})$, $2(\overline{\mathbf{3}}, \mathbf{1})$, $(\mathbf{1}, \mathbf{2})$ and $2(\mathbf{1}, \mathbf{1})$. Thus, this model has three left-handed quark doublets, a net number of six $(15 - 9 = 6)$ right-handed quark singlets and many lepton-like particles.

- Localization of the three generations:

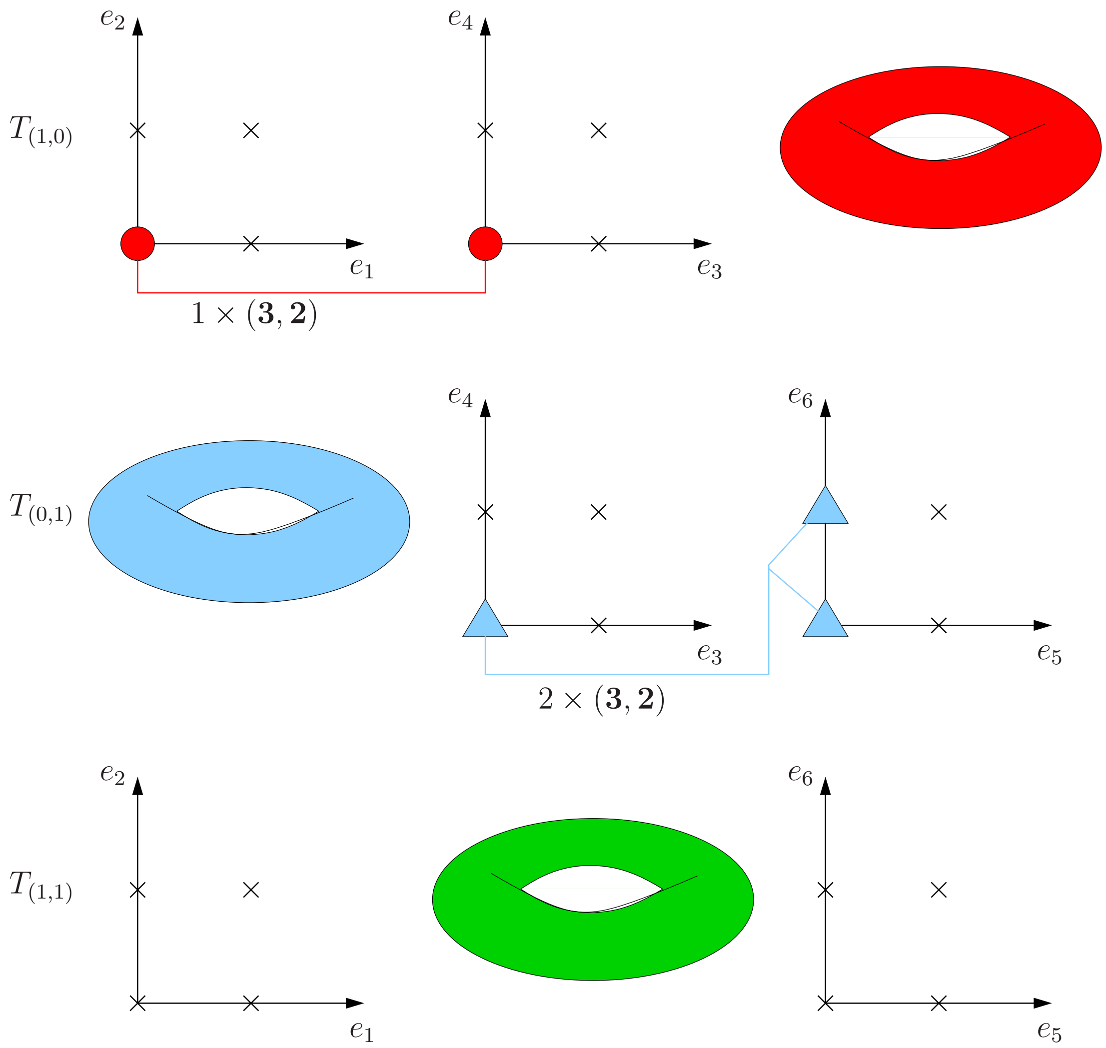


Figure E.2: Localization of the left-handed quarks.

E.3 Three Generations of $\mathbf{SU}(4) \times \mathbf{SU}(2) \times \mathbf{SU}(2)$

- Standard embedding with Wilson lines:

$$
\begin{aligned}
A_1 &= \left(0^4, {\tfrac{1}{2}}^2, 0^2\right)\left({\tfrac{1}{2}}^2, 1, 0^5\right) & A_2 &= \left(0^6, {\tfrac{1}{2}}^2\right)\left(0^3, {\tfrac{1}{2}}^2, 1, 0^2\right) \\
A_3 &= \left(0^8\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0^3, {\tfrac{1}{2}}^2\right) & A_4 &= \left(0^8\right)\left(0^2, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, -\tfrac{1}{2}\right) \\
A_5 &= \left(0^8\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0^3, {\tfrac{1}{2}}^2\right) & A_6 &= \left(0^8\right)\left(0^8\right)
\end{aligned}
$$

- Gauge group: $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)'^8$

- Massless spectrum:

untwisted sector:	
6	$(\mathbf{6, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1})$
12	$(\mathbf{1, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1		2	3	$(\mathbf{4, 2, 1; 1, 1, 1, 1, 1, 1, 1, 1})$
1		2	3	$(\overline{\mathbf{4}}, \mathbf{1, 2; 1, 1, 1, 1, 1, 1, 1, 1})$
1		2	3	$(\mathbf{1, 2, 2; 1, 1, 1, 1, 1, 1, 1, 1})$
1	4	2	7	$(\mathbf{6, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1})$
1			1	$(\mathbf{4, 1, 1; 1, 1, 1, 1, 2, 1, 1, 1})$
1			1	$(\mathbf{4, 1, 1; 1, 1, 1, 1, 1, 1, 1, 2})$
1			1	$(\overline{\mathbf{4}}, \mathbf{1, 1; 2, 1, 1, 1, 1, 1, 1, 1})$
1			1	$(\overline{\mathbf{4}}, \mathbf{1, 1; 1, 1, 1, 1, 1, 2, 1, 1})$
1			1	$(\mathbf{1, 2, 1; 2, 1, 1, 1, 1, 1, 1, 1})$
1			1	$(\mathbf{1, 2, 1; 1, 1, 1, 1, 2, 1, 1, 1})$
1			1	$(\mathbf{1, 2, 1; 1, 1, 1, 1, 1, 2, 1, 1})$
1			1	$(\mathbf{1, 2, 1; 1, 1, 1, 1, 1, 1, 1, 2})$
1			1	$(\mathbf{1, 1, 2; 2, 1, 1, 1, 1, 1, 1, 1})$
1			1	$(\mathbf{1, 1, 2; 1, 1, 1, 1, 2, 1, 1, 1})$
1			1	$(\mathbf{1, 1, 2; 1, 1, 1, 1, 1, 2, 1, 1})$
1			1	$(\mathbf{1, 1, 2; 1, 1, 1, 1, 1, 1, 1, 2})$
6	24	12	42	$(\mathbf{1, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1})$
				hidden matter:
	4		4	$(\mathbf{1, 1, 1; 1, 2, 1, 1, 1, 1, 1, 2})$
	4		4	$(\mathbf{1, 1, 1; 1, 1, 1, 1, 1, 2, 2, 1})$
1		2	3	$(\mathbf{1, 1, 1; 2, 1, 1, 1, 2, 1, 1, 1})$
1		2	3	$(\mathbf{1, 1, 1; 1, 1, 1, 1, 1, 2, 1, 2})$
1			1	$(\mathbf{1, 1, 1; 2, 1, 1, 1, 1, 2, 1, 1})$
1			1	$(\mathbf{1, 1, 1; 2, 1, 1, 1, 1, 1, 1, 2})$
1			1	$(\mathbf{1, 1, 1; 1, 2, 2, 1, 1, 1, 1, 1})$
1			1	$(\mathbf{1, 1, 1; 1, 2, 1, 2, 1, 1, 1, 1})$

... continuation

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1})$
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1})$
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1})$
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1})$
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1})$
1			1	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1};\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2},\ \mathbf{1},\ \mathbf{1},\ \mathbf{2})$

- One generation contains a $(\mathbf{4},\mathbf{2},\mathbf{1})$ and a $(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$.
- Localization of the three generations:

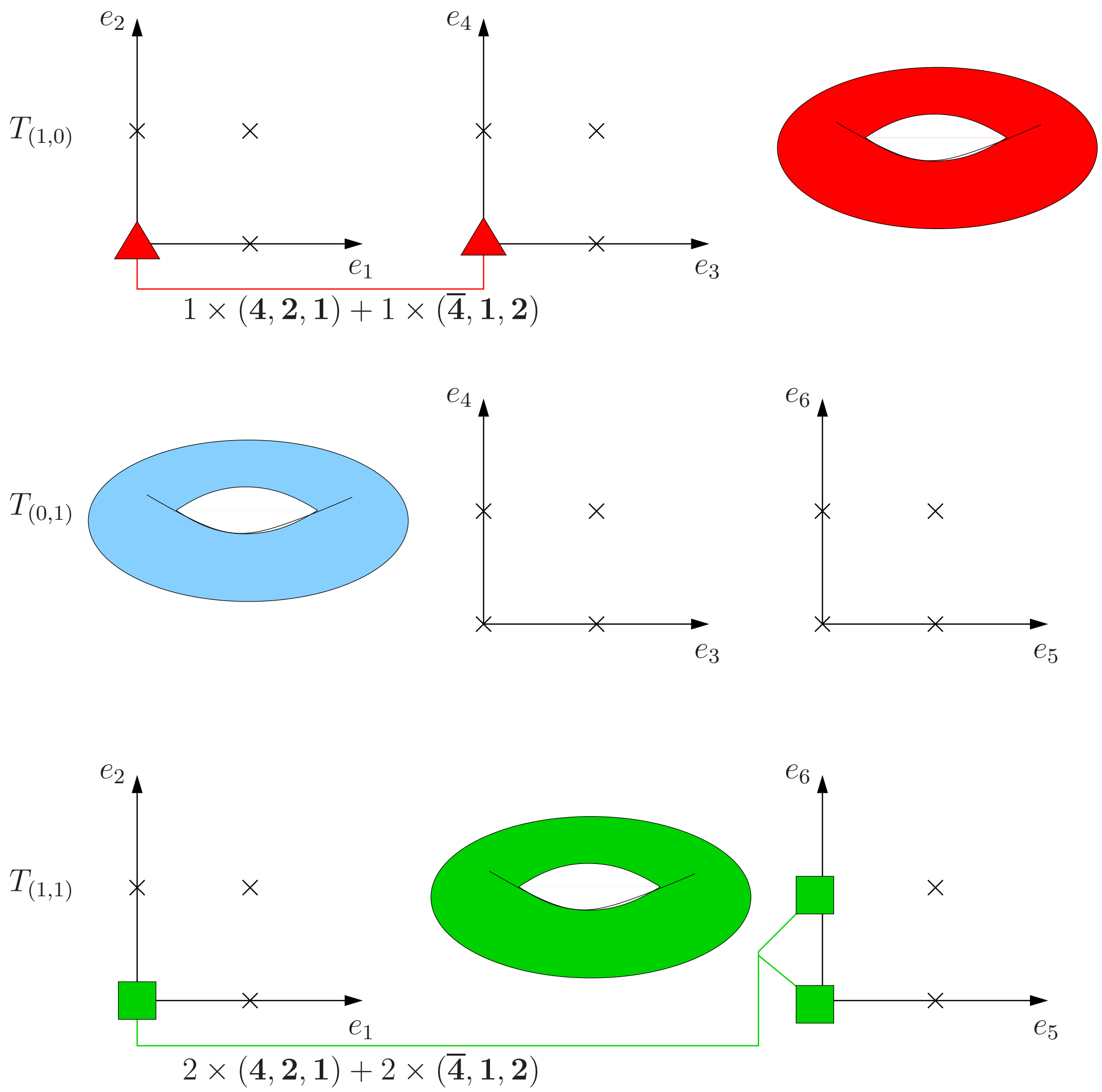


Figure E.3: Localization of the generations with Pati-Salam gauge group.

E.4 Three Generations of SU(5)

- Standard embedding with Wilson lines:

$$A_1 = \left(0^6, \tfrac{1}{2}^2\right)\left(1, \tfrac{1}{2}^2, 0^5\right) \qquad A_2 = \left(\tfrac{1}{2}^3, 0^3, \tfrac{1}{2}, 0\right)\left(1, -\tfrac{1}{2}, 0, \tfrac{3}{2}, \tfrac{1}{2}^2, 1, 0\right)$$
$$A_3 = (0^8)\left(1^2, -1, 0, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{3}{2}\right) \qquad A_4 = \left(0^6, \tfrac{1}{2}^2\right)\left(\tfrac{1}{2}, 0^2, -\tfrac{1}{2}, 0^3, 1\right)$$
$$A_5 = (0^8)\left(1^2, -1, 0, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{3}{2}\right) \qquad A_6 = (0^8)\,(0^8)$$

- Gauge group: $\mathrm{SU}(5) \times \mathrm{U}(1)^4 \times \mathrm{SU}(2)'^4 \times \mathrm{U}(1)'^4$

- Massless spectrum:

untwisted sector:	
3	$(\overline{\mathbf{5}};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
6	$(\mathbf{1};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
3	$(\mathbf{5};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
6		4	10	$(\mathbf{5};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
1	4	2	7	$(\overline{\mathbf{5}};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
1		2	3	$(\overline{\mathbf{10}};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
27	44	14	85	$(\mathbf{1};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
				hidden matter:
8		4	12	$(\mathbf{1};\, \mathbf{2},\, \mathbf{1},\, \mathbf{1},\, \mathbf{1})$
8		4	12	$(\mathbf{1};\, \mathbf{1},\, \mathbf{2},\, \mathbf{1},\, \mathbf{1})$
8		4	12	$(\mathbf{1};\, \mathbf{1},\, \mathbf{1},\, \mathbf{2},\, \mathbf{1})$
8		4	12	$(\mathbf{1};\, \mathbf{1},\, \mathbf{1},\, \mathbf{1},\, \mathbf{2})$
1			1	$(\mathbf{1};\, \mathbf{2},\, \mathbf{1},\, \mathbf{1},\, \mathbf{2})$
1			1	$(\mathbf{1};\, \mathbf{1},\, \mathbf{2},\, \mathbf{2},\, \mathbf{1})$
	4		4	$(\mathbf{1};\, \mathbf{2},\, \mathbf{1},\, \mathbf{2},\, \mathbf{1})$

- One generation contains a $\mathbf{5}$ and a $\overline{\mathbf{10}}$. Thus, this model has a net number of three $(10 - 7 = 3)$ $\mathbf{5}$s and three $\overline{\mathbf{10}}$s.

- Localization of the three generations:

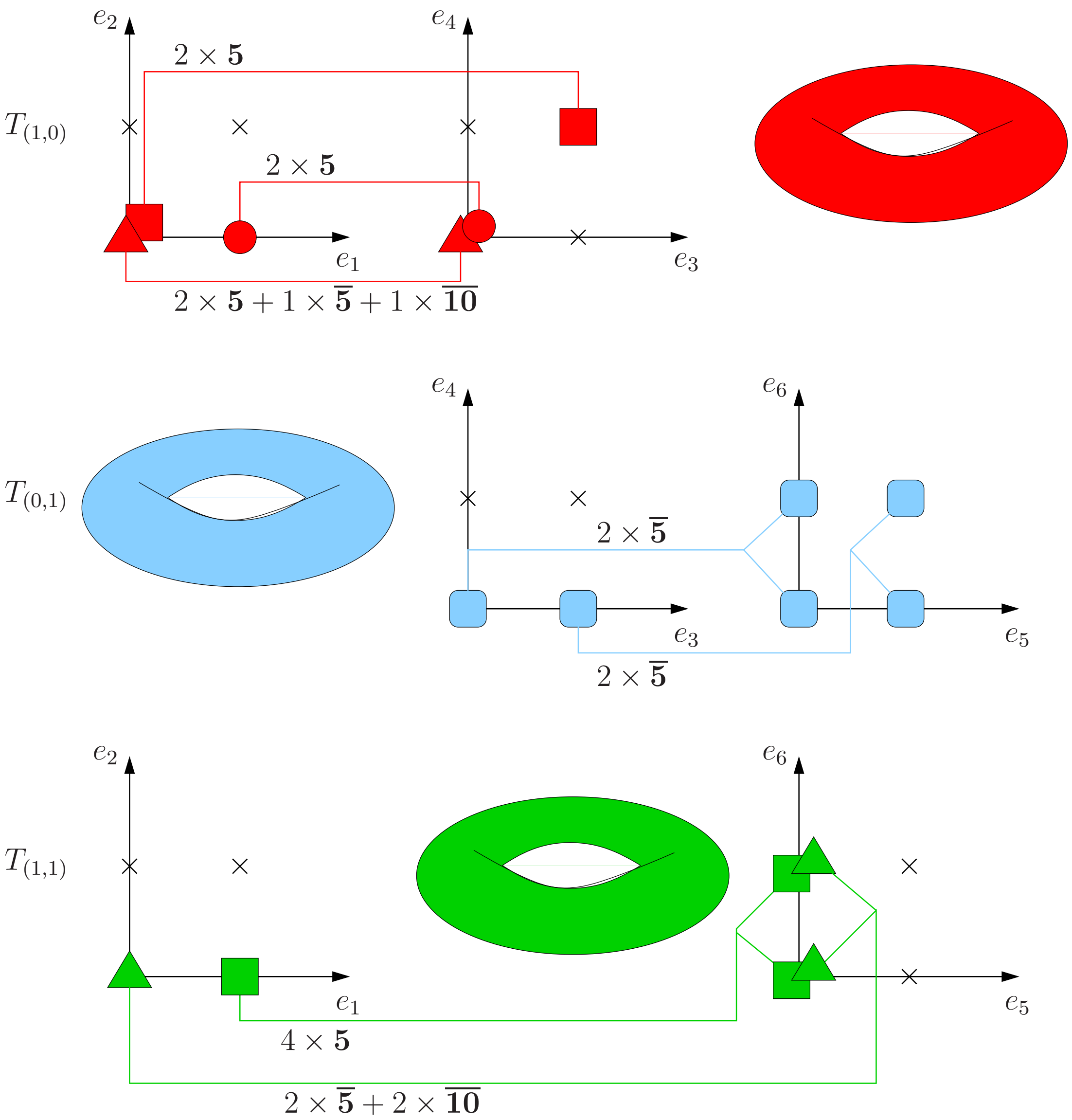


Figure E.4: Localization of the generations with SU(5) gauge group.

E.5 Three Generations of SO(10) - Model 1

- Standard embedding with Wilson lines:

$$A_1 = (0^7, 1)\,(1, 0^7) \qquad A_2 = (0^8)\left(0, 1, \tfrac{1}{2}^4, 0^2\right)$$
$$A_3 = (0^8)\left(0^2, \tfrac{1}{2}, -\tfrac{1}{2}^2, \tfrac{1}{2}, -1, 0\right) \qquad A_4 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$
$$A_5 = (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_6 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$

- Gauge group: $\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{U}(1)'^8$
- Massless spectrum:

untwisted sector:	
6	$(\mathbf{10})$
12	$(\mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
2	2	1	5	$(\mathbf{10})$
2		1	3	$(\overline{\mathbf{16}})$
60	60	54	174	$(\mathbf{1})$

- The $\overline{\mathbf{16}}$ counts as a generation.

- Localization of the three generations:

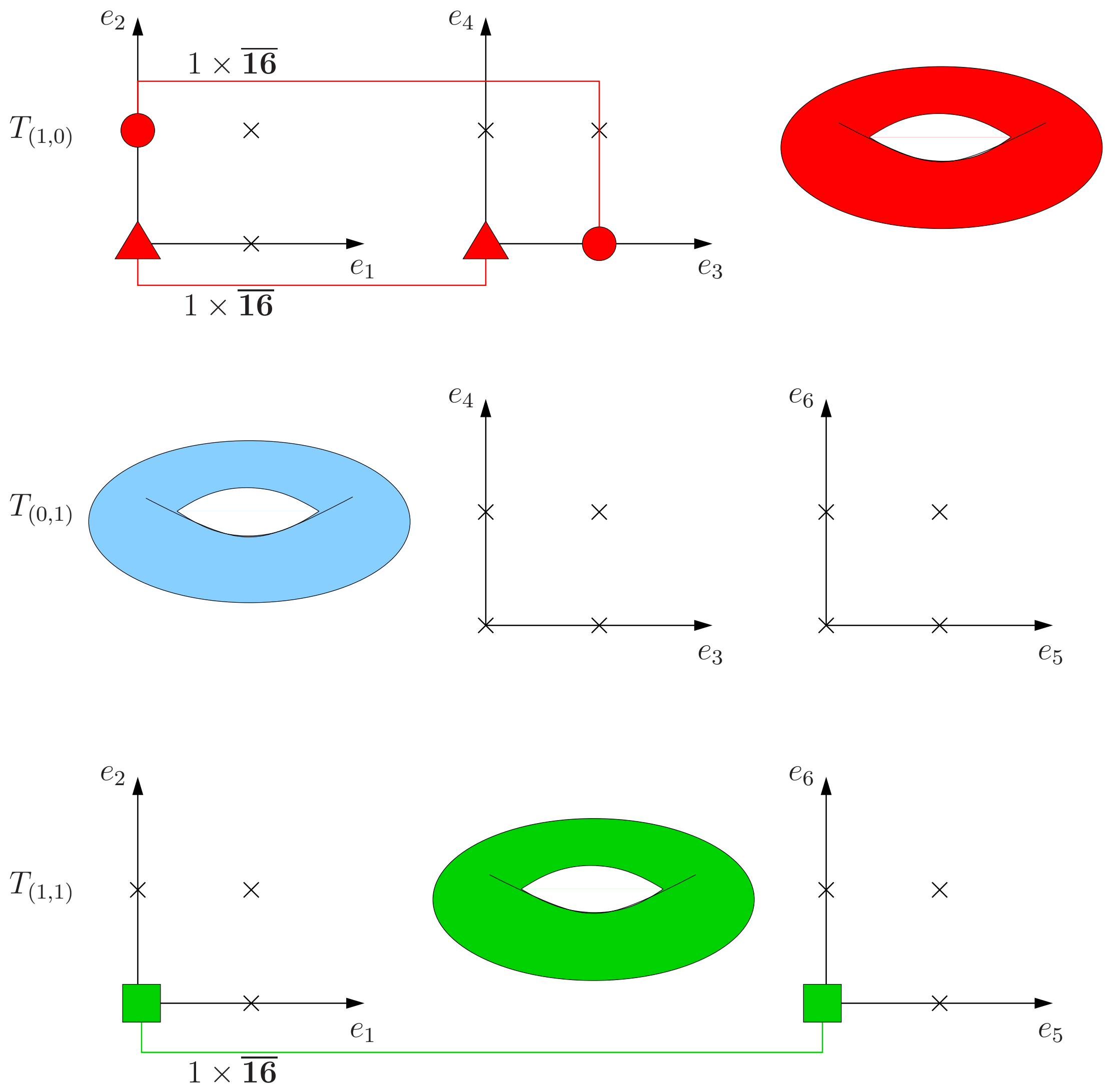


Figure E.5: Localization of the generations with SO(10) gauge group.

E.6 Three Generations of SO(10) - Model 2

- Standard embedding with Wilson lines:

$$A_1 = \left(0^3, \tfrac{1}{2}^4, 1\right)(0^8) \qquad A_2 = (0^8)\left(0, 1, \tfrac{1}{2}^4, 0^2\right)$$
$$A_3 = \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_4 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$
$$A_5 = (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_6 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$

- Gauge group: $\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)'^8$
- Massless spectrum:

untwisted sector:	
6	(**10**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
12	(**1**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1	2	1	4	(**16**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
1			1	($\overline{\mathbf{16}}$; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
2	2	1	5	(**10**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
12	12	6	30	(**1**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
				hidden matter:
2		1	3	(**1**; **2**, **1**, **2**, **1**, **1**, **1**, **1**, **1**)
2		1	3	(**1**; **1**, **2**, **1**, **1**, **1**, **1**, **2**, **1**)
2		1	3	(**1**; **1**, **1**, **1**, **2**, **1**, **2**, **1**, **1**)
2		1	3	(**1**; **1**, **1**, **1**, **1**, **2**, **1**, **1**, **2**)
	2	1	3	(**1**; **1**, **2**, **1**, **1**, **1**, **2**, **1**, **1**)
	2	1	3	(**1**; **1**, **1**, **1**, **2**, **1**, **1**, **2**, **1**)
	2		2	(**1**; **2**, **2**, **1**, **1**, **1**, **1**, **1**, **1**)
	2		2	(**1**; **2**, **1**, **1**, **1**, **1**, **2**, **1**, **1**)
	2		2	(**1**; **1**, **1**, **2**, **2**, **1**, **1**, **1**, **1**)
	2		2	(**1**; **1**, **1**, **2**, **1**, **1**, **1**, **2**, **1**)
		1	1	(**1**; **2**, **1**, **1**, **1**, **2**, **1**, **1**, **1**)
		1	1	(**1**; **2**, **1**, **1**, **1**, **1**, **1**, **1**, **2**)
		1	1	(**1**; **1**, **2**, **1**, **2**, **1**, **1**, **1**, **1**)
		1	1	(**1**; **1**, **1**, **2**, **1**, **2**, **1**, **1**, **1**)
		1	1	(**1**; **1**, **1**, **2**, **1**, **1**, **1**, **1**, **2**)
		1	1	(**1**; **1**, **1**, **1**, **1**, **1**, **2**, **2**, **1**)

- The **16** counts as a generation.

- Localization of the three generations:

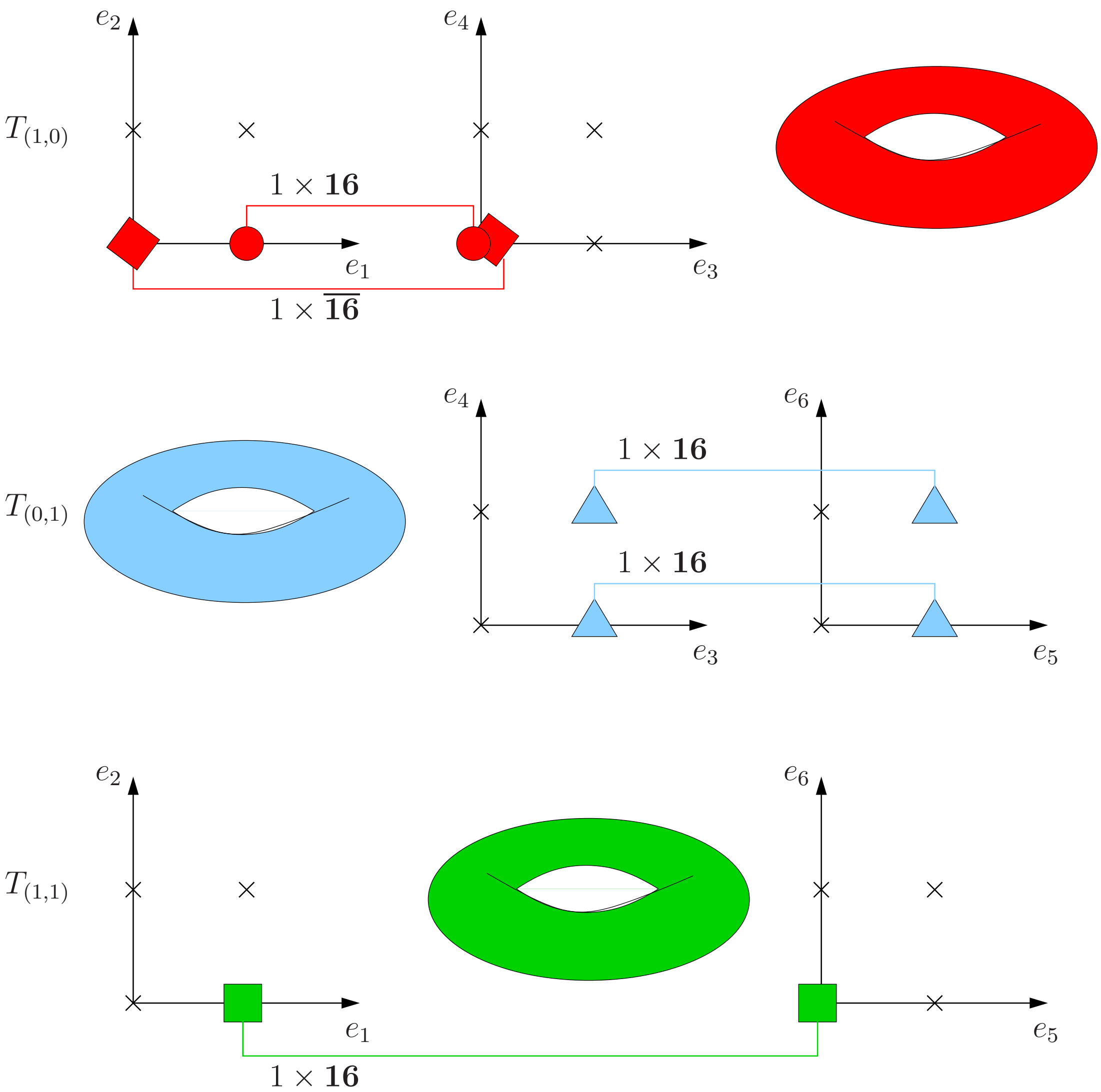


Figure E.6: Localization of the generations with SO(10) gauge group.

E.7 Three Generations of SO(10) - Model 3

- Standard embedding with Wilson lines:

$$A_1 = \left(0^3, \tfrac{1}{2}^4, 1\right)(0^8) \qquad A_2 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$
$$A_3 = (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_4 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$
$$A_5 = \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_6 = (0^8)\left(0, 1, \tfrac{1}{2}^4, 0^2\right)$$

- Gauge group: $\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)'^8$
- Massless spectrum:

untwisted sector:	
6	$(\mathbf{10}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
12	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
2	1	1	4	$(\mathbf{16}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
		1	1	$(\overline{\mathbf{16}}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
2	1	2	5	$(\mathbf{10}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
12	6	12	30	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
				hidden matter:
2	1		3	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
2	1		3	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
2			2	$(\mathbf{1}; \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
2			2	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
2			2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
2			2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
	1	2	3	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1	2	3	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
	1	2	3	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
	1	2	3	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
	1		1	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1		1	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$

In the sum this spectrum is equal to SO(10) model 2 [E.6]. However, the three generations live on different fixed tori.

- The **16** counts as a generation. Thus, this model has a net number of three $(4-1=3)$ generations.
- Localization of the three generations:

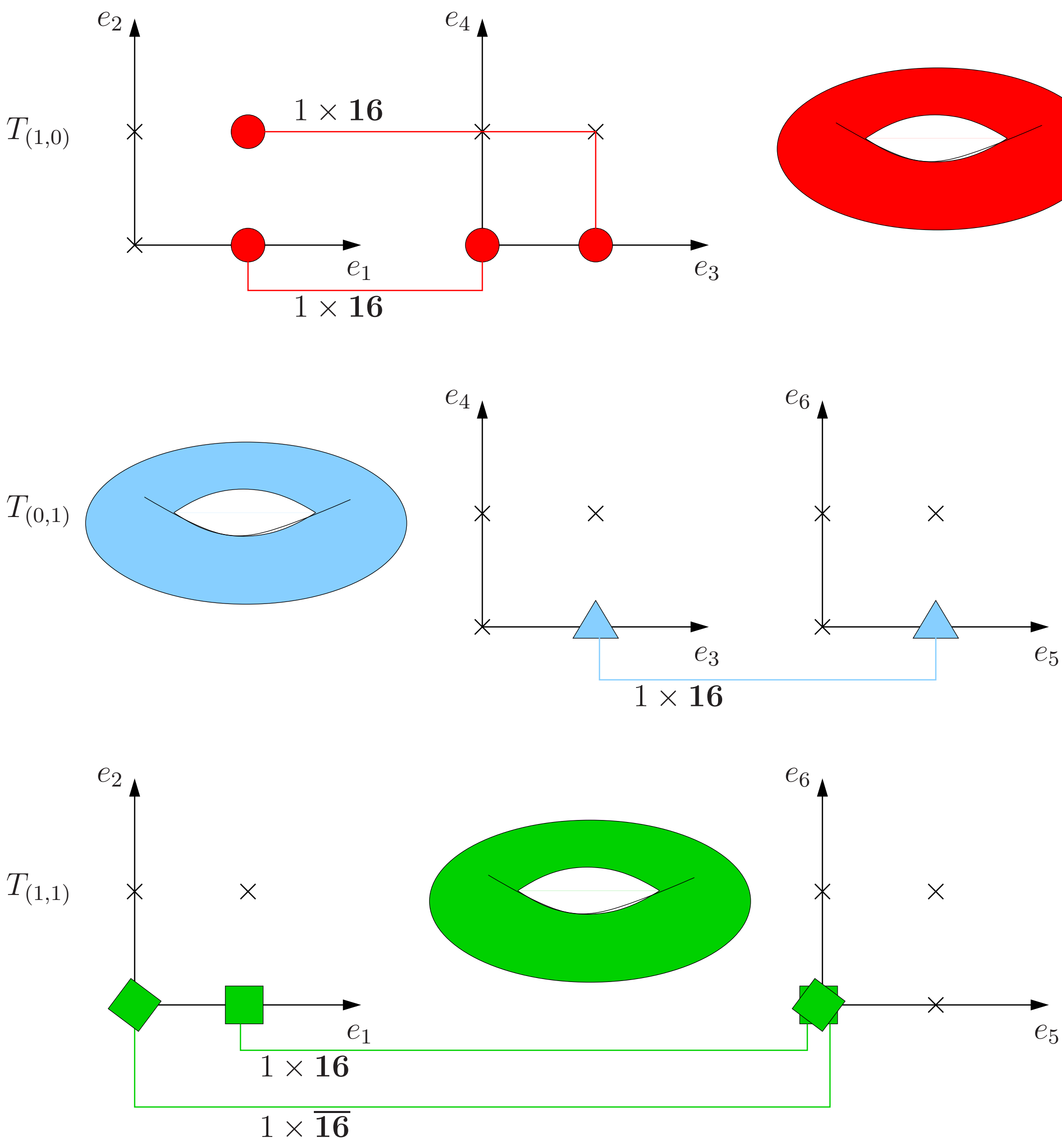


Figure E.7: Localization of the generations with SO(10) gauge group.

E.8 Three Generations of SO(10) - Model 4

- Standard embedding with Wilson lines:

$$A_1 = (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_2 = (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right)$$
$$A_3 = \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_4 = (0^8)\left(0, 1, \tfrac{1}{2}^4, 0^2\right)$$
$$A_5 = (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) \qquad A_6 = \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, 1, \tfrac{1}{2}^4, 0^2\right)$$

- Gauge group: $\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)'^8$

- Massless spectrum:

untwisted sector:	
6	(**10**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
12	(**1**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1	3		4	(**16**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
	1		1	($\overline{\mathbf{16}}$; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
1	4	2	7	(**10**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
6	24	12	42	(**1**; **1**, **1**, **1**, **1**, **1**, **1**, **1**, **1**)
				hidden matter:
1	4		5	(**1**; **1**, **1**, **1**, **2**, **1**, **2**, **1**, **1**)
1	4		5	(**1**; **1**, **1**, **1**, **1**, **2**, **1**, **1**, **2**)
1		2	3	(**1**; **2**, **2**, **1**, **1**, **1**, **1**, **1**, **1**)
1		2	3	(**1**; **1**, **1**, **2**, **1**, **1**, **1**, **2**, **1**)
1			1	(**1**; **2**, **1**, **2**, **1**, **1**, **1**, **1**, **1**)
1			1	(**1**; **2**, **1**, **1**, **1**, **1**, **1**, **2**, **1**)
1			1	(**1**; **1**, **2**, **2**, **1**, **1**, **1**, **1**, **1**)
1			1	(**1**; **1**, **2**, **1**, **1**, **1**, **1**, **2**, **1**)
1			1	(**1**; **1**, **1**, **1**, **2**, **2**, **1**, **1**, **1**)
1			1	(**1**; **1**, **1**, **1**, **2**, **1**, **1**, **1**, **2**)
1			1	(**1**; **1**, **1**, **1**, **1**, **2**, **2**, **1**, **1**)
1			1	(**1**; **1**, **1**, **1**, **1**, **1**, **2**, **1**, **2**)
		2	2	(**1**; **2**, **1**, **1**, **1**, **1**, **1**, **1**, **2**)
		2	2	(**1**; **1**, **2**, **1**, **1**, **1**, **1**, **1**, **2**)
		2	2	(**1**; **1**, **1**, **2**, **1**, **2**, **1**, **1**, **1**)
		2	2	(**1**; **1**, **1**, **1**, **1**, **2**, **1**, **2**, **1**)

- The **16** counts as a generation. Thus, this model has a net number of three ($4-1=3$) generations.

- Localization of the three generations:

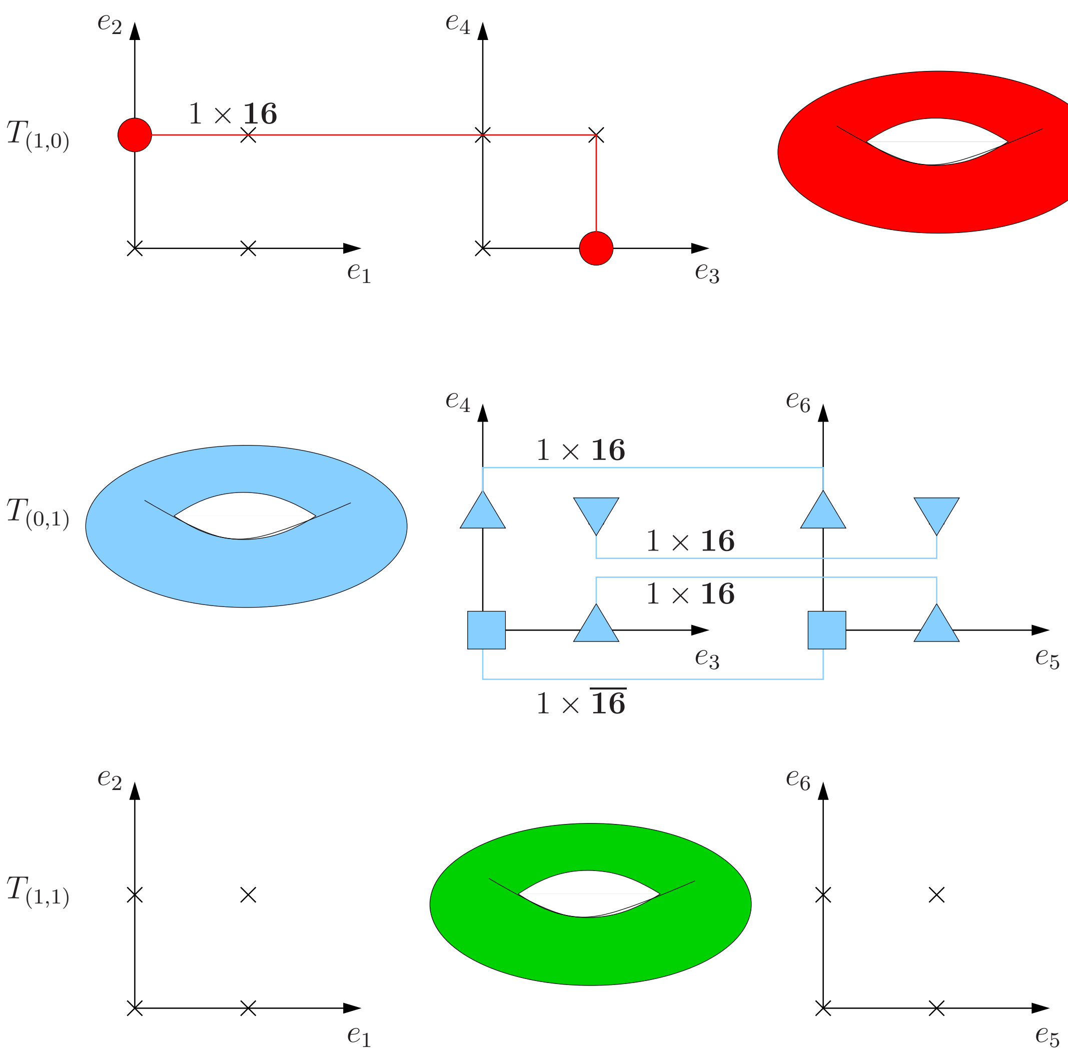


Figure E.8: Localization of the generations with SO(10) gauge group.

E.9 Three Generations of SO(10) - Model 5

- Standard embedding with Wilson lines:

$$\begin{aligned}
A_1 &= (0^8)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right) & A_2 &= \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, 1, \tfrac{1}{2}^4, 0^2\right) \\
A_3 &= (0^8)\left(0, 1, \tfrac{1}{2}^4, 0^2\right) & A_4 &= \left(0^3, \tfrac{1}{2}^4, 1\right)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right) \\
A_5 &= (0^8)\left(1, 0^3, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}^2\right) & A_6 &= \left(0^3, \tfrac{1}{2}^4, 1\right)\left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right)
\end{aligned}$$

- Gauge group: $\mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SU}(2)'^8$
- Massless spectrum:

untwisted sector:	
6	$(\mathbf{10}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
12	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

twisted sectors:

$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	$\sum$	
1	1	1	3	$(\mathbf{16}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1	1	1	3	$(\mathbf{10}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
6	6	6	18	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
				hidden matter:
1	1		2	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1	1		2	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
1	1		2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
1	1		2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
1		1	2	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
1		1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
1		1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1		1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
1			1	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1			1	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
1			1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
1			1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$
	1	1	2	$(\mathbf{1}; \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1	1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
	1	1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1	1	2	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$
	1		1	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$
	1		1	$(\mathbf{1}; \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
		1	1	$(\mathbf{1}; \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$

... continuation

$T_{(0,1)}$	$T_{(1,0)}$	$T_{(1,1)}$	$\sum$	
		1	1	$(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
		1	1	$(\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
		1	1	$(\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$

- The **16** counts as a generation.
- Localization of the three generations:

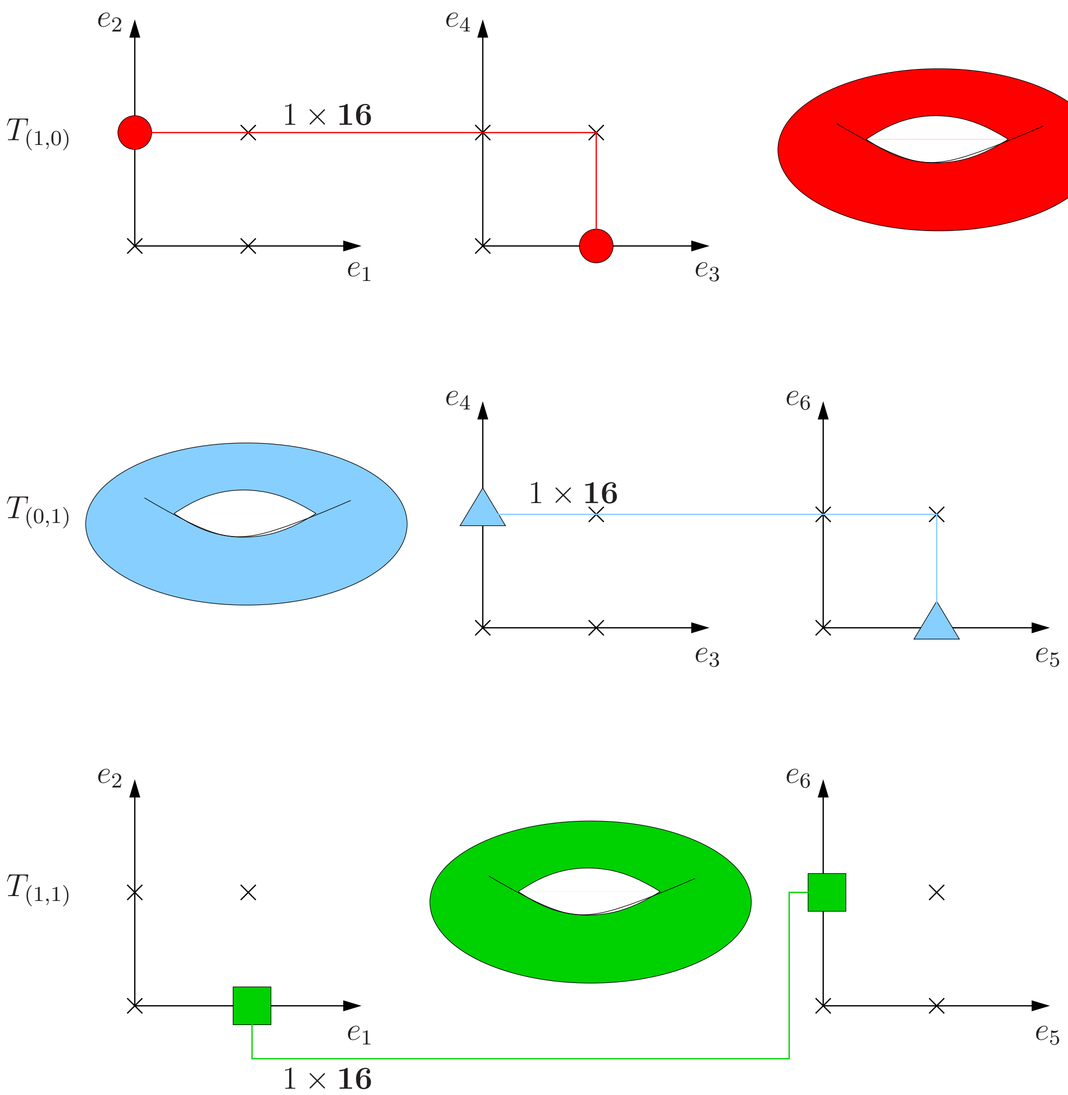


Figure E.9: Localization of the generations with SO(10) gauge group.

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