Local Anomaly Cancellation in Heterotic $E_8 \times E_8$ Orbifold Compactifications with Wilson Line Backgrounds

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Abstract

In the context of physics with extra dimensions the concept of brane worlds has triggered a lot of research over the past few years. Brane world like models were firstly found in type II string theory compactifications on singular manifolds. Inspite of the singularities, string theory on these manifolds is well–defined. The branes within a brane world live at the singularities and are $p+1$ dimensional extended objects that overlap in the uncompactified dimensions. In internal space they can overlap, intersect or be seperated from each other. Each brane has its own dimension, carries its own gauge group and particle spectrum and thus constitutes a potential candidate to describe our world. Therefore, the phenomenology depends on the brane we are living at, or in other words, where we are localized in internal space. Due to the large amount of symmetries contained in string theories, the type II brane worlds are highly constrained. These constraints are expected to render the low energy effective field theories appearing on the branes consistent. In fact there have been several crosschecks to verify this expectation, e.g. all anomalies are cancelled locally at the branes.

Triggered by this development brane world like models have been studied in field theories, e.g. in the context of Grand Unification from extra dimensions. However, unlike string theory, field theory on a singular manifold is potentially inconsistent. This inconsistency is often signalled by the appearance of anomalies at the singularities. Inspired by the brane fields of the string scenario one tries to cure the theories by the introduction of additional fields at the singularities. These additional fields are then chosen in such a way, that all anomalies cancel. However, this choice is not unique. Therefore the hope is to embed the field theory model in string theory completely and so profit from stringy symmetries. Most conveniently, such an embedding is achieved in heterotic theories because there the relation to Grand Unified Field Theories is the clearest.

Interestingly, heterotic string theory has not been studied from a local perspective in detail before. This thesis fills the gap and paves the way for a better understanding of field theory brane worlds. We consider several examples of a special class of heterotic compactifications, where non–trivial local physics is expected, i.e. heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ orbifolds with Wilson line backgrounds. By developing a local perspective we show that indeed a brane world like picture emerges. As an important result we prove that the local massless spectrum at such a brane can always be traced back to the global spectrum of a (different) orbifold without Wilson lines. One particular implication of this result is that the use of (discrete) Wilson lines for the construction of phenomenologically interesting models has to be rethought. We show that stringy constraints render the brane spectra consistent. Using our local picture we are able to compute the local anomalies appearing at the different branes for our examples and show that they can all be cancelled by a local version of the Green–Schwarz mechanism at the same time.

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Contents

Chapter 1

Introduction

Nowadays, theoretical elementary particle physics is able to explain many physical effects, mostly in overwhelming agreement with experimentally observed data. Nevertheless the theoretical models providing these explanations sometimes suffer from conceptual problems or are even incompatible with others, indicating that we are not at the end of the story. Three of these indications are now discussed:

Firstly, theoretical physics is based on two different concepts, quantum field theory and general relativity, that seem to rival each other. Quantum field theory is an essential ingredient of the standard model of elementary particles (SM) that has passed a vast number of experimental tests with astonishing precision up to now. As a quantum theory its effects get more and more important the smaller the scale gets. In other words, the concept of quantum field theories is indispensible for an exact description of low scale (or equivalently high energy) physics. On the other hand Einstein's theory of general relativity is needed for a description of physics at large scales (or equivalently low energies). It also passed a lot of experimental tests e.g. in astronomy and cosmology. General relativity is a classical field theory like Maxwell's electrodynamics. The problem is, that unlike electrodynamics it can not be quantized in a consistent manner. Therefore a formulation of general relativity as a quantum field theory is impossible. Still its accordance with nature suggests not to discard it, a conceptual mismatch.

Secondly, there is another trend in today's theoretical elementary particle physics, the concept of unification. Beside gravity there were formerly three different theories describing electromagnetic, weak and strong forces. The quantized theories corresponding to electromagnetic and strong forces are called quantum electro and quantum color dynamics (QED and QCD), respectively. By Glashow, Weinberg and Salam (GWS) it was shown that the theories of electromagnetism and weak interactions can be incorporated much more elegantly in one single theory, the so called electroweak quantum field or GWS theory. It suggests that at scales above roughly 100 GeV only one interaction (the electroweak one) is visible and that at scales below the symmetry is broken down to the original ones. This breaking is implemented by the well–known Higgs mechanism. Again this concept passed many experimental tests although the Higgs particle has not yet been found, but moreover it is theoretically very attractive as a first step of approaching the goal of a formulation of a so called theory of everything (TOE). A TOE should contain all known forces above a certain scale, the so called Planck scale $M_P \sim 10^{19}$ GeV. Below that scale its interaction then splits into gravitational and other interactions. At a further so called GUT (Grand Unified Theory) scale $M_{\mathrm{GUT}} \sim 10^{15}$ GeV the strong forces should seperate and we would arrive at the GWS model. Clearly a GUT theory has to contain the SM gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ so that the described cascade could be realized by a chain of symmetry breakings. Possible GUT groups are given by SU(5), SO(10), E_6, E_7 and E_8 (every group is a real subgroup of the successor).

Thirdly, there is the so called hierarchy problem, which is connected to unification. The electroweak symmetry breaking scale M_W is connected to the mass of the Higgs particle. However, in the quantum theory this mass gets quadratically divergent quantum corrections of the order of the cut–off scale. But in a GUT the cut–off scale would naturally be the M_{GUT}, or if gravity is included even M_P. But these corrections then destroy the electroweak scale, which has been fixed by experiments. One possbile way out of this quandary is provided by supersymmetry (SUSY), a symmetry that assigns a fermionic degree of freedom (d.o.f.) to every bosonic one in the theory and vice versa. In supersymmetric theories quadratically divergent corrections are absent and only logarithmic corrections survive. Although then the severe problem of quadratically divergent quantum corrections is solved it is traded for the problem that SUSY has not been observed in experiments. Therefore SUSY has to be broken at scales of order TeV as well. There are different proposals how such a breaking could be realized in accordance with experiments, although it is very hard to verify or falsify these proposals experimentally.

String theory extends the concept of point particles to one dimensional extended objects, the strings. In its quantized version it incorporates both, quantum field theory and general relativity in a consistent manner. Therefore the problem of quantizing general relativity for point particles is no longer present. By enforcing the spectrum to contain fermions as well as bosons free string theory predicts SUSY and therefore already contains a possible solution to the hierarchy problem as stated above, a very nice feature. On first sight less attracive, string theory also predicts that we should live in a ten $(9 + 1)$ dimensional world. Have we traded two problems for another? Independent of string theory Kaluza and Klein (KK) invented the idea of extra dimensions in the 1920's and showed that by curling up those extra dimensions on very tiny scales (a so called compactification) one does not only get rid of them but also produces symmetries that could serve as gauge groups. However, it was realized that in such theories every massless state is accompanied by infinitely many massive states (forming the so called

KK tower), whose mass scale is given by the inverse of the compactification size. On the other hand, if this size is small enough the masses are sufficiently heavy so that all massive states effectively decouple from the effective four dimensional physics. This is called the zero–mode limit, because in this limit only modes which are constant over the whole compactification manifold survive. Applied to string theory sometimes the symmetries arising upon compactification are big enough to contain GUT groups. Therefore the appearance of extra dimensions can be seen as an advantage, since the unification picture as described above is likely to appear as a consequence of compactification. In summary it seems that string theory provides not only an appropriate framework to tackle today's theoretical physics problems, but already contains some solutions.

There are five different free supersymmetric string theories known, type I theory with gauge group[1] $\mathrm{Spin}(32)/\mathbb{Z}_2$, type IIA and type IIB [2, 3], heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ and heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ string theory [4, 5, 6], all living in ten dimensions. The type I and heterotic theories possess $\mathcal{N} = 1$ SUSY and are already equipped with some gauge groups, the type II theories possess $\mathcal{N} = 2$ SUSY and come without gauge groups. As has become apparent over the last decade, all these string theories are connected to each other by a web of so–called dualities. E.g. one can start with a compactification of type IIA theory on a circle and apply a discrete symmetry transformation called T–duality [7, 8]. This symmetry maps the circle on another circle that then can be decompactified again and one reaches the type IIB theory. Further on there is another duality transformation that connects a weakly coupled description of one theory with the strongly coupled of another, the so–called S–duality [9, 10]. E.g. the heterotic and type I theories with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$ both compactified on a circle are S–dual to each other [10]. Interestingly in this web of dualities one encounters another (non–string) theory $\mathcal{N} = 1$ supergravity (SUGRA) in eleven dimensions [11, 12]. Compactified on a circle it reproduces the low energy description of type IIA string theory [13, 9]. Therefore it was proposed that eleven dimensional SUGRA should actually provide the low energy description of a yet to be determined eleven dimensional theory, termed M–theory (see e.g. [14, 15] and references therein, for a texbook introduction see [16]). Then by dualities and compactifications string theories appear as different effective descriptions of M–theory, today's only serious candidate of a TOE. By studying string theories we hope to explore some regions and properties of M–theory.

The fact that M–theory is the only known potential TOE does not mean that it has to be realized in nature. Therefore it is essential not to loose contact to phenomenology. In fact there are many restrictions on compactification schemes that can be drawn. As we have seen, the compactified four dimemsional theory should provide the SM or at least a GUT gauge group. Further on, if SUSY

[1] This gauge group (and the one for the corresponding heterotic theory) is often incorrectly stated as SO(32) in the literature. For a detailed account see [1].

should be present at all, then the SM particle content is only embeddable in a theory with $\mathcal{N} = 1$ SUSY in $D = 4$. The amount of supersymmetries in string theories is always larger, so at least some SUSY must be broken. It turned out that all known compactifications that break SUSY completely lead to tachyons in the spectra. These tachyons lead to enormous problems for model building (see e.g. [17] and references therein). Then by enforcing $\mathcal{N} = 1, D = 4$ SUSY one gets restrictions on the curvature of the manifold on which a given string or M–theory is compactified. For M–theory (or to be precise its low energy description $\mathcal{N} = 1, D = 11$ SUGRA) the compactification manifold must possess the holonomy group G_2, for a heterotic theory the corresponding manifold is enforced to be a Calabi–Yau (such manifolds possess SU(3) holonomy) [18, 19]. For the other string theories the situation is more involved. Although some topological information is known about such manifolds, a detailed geometrical descripition could not yet be given. Therefore the accessible amount of four dimensional physics is very limited.

Fortunately it has been realized that so called orbifolds appear as certain limits of Calabi–Yau's [20, 21]. Orbifolds are basically smooth manifolds with some isolated singular points, so called orbifold singularities. Compared to Calabi–Yau's orbifolds are much better understood. Therefore computations of compactified physics get either simpler or even possible. Even despite the singular points string theory compactifications on orbifolds are well–defined. This follows from the fact that the conformal field theories (CFTs), on which the string construction is based, are not singular at all, even if an orbifolding has been performed. In the evaluation of string spectra on orbifolds it has been shown that the associated states fall into basically two categories. Firstly, there are so called untwisted states. Their center of mass (CMS) is free to move over the whole orbifold. Untwisted states are very similar to states one would expect for compactifications on smooth manifolds. In fact they can be derived using pure field theory considerations. But secondly string theory requires the existence of so called twisted states. Unlike the untwisted states, the twisted states are not allowed to propagate freely but are confined to the orbifold singularities. The prediction of twisted states is due to a very strong symmetry called modular invariance. Modular invariance arises as a symmetry of conformal field theories (CFTs), on which string constructions are heavily based.

Orbifold compactifications of heterotic string theories had been studied extensively long before a possible existence of M–theory was actually known. By using an appropriate embedding, the orbifold does not only break SUSY but also gauge symmetry. Then the aim was to choose the embedding in such a way, that the gauge symmetry was directly broken to the SM gauge group or at least to a GUT group. There were several possibilities to control such a breaking, one of which was provided by Wilson lines. Wilson lines are gauge transformations that are assigned to non–vanishing one–cycles and therefore only contribute in topological non–trivial situations. In fact by taking Wilson lines into account SM gauge groups could be reconstructed on the zero–mode level [22, 23, 24]. But there were still other problems with these theories that could not be overcome: It is e.g. notoriously hard to provide the observed weak mixing angle (see e.g. [25] and references therein).

However, by considerations of heterotic orbifolds it was realized that by mathematical abstraction the notion of orbifold singularities could be disentangled completely from the Calabi–Yau's (see e.g. [26, 27] and references therein). Meanwhile orbifold singularities or generalized relatives appear in almost all phenomenologically interesting string compactifications, no matter which string theory was considered as starting point. In a generalized construction starting from the low energy description of M–theory orbifold singularities (and so called conical singularities) were recently placed on seven dimensional manifolds to yield $\mathcal{N} = 1, D = 4$ SUSY, non–Abelian gauge groups and chiral fermions directly from the low energy description of M–theory, a very non–trivial task [28]. Since no string theory computations were possible in this regime field theory arguments had to be used. The problem was that unlike string theory field theory does not uniquely predict twisted states (or in this specific example generalized analogs). This is due to the fact that modular invariance, the symmetry that proved so powerful in string theories, is not a symmetry of field theories. However, a way out was provided by local anomaly considerations. It has been shown that the untwisted states (that are calculable from pure field theory) induce anomalies at the singularities. With these anomalies the theory would have been inconsistent. Therefore a minimal set of states is conjectured to live at the singularties, where "minimal" is meant in the sense that all anomalies cancel. These states are then regarded as analogs of the twisted states that appear in the string constructions. The weak point in this argumentation is of course that anomaly considerations can never predict non–anomalous states or combinations of states.

Other orbifold relatives, so called orientifolds provide compactification manifolds appropriate to yield $\mathcal{N} = 1, D = 4$ SUSY from type II theories [29, 30, 31]. In these compactifications the type II and type I theories were shown to be closely related in the so called D–brane picture [32]. D–branes are very interesting because they are defined as the set of points where open strings can end. They are dynamical extended objects of the theory and couple to so called RR–potentials. In orientifold models the existence of D–branes is often enforced by tadpole cancellation conditions. D–branes give rise to a vast number of physical effects, but here we want to stress one aspect only. In orientifold models different spectra arise at different D–branes. There is no gauge group right from the start that just has to be broken. Each brane has its own dimensionality, generates and supports its own gauge group and carries its own particle spectrum (of course within given consistency conditions). Again this is a very local picture. If such a scenario was realized in nature our world definitely would depend on the brane on which we are localized or in other words on the position of our universe in the compactified space.

The local picture has influenced many aspects of string and string inspired phenomenology. As a last example we would like to consider field theory orbifold GUTs, where usually a five [33, 34, 35, 36, 37] or six dimensional [38, 39] SUSY gauge theory is considered as starting point (for a discussion in arbitrary dimensions see [40]). Then, very similarly as in the original heterotic orbifold compactifactions again e.g. Wilson lines are used to break to the SM gauge group. Local anomaly considerations partially replace the string computation of twisted states.

We are now in the position to put this thesis into the right context: There are two reasons why the study of heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory orbifolds is interesting. Firstly (top–down perspective), since the free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory is S–dual to M–theory on $S^1/\mathbb{Z}_2$ all its orbifold compactifications should appear as special cases of the M–theory compactifications considered above. In fact the heterotic orbifolds are more special because the compactifications start from ten and not from eleven dimensions, on the other hand they are more restrictive since we can use the power of string theory to compute twisted states explicitly. Therefore the cancellation of local anomalies for heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ orbifolds provide a crosscheck of the field theory reasoning given in [28]. The second reason for the study of heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory orbifolds (bottom–up approach) was, as pointed out above, their similarity to field theory orbifold GUTs. Again the hope was that an analysis of anomaly cancellations within string theory helps for the understanding of field theory orbifolds

From the discussion above it could have been suspected that all local anomalies cancel due to the large amount of symmetries incorporated in string theories. In fact for the zero–mode spectra of heterotic orbifolds without Wilson lines global anomaly cancellation has already been shown in 1987 by the authors of [41]. However, by inclusion of Wilson lines the situation gets more involved. In [42] the zero–mode spectrum of a model with Wilson lines was presented in which local anomaly cancellation seemed spoiled. This possible contradiction actually triggered parts of our work.

The thesis is organized as follows: In chapter 2 we develop a new local perspective on heteretic orbifolds. Starting point is a very brief discussion of free heterotic string theory, followed by generalities about compactification manifolds. We then move on to the specific realization of heterotic orbifold models with Wilson lines and discuss many of their aspects in detail. In the analysis of orbifold constraints, we show as a byproduct that the so called level matching condition is often stated incorrectly in the literature and how it has to be corrected. We then introduce a set of examples that will repeatedly be reconsidered throughout the thesis. Up to this point the whole discussion of heterotic orbifolds is still on the (old) level of global zero–modes. We close the first chapter with the development of a new local brane–like picture for heterotic orbifolds and show, that at such a brane an orbifold model with Wilson lines locally always looks exactly like another orbifold model without Wilson lines. Therefore, as long as the compactification radii are fixed and there is no communication between different branes, an observer can never judge wether he lives in a model with or without Wilson lines.

Equipped with the new local picture we move on to the computation of anomalies in chapter 3. We start with a brief introduction to the subject and continue with the calculation of global zero–mode anomalies for all different examples. We then move on to another central part of this work, where we discuss how anomalies on orbifolds are localized and collect all local contributions. We are able to show, that in accordance with the local picture developed in chapter 2, all local anomalies can be related to global anomalies of (possibly) different models. All global zero–mode anomalies have been shown to cancel by the so called Green–Schwarz mechanism. We introduce this mechanism and check zero–mode anomaly cancellation for our models by direct calculations. We then introduce a new local version of the Green–Schwarz mechanism and show that it is really appropriate to cancel all local anomalies present. We conclude in chapter 4 and give a brief outlook. The appendices A and B contain information on group theory and SUSY multiplets. They are meant for supplementary reading.

Some results of this work have been published in

- F. Gmeiner, S. Groot Nibbelink, H.P. Nilles, M.G.A. Walter, "Localized Anomalies in Heterotic Orbifolds" *Nucl. Phys.* **B648** (2003) 35-68 [hep–th/0208146].

- S. Groot Nibbelink, H.P. Nilles, M. Olechowski, M.G.A. Walter, "Localized Tadpoles of Anomalous Heterotic U(1)'s" *Nucl. Phys.* **B665** (2003) 236-272 [hep–th/0303101].

- S. Groot Nibbelink, M. Hillenbach, T. Kobayashi, M.G.A. Walter, "Localization of Heterotic Anomalies on Various Hyper Surfaces of $T^6/\mathbb{Z}_4$" *to appear in Phys. Rev.* **D** (2004) [hep–th/0308076].

Chapter 2

Heterotic Orbifolds

In this chapter we examine a special class of heterotic $E_8 \times E_8$ orbifolds, namely those with non–trivial constant gauge field backgrounds, so–called Wilson lines. To do so, we start with a brief review of heterotic $E_8 \times E_8$ string theory [4] in section 2.1, mainly to set up some conventions. In section 2.2 we follow [43] and consider generalities about compactification manifolds. We rederive some well–known restrictions on them. In the next two sections we introduce geometrical orbifolds [20, 21], a key ingredient for our work, and show how heterotic $E_8 \times E_8$ string theory can be defined on them, in particular with inclusion of Wilson lines. For a very complete account on this topic see [44] and references therein. We continue with an extensive discussion of orbifold Hilbert spaces (following [43] again) in section 2.5. In section 2.7 we derive string theoretical constraints on the defining parameters of heterotic orbifolds with Wilson lines. Here we use techniques of [16, 45] and comment on constraints from level–matching that are usually not considered in the literature but are relevant for non–prime orbifolds. We then introduce three examples of heterotic orbifolds that we will discuss throughout the whole thesis and derive their massless particle content. Section 2.9 is the heart of the first chapter and devoted to the concept of so–called brane (or local equivalent) models, wich can be used to decompose heterotic orbifolds with Wilson lines in terms of simpler ones without. The concept of local equivalent models has first been introduced in [46, 47, 48] on basically field theoretic grounds. The introduction given here is more stringy.

Useful basics about group theory are collected in appendix A, some supersymmetric (SUSY) multiplets are listed in appendix B. Please note, that some conventions used in this thesis are only introduced there.

2.1 The Free Heterotic String

This section is devoted to a brief discussion of free heterotic string theory [4]. However, it is far from being complete. We assume, that the reader is already familiar with heterotic and/or superstring theory constructions. The main purpose is to review some facts needed later in this work and to introduce notation. In the construction we follow [49, 16].

We start this section by stating the setup of the superconformal field theory (SCFT) construction of the free heterotic string. Then we move on to a discussion of the Neveu–Schwarz/Ramond (NS/R) formulation of a set of fermionic SCFT's. This part will be more detailed, since many of the steps performed there are relevant for the orbifold discussion as well. Finally, we derive the massless spectrum of free heterotic $E_8 \times E_8$ string theory explicitly and interpret the result.

2.1.1 Setup

With D we denote the critical dimension of a string theory or the number of spacetime dimensions, in which a field theory is defined. For later convenience we define d to label the number of uncompactified dimensions. Then we take lowercase greek indices $\mu, \nu, \ldots$ to run from 0 to $d-1$ and capital latin indices $M, N, \ldots$ to run from 0 to $D-1$. For the discussion of the massless spectra only transverse dimensions give rise to physical excitations (light–cone gauge). In this case we use the same indices as before, but restrict them to run from 2 to $D-1$ ($d-1$), only. Compactified indices will be denoted by $m, n, \ldots$ and take values between d and $D-1$. For the heterotic string there are additional gauge degrees of freedom (d.o.f.), which will be labeled by $I, J, \ldots$. We use mostly plus spacetime metrics. The worldsheet coordinates of the string will be denoted either by a spacelike σ and a timelike τ, by $w = \sigma + i\tau$ and $\bar{w}$ or $z = \exp(-iw)$ and $\bar{z}$. We are dealing with closed strings only and take their length to be 2π.

The free heterotic string is built from $D = 10$ copies of free bosonic worldsheet fields $X^\mu_L(z)$ on the leftmoving side (i.e. holomorphic in z) and $D = 10$ copies of free bosonic fields $X^\mu_R(\bar{z})$ together with free fermionic fields $\tilde{\psi}^\mu(\bar{z})$ on the rightmoving side (antiholomorphic in z). The sums $X^\mu(z, \bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$ are exactly the constituents describing the closed bosonic string. Additionally, there is an internal symmetry, provided by 32 leftmoving fermionic[1] fields $\lambda^I(z)$. This is the full (CFT) field content of the free heterotic string. The total central charges vanish on both sides (ghosts included). The setup is schematically shown in figure 2.1.

[1] This is called the fermionic construction.

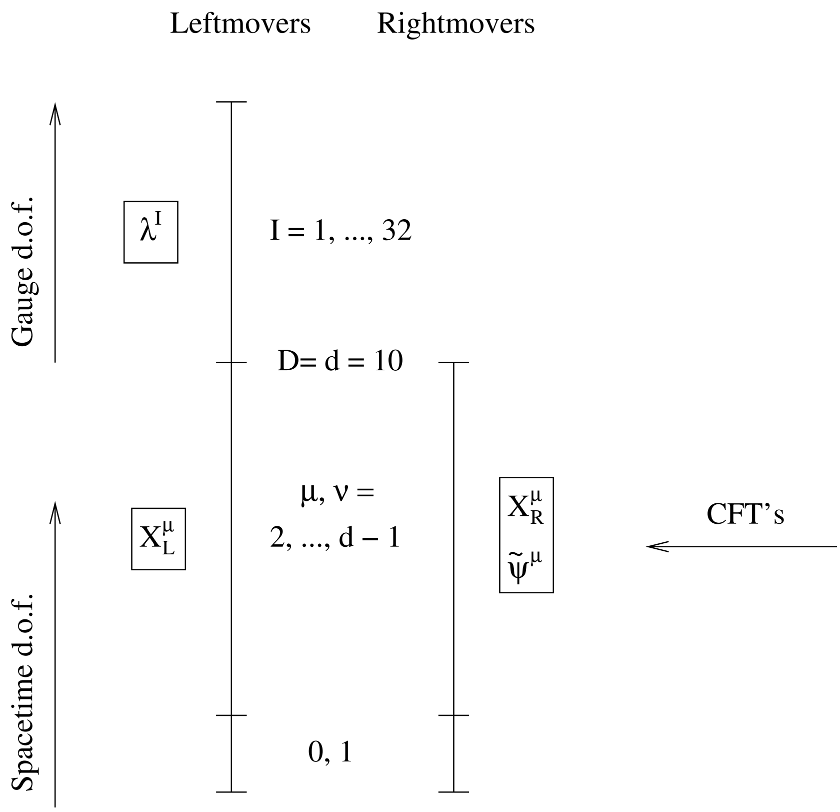

Figure 2.1: Sketch of the CFT construction of the free heterotic string, living in $D = d = 10$ critical and non–compact dimensions. Left– and rightmoving fields are shown in boxes.

2.1.2 Mode expansions and the NS/R Formulation

Consider the bosonic worldsheet fields X_L^μ and X_R^μ and combine them into complex pairs like

$$\begin{aligned} Z^i_{L/R} &\equiv X^{2i}_{L/R} + iX^{2i+1}_{L/R}\,, \qquad & Z^{\bar{\imath}}_{L/R} &\equiv X^{2i}_{L/R} - iX^{2i+1}_{L/R}\,, \\ Z^i &\equiv Z^i_L + Z^i_R\,, & Z^{\bar{\imath}} &\equiv Z^{\bar{\imath}}_L + Z^{\bar{\imath}}_R\,. \end{aligned}$$

Then, by using (anti–)holomorphicity[2], the most general mode expansions for these bosons are given by

$$\begin{aligned} Z^i_L(z) &= z^i_L - i\frac{\alpha'}{2}p^i_L \ln z + i\sum_{r_i\neq 0}\frac{\alpha^i_{r_i}}{r_i z^{r_i}}\,, \\ Z^{\bar{\imath}}_L(z) &= z^{\bar{\imath}}_L - i\frac{\alpha'}{2}\bar{p}^i_L \ln z + i\sum_{s_i\neq 0}\frac{\alpha^{\bar{\imath}}_{s_i}}{s_i z^{s_i}}\,, \\ Z^i_R(\bar{z}) &= z^i_R - i\frac{\alpha'}{2}p^i_R \ln \bar{z} + i\sum_{\tilde{r}_i\neq 0}\frac{\tilde{\alpha}^i_{\tilde{r}_i}}{\tilde{r}_i \bar{z}^{\tilde{r}_i}}\,, \\ Z^{\bar{\imath}}_R(\bar{z}) &= z^{\bar{\imath}}_R - i\frac{\alpha'}{2}p^{\bar{\imath}}_R \ln \bar{z} + i\sum_{\tilde{s}_i\neq 0}\frac{\tilde{\alpha}^{\bar{\imath}}_{\tilde{s}_i}}{\tilde{s}_i \bar{z}^{\tilde{s}_i}}\,. \end{aligned} \tag{2.1.1}$$

In these equations the $z^i_{L/R}$ constitute the (complex) center of mass of the string $z^i = z^i_L + z^i_R$, and the $p^i_{L/R}$ the conjugated center–of–mass or zero–mode momenta. The α's are leftmoving oscillators with mode numbers r_i and s_i (for the rightmovers everything with tilde). The mode numbers are integer spaced, but can have an offset w.r.t. $\mathbb{Z}$. We will go into more details about this later. The first two terms in each equation are usually called zero–mode part, the rest oscillator part of the mode expansion.

Going once around the string corresponds to taking $\sigma \to \sigma + 2\pi$ or equivalently $z \to z\exp(-2\pi i)$. A relation between a field at these two points is called monodromy condition. Since we want to describe the free heterotic string first, the monodromy following from closure of the strings is particularly simple, namely single–valuedness of $Z^{i/\bar{\imath}}$. This implies, that in (2.1.1)

$$p^i_L = p^i_R \equiv p^i\,, \quad r_i, s_i, \tilde{r}_i, \tilde{s}_i \in \mathbb{Z}\,. \tag{2.1.2}$$

Using operator product expansions (OPEs) it follows, that the leftmoving oscillators $\alpha^i_{r_i}$ and $\alpha^{\bar{\imath}}_{s_i}$ (and rightmoving ones) fulfill

$$[\alpha^i_{r_i}, \alpha^{\bar{\jmath}}_{s_j}] = r_i \delta_{r_i,-s_j}\eta^{i\bar{\jmath}} \tag{2.1.3}$$

[2]To be precise $\bar{\partial}Z$ and ∂Z are holomorphic and antiholomorphic, respectively, and therefore Laurent expanded. The $z_{L/R}$ appear as integration constants.

($\eta_{i\bar{\jmath}}$ being the flat Minkowski metric) and can be used as raising and lowering operators in the usual way. By taking the oscillators $\alpha^i_{r_i}$ with $r_i > 0$ as lowering operators and $\alpha^i_{r_i}$ with $r_i < 0$ as raising operators, and for the rightmovers analogously, a vacuum $|\,0; k^i\,\rangle = |\,0; k^i_L\,\rangle \otimes |\,0; k^i_R\rangle$ for direction i carrying a momentum k^i can be defined by enforcing, that it is annihilated by all lowering operators. Correspondingly states can be obtained by acting with raising operators on the vacua.

For fermionic worldsheet fields the situation is more involved, i.e. the field does not necessarily have to be invariant going once around the string, but can pick up a factor of -1 instead, so

$$\begin{aligned} \lambda^I(\sigma + 2\pi) &= e^{2\pi i\nu}\lambda^I(\sigma)\ , \\ \tilde{\psi}^\mu(\sigma + 2\pi) &= e^{-2\pi i\tilde{\nu}}\tilde{\psi}(\sigma)\ . \end{aligned} \tag{2.1.4}$$

A boundary condition with $\nu = 0$ ($\tilde{\nu} = 0$) is called Ramond (R/$\widetilde{\mathrm{R}}$) boundary condition, the one with $\nu = 1/2$ ($\tilde{\nu} = 1/2$) Neveu–Schwarz (NS/$\widetilde{\mathrm{NS}}$) boundary condition. Transformed to the $(z, \bar{z})$ frame, the conditions (2.1.4) read[3]

$$\begin{aligned} \lambda^I(ze^{-2\pi i}) &= e^{2\pi i(\nu+1/2)}\lambda^I(z)\ , \\ \tilde{\psi}^\mu(\bar{z}e^{2\pi i}) &= e^{-2\pi i(\tilde{\nu}+1/2)}\tilde{\psi}^\mu(\bar{z})\ . \end{aligned} \tag{2.1.5}$$

Using these monodromies and the (anti-)holomorphicity again, we arrive at the mode expansions

$$\begin{aligned} \lambda^I(z) &= \sum_{r\in\mathbb{Z}+\nu} \frac{\lambda^\mu_r}{z^{r+1/2}}\ , \\ \tilde{\psi}^\mu(\bar{z}) &= \sum_{s\in\mathbb{Z}-\tilde{\nu}} \frac{\tilde{\psi}^\mu_s}{\bar{z}^{r+1/2}}\ . \end{aligned} \tag{2.1.6}$$

By using OPEs once more one finds for the fermionic oscillators

$$\{\lambda^I_r, \lambda^J_s\} = \delta^{IJ}\delta_{r,-s}\ , \quad \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}\ , \tag{2.1.7}$$

with δ_{IJ} (Kronecker delta) the Euclidean metric.

Although in principle we could think about assigning different boundary conditions for every spacetime/internal direction, there are restrictions on this: For the rightmovers this would imply a breakdown of ten dimensional Lorentz invariance, which we do not want, for the leftmovers a breakdown of reparametrization invariance of the worldsheet. (There are different ways to find restrictions for the leftmoving d.o.f., in fact we will see one in subsection 3.3.1). The outcome is, that for the rightmovers we have to assign identical boundary conditions for

[3]The additional minus sign stems from the fact, that the fermionic worldsheet fields transform with half the conformal weight of a vector, e.g. $\lambda(z) = \sqrt{\partial_z w}\lambda(w)$.

all spacetime directions μ, either $\widetilde{\mathrm{R}}$ or $\widetilde{\mathrm{NS}}$. We will call the fermions to lie in the $\widetilde{\mathrm{R}}$ or $\widetilde{\mathrm{NS}}$ sector correspondingly. For the leftmovers we split the 32 λ^I in two groups (I running from 1 to 16 and 17 to 32 respectively), with identical boundary conditions within each group (but indepent from each other), denoting them as NSNS, NSR, RNS and RR sectors.

Consider the $\widetilde{\mathrm{R}}$ sector for definiteness. All results will straightforwardly carry over to the leftmoving sectors. There are zero–modes which anticommute with the annihilation operators, leading to the well–known degeneracy of the $\widetilde{\mathrm{R}}$ sector ground state. Since from (2.1.7) it is clear, that ($\sqrt{2}$ times) the zero–modes fulfill a Clifford algebra, the ground state must form a representation of the corresponding group. Then, by using this equivalence one can proceed parallel to appendix A.2 and finds, that the $\widetilde{\mathrm{R}}$ sector ground state transforms as **32** which splits into a **16** plus a **16**$'$, with postive and negative chirality, corresponding to an equal and odd number of minus signs, respectively. We label the possible ground states in the spin basis again. They read

$$|\underbrace{\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}}_{\#(-)\text{ even}}\rangle_{\widetilde{\mathrm{R}}} \oplus |\underbrace{\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}}_{\#(-)\text{ odd}}\rangle_{\widetilde{\mathrm{R}}}\ , \tag{2.1.8}$$

with $\#(-)$ the number of minus signs[4]

For a fermion with NS ($\widetilde{\mathrm{NS}}$) boundary conditions the definition of a vacuum $|\,0\,\rangle_{\mathrm{NS}/\widetilde{\mathrm{NS}}}$ is straightforward by choosing the positive modes to be annihilation and the negative modes raising operators.

As is well–known, if we would build up the spectrum as such, the string theory would be incosistent (non–locality of vertex operators, divergences in one–loop amplitudes, etc...). We have to take an additional projection into account, the Gliozzi–Scherk–Olive (GSO) projection. It can be seen as a generalization of the chirality operator (A.2.10). Consider the $\widetilde{\mathrm{R}}$ sector again. The chirality operator was introduced by enforcing that it anticommutes with all $\Gamma^\mu \cong \sqrt{2}\tilde{\psi}^\mu_0$, so with the zero–modes of the worldsheet fermion. Extending this to non–zero–modes as well, one searches for an operator that anticommutes with all $\tilde{\psi}^\mu$. In analogy to (A.2.8) (re)define

$$S_i \equiv i^{\delta_{i,0}}\Sigma^{2i,2i+1} \quad \text{with} \quad \Sigma^{\mu\nu} \equiv -\frac{i}{2}\sum_{r\in\mathbb{Z}+\tilde{\nu}} [\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_r]\ . \tag{2.1.9}$$

Then, the operator

$$\exp(\pi i \tilde{F}) \quad \text{with} \quad \tilde{F} \equiv \sum_{i=0}^{4} S_i \tag{2.1.10}$$

[4]We labeled the spin ket on the RHS with an index $\widetilde{\mathrm{R}}$ to emphasize, that it was obtained from the $\widetilde{\mathrm{R}}$ sector. We will do the same in the $\widetilde{\mathrm{NS}}$ sector. However, please note that this is just a 'reminder'. We will always refer to the same basis and the states will always form representations of one and the same group (here $\mathrm{SO}(9,1)$ for the rightmovers), no matter where they stemmed from.

does the job. The eigenvalue of $\tilde{F}$ is called rightmoving worldsheet fermion number. The leftmoving worldsheet fermion numbers F_1 and F_2 (corresponding to the two groups) is defined analogous. Clearly these numbers are defined modulo 2. With the above definitions it is straightforward to show, that for $i \neq 0$ (we will not need the case $i = 0$ as will become apparent later) and $j \neq i$

$$\begin{aligned} S_i(\tilde{\psi}_r^{2i} \pm i\tilde{\psi}_r^{2i+1}) &= (\tilde{\psi}_r^{2i} \pm i\tilde{\psi}_r^{2i+1})(S_i \pm 1) \,, \\ S_i(\tilde{\psi}_r^{2j} \pm i\tilde{\psi}_r^{2j+1}) &= (\tilde{\psi}_r^{2j} \pm i\tilde{\psi}_r^{2j+1})S_i \,. \end{aligned} \tag{2.1.11}$$

Therefore the complex combinations $(\tilde{\psi}_r^{2i} \pm i\tilde{\psi}_r^{2i+1})$ are nothing but raising and lowering operators w.r.t. the spin quantum numbers s_i. Due to (2.1.10) the application of such a ladder operator changes the worldsheet fermion number by one. Since all states are obtained by acting with creation operators on a groundstate, all worldsheet fermion numbers are fixed, once they are fixed for the groundstates. Moreover, let $|\,\tilde{q}\,\rangle$ denote a rightmoving state obtained in such a way, then it is clear that it has to lie in some lattice $\tilde{\Gamma}$. Analogous a leftmoving state $|\,q\,\rangle$ has to lie in a lattice Γ.

Please note, that eqns. (2.1.9) – (2.1.11) do not depend on the fermions boundary condition. It is therefore convenient to extend this formalism to the NS ($\widetilde{\mathrm{NS}}$) sectors as well. Since there are no zero–modes within (2.1.9) for the NS/$\widetilde{\mathrm{NS}}$ sector, $\Sigma^{\mu\nu}$ annihilates $|\,0\,\rangle_{\widetilde{\mathrm{NS}}}$ identically, leaving it as a Lorentz singlet. As a zero dimensional representation it is clear that in the spin basis the groundstate corresponds to

$$|\,0\,\rangle_{\widetilde{\mathrm{NS}}} \cong |\,0,0,0,0,0\,\rangle_{\widetilde{\mathrm{NS}}} \,. \tag{2.1.12}$$

We close this subsection with some notes about bosonization, which will prove convenient and useful for the rest of this work. For definiteness consider the rightmovers again and complexify the fermionic fields according to (i running from 1 to 4 here)

$$(\tilde{\psi}^{2i} \pm i\tilde{\psi}^{2i+1}) \to (\tilde{\psi}^{i}(\bar{z}), \tilde{\psi}^{\bar{\imath}}(\bar{z})) \,. \tag{2.1.13}$$

Then bosonization means that we can consistently identify

$$\tilde{\psi}^{i}(\bar{z}) \cong e^{i\tilde{H}^i(\bar{z})} \,, \quad \tilde{\psi}^{\bar{\imath}}(\bar{z}) \cong e^{-i\tilde{H}^i(\bar{z})} \,, \tag{2.1.14}$$

where $\tilde{H}^i(\bar{z})$ is a real bosonic field compactified on a circle with mode expansion

$$\tilde{H}^i(\bar{z}) = \tilde{H}_0^i - i\tilde{q}^i \ln \bar{z} + i \sum_{n\in(\mathbb{Z}-\{0\})} \frac{\tilde{\gamma}_n^i}{n\bar{z}^n} \,. \tag{2.1.15}$$

Now consider the generalized monodromies

$$\tilde{\psi}^{i}(\bar{z}e^{2\pi i}) = e^{2\pi i(\tilde{\nu}+1/2)}\tilde{\psi}^{i}(\bar{z}) \,, \tag{2.1.16}$$

where $\tilde{\nu}$ is allowed to be real (we can use this more general form later). Then the form of the mode expansions (2.1.6) is unchanged (the oscillators now being complex of course)

$$\tilde{\psi}^i(\bar{z}) = \sum_{r\in\mathbb{Z}-\tilde{\nu}} \frac{\tilde{\psi}^i_r}{\bar{z}^{r+1/2}} \,, \quad \tilde{\psi}^{\bar{\imath}}(\bar{z}) = \sum_{s\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^{\bar{\imath}}_s}{\bar{z}^{s+1/2}} \,, \quad \{\tilde{\psi}^i_r, \tilde{\psi}^{\bar{\jmath}}_s\} = \delta^{i\bar{\jmath}}\delta_{r,-s} \,. \qquad (2.1.17)$$

Again we have to define a vacuum (which depends on the boundary conditions, and therefore $\tilde{\nu}$) by imposing

$$\tilde{\psi}^i_{n+\nu}|\,0\,\rangle_{\tilde{\nu}} = \tilde{\psi}^{\bar{\imath}}_{n+1-\tilde{\nu}}|\,0\,\rangle_{\tilde{\nu}} = 0 \qquad (2.1.18)$$

for all[5] non–negative n. Now, by making use of the operator state correspondence, a state built from the above vacuum, corresponds to local operators $\mathcal{A}^i_{\tilde{\nu}}$ at $\bar{z} = 0$. On the other hand, from the OPE of $\mathcal{A}^i_{\tilde{\nu}}$ with $\tilde{\psi}^i$ one can read off, that $\mathcal{A}^i_{\tilde{\nu}}$ corresponds to the bosonic winding vertex operator $\exp(i\tilde{q}_i\tilde{H}^i)$ with $\tilde{q}_i = (-\tilde{\nu} + 1/2)$.

Now let $\tilde{\nu} = 1/2$ ($\widetilde{\mathrm{NS}}$ sector) again. The groundstate is non–degenerate and $\tilde{q}_i = 0$ for all (complex) i. For $\tilde{\nu} = 0$ ($\widetilde{\mathrm{R}}$ sector) we find $\tilde{q}_i = \pm 1/2$ in accordance with (2.1.8). This is due to the identification

$$|\,\mathbf{s}\,\rangle \cong e^{is_i\tilde{H}^i} \,, \qquad (2.1.19)$$

following from the bosonic state operator correspondence again.

2.1.3 Particle Interpretation of the Massless Spectrum

Up to now, we have completely ignored the constraints coming from the Virasoro generators. At this point we will just quote some results and refer the reader to the literature instead.

First of all we will only be interested in massless spactra here and throughout this work. Therefore states will be classified in terms of the massless little group SO(8) of SO(9, 1). We will choose a reference momentum $k_0 = k_1$. We start with the fermionic construction again. As can be seen in several ways (light–cone gauge, BRST quantization, etc...) only transversal excitations $\mu, \nu, \dots = 2 \dots d{-}1$ give rise to physically meaningful states, allowing to ignore ghosts. This is valid for all bosonic excitations as well as those of the rightmoving $\widetilde{\mathrm{NS}}$ vacua (fixing the eigenvalue of S_0 to be 0 then). For the rightmoving $\widetilde{\mathrm{R}}$ sector, an analog of the massless Dirac equation fixes the eigenvalues of S_0 to be $+1/2$, as can easily be seen from $k_\mu\Gamma^\mu = -2k_1\Gamma^0(S_0 - 1/2)$, which is valid in our reference frame. Since with this s_0 is fixed for every rightmoving state, it will be omitted

[5] Note, that one has to enforce $0 \le \tilde{\nu} \le 1$ in order not to arrive at excited states (spectral flow)

in the following. Please note that there are no fermionic constraints on the leftmoving side.

The only Virasoro constraint left up to now, is those determining the mass. At this point it is convenient to bosonize the transversal fermions, because then all contributions to the mass can be summarized within the bosonic zero–mode Virasoro constraints L_0 and $\tilde{L}_0$ which corresponds to the left– and rightmoving bosonic energy–momentum tensors. One finds

$$
\begin{aligned}
L_0 &= \frac{\alpha'}{4}p_L^2 + \sum_i \left(\sum_{r_i<0}(-r_i)N_{r_i}^{\alpha^i} + \sum_{s_i<0}(-s_i)N_{s_i}^{\alpha^{\bar{\imath}}} \right) \\
&\quad + \frac{1}{2}q^2 + \sum_I \sum_{n>0} nN_n^{\gamma^I} - 1 \,, \qquad (2.1.20) \\
\tilde{L}_0 &= \frac{\alpha'}{4}p_R^2 + \sum_i \left(\sum_{\tilde{r}_i<0}(-\tilde{r}_i)N_{\tilde{r}_i}^{\tilde{\alpha}^i} + \sum_{\tilde{s}_i<0}(-\tilde{s}_i)N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{\imath}}} \right) \\
&\quad + \frac{1}{2}\tilde{q}^2 + \sum_i \sum_{n>0} nN_n^{\tilde{\gamma}^i} - \frac{1}{2} \,, \qquad (2.1.21)
\end{aligned}
$$

with the N's denoting the occupation numbers of the indexed modes. The contributions -1 and $-1/2$ are zero–mode energies. By imposing these constraints on left– and rightmoving states, the masses $m_L^2 = -k_L^2$ and $m_R^2 = -k_R^2$ can be read off. It is clear, that acting with an oscillator with a given mode number, say $-n$, increases the corresponding mass–squared (times $\alpha'/4$) by n. A state of mass m^2 can be obtained by tensoring left– and rightmovers with $m_L^2 = m_R^2 = m^2$ together.

We are now in the position to give the final definitions for heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory [5, 6]. All fermionic groundstates together with their worldsheet fermion numbers read:

$$
\begin{array}{lcll}
|\,0\,\rangle_{NSNS} & \equiv & |\,0^8\,|\,0^8\,\rangle & F = (0|0) \\
|\,0\,\rangle_{RNS} & \equiv & |\,\tfrac{1}{2}^8\,|\,0^8\,\rangle & F = (0|0) \\
|\,0\,\rangle_{NSR} & \equiv & |\,0^8\,|\,\tfrac{1}{2}^8\,\rangle & F = (0|0) \\
|\,0\,\rangle_{RR} & \equiv & |\,\tfrac{1}{2}^8\,|\,\tfrac{1}{2}^8\,\rangle & F = (0|0) \\
|\,0\,\rangle_{\widetilde{NS}} & \equiv & |\,0^4\,\rangle & \tilde{F} = 1 \\
|\,0\,\rangle_{\tilde{R}} & \equiv & |\,\tfrac{1}{2}^4\,\rangle & \tilde{F} = 0
\end{array}
\qquad (2.1.22)
$$

where we used the collective notation $F = (F_1|F_2)$ for the two independent woldsheet fermion numbers of the leftmovers. Notations like e.g. 0^4 denote repetitions here, i.e. $0, 0, 0, 0$. The GSO projections are defined by[6]

$$\exp(\pi i F_1) = \exp(\pi i F_2) = \exp(\pi i \tilde{F}) = 1 \tag{2.1.23}$$

in each combination of sectors. Some comments are in order: Firstly the worldsheet fermion numbers of the groundstates in (2.1.22) seem to disagree with definition (2.1.10) for both rightmoving vacua (note that we supressed s_0). This is due to the ghosts, which contribute a factor of -1 in the $\widetilde{\mathrm{NS}}$ and $-i$ in the $\widetilde{\mathrm{R}}$ sector to the GSO, but can be omitted in lightcone gauge otherwise. Secondly, it is interesting to see how the GSO projection works, e.g. in the $\widetilde{\mathrm{R}}$ sector it removes the $\mathbf{16'}$ from the possible groundstates (2.1.8). Therefore the choice of signs within the different R sector groundstates of (2.1.22) is compatible with the GSO's.

More generally, the GSO restricts the lattices Γ and $\tilde{\Gamma}$, which are generated from the vacua. We have seen in the last subsection that we can raise and lower the spin eigenvalues by one, changing the worldsheet fermion number by one as well. Consider the $\widetilde{\mathrm{NS}}$ groundstate: Then, according to the GSO, we can generate the lattice $\Gamma_{\mathrm{SO}(8)_v} \equiv \Gamma_{\mathrm{SO}(8)} + (1, 0^3)$, where $\Gamma_{\mathrm{SO}(8)} \equiv \{(n_1, n_2, n_3, n_4) \,|\, n_i \in \mathbb{Z}, \sum n_i = 0 \mod 2\}$ from it. Analogously from the $\widetilde{\mathrm{R}}$ groundstate we can generate the lattice $\Gamma_{\mathrm{SO}(8)} + (\frac{1}{2}^4)$. Therefore we identify

$$\tilde{\Gamma} = \left[(1, 0^3) + \Gamma_{\mathrm{SO}(8)}\right] \oplus \left[(\tfrac{1}{2}^4) + \Gamma_{\mathrm{SO}(8)}\right] , \tag{2.1.24}$$

which in the language of appendix A.3 is nothing but the direct sum of the weight lattices of the vector and the positive chirality spinor representation of SO(8). Now consider the leftmovers of the first internal group (both internal groups of fields are treated identically): From the NS_1 groundstate we generate the lattice $\Gamma_{\mathrm{SO}(16)} \equiv \{(n_1, n_2, \ldots, n_8) \,|\, n_i \in \mathbb{Z}, \sum n_i = 0 \mod 2\}$, and from the R_1 groundstate the lattice $\Gamma_{\mathrm{SO}(16)} + (\frac{1}{2}^8)$. Combining this with the lattices from the second group we find

$$\Gamma = \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8} \qquad \text{with} \quad \Gamma_{\mathrm{E}_8} = \Gamma^+_{\mathrm{SO}(16)} \equiv \Gamma_{\mathrm{SO}(16)} \oplus \left[(\tfrac{1}{2}^8) + \Gamma_{\mathrm{SO}(16)}\right] , \tag{2.1.25}$$

which again in the language of appendix A.3 is the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8$.

Now the stage is set to obtain the massless spectrum of free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory. We start with the rightmoving zero-mode Virasoro generator (2.1.21). It is easy to see, that exciting with a bosonic $\tilde{\alpha}$ oscillator would already lead to massive states $(-1/2 + 1 = +1/2)$, so the only contributions can come from the fermions. The freedom to excite $\tilde{\gamma}^i$ oscillators has already been used to build the lattice $\tilde{\Gamma}$. Therefore the possible rightmoving massless states are just

[6] There is another possibility of defining GSO projections that leads to the heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. However, it will not be considerd in this work. For a detailed description of its structure we refer to [1] and references therein.

given by $\tilde{q} \in \tilde{\Gamma}$ with $\tilde{q}^2 = 1$. They split into two times eight states according to the direct sum in (2.1.24), and give rise to an $\mathbf{8_v} + \mathbf{8_s}$ of SO(8). We write

$$| \underline{\pm 1, 0, 0, 0} \rangle_{\widetilde{\mathrm{NS}}} \,, \quad | \underbrace{\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}}_{\#\,(-)\text{ even}} \rangle_{\tilde{\mathrm{R}}} \,. \tag{2.1.26}$$

For the leftmovers there are two possibilities in (2.1.20) to obtain massless states[7]. Either with $q^2 = 2$ and no bosonic oscillator excited, or with $q^2 = 0$ (note, that this is contained in Γ) and one bosonic oscillator excited. The first set of states clearly gives rise to the roots of $\mathrm{E}_8 \times \mathrm{E}_8$, which form the adjoint representation $(\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{248})$ of $\mathrm{E}_8 \times \mathrm{E}_8$ and transform as a singlet under the (spacetime) little group SO(8). The other set of states is an $\mathrm{E}_8 \times \mathrm{E}_8$ singlet, which transforms as a vector under SO(8).

It remains to tensor left- and rightmovers together. We label the states by their $\mathrm{SO}(8) \times \mathrm{E}_8 \times \mathrm{E}_8$ quantum numbers again (the rightmovers clearly do not transform under $\mathrm{E}_8 \times \mathrm{E}_8$):

$$\begin{aligned} &(\mathbf{1}; \mathbf{248}, \mathbf{1}) + (\mathbf{1}; \mathbf{1}, \mathbf{248}) + (\mathbf{8_v}; \mathbf{1}, \mathbf{1}) \times (\mathbf{8_v} + \mathbf{8_s}; \mathbf{1}, \mathbf{1}) = \\ &\quad (\mathbf{8_v}; \mathbf{248}, \mathbf{1}) + (\mathbf{8_s}; \mathbf{248}, \mathbf{1}) + (\mathbf{8_v}; \mathbf{1}, \mathbf{248}) + (\mathbf{8_s}; \mathbf{1}, \mathbf{248}) + \\ &\quad + (\mathbf{1}; \mathbf{1}, \mathbf{1}) + (\mathbf{28}; \mathbf{1}, \mathbf{1}) + (\mathbf{35}; \mathbf{1}, \mathbf{1}) + (\mathbf{56}; \mathbf{1}, \mathbf{1}) + (\mathbf{8_c}; \mathbf{1}, \mathbf{1}) \end{aligned} \tag{2.1.27}$$

Using the table of appendix B.1 we can identify the massless states with representations of the $D = 10, \mathcal{N} = 1$ SUSY algebra, i.e. those of the second line of (2.1.27) fill an $\mathrm{E}_8 \times \mathrm{E}_8$ vector/gauge multiplet and those of the last line a $D = 10, \mathcal{N} = 1$ supergravity (SUGRA) multiplet. This ends our discussion of the massless spectrum of free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory.

2.2 Compactification Manifolds

As already stated in the introduction we seem to live in a three plus one dimensional Minkowskian world, so that, if heterotic string theory (or in fact any string theory) should describe our world, we have to compactify some of the critical ten dimensions on a compact manifold X, which we choose to be Riemannian and orientable. Even more we assume that the effective four dimensional theory should possess $\mathcal{N} = 1$ SUSY. In this section we take the massless spectrum of free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory derived in the last section as a low energy effective field theory and collect some generalities, that give restrictions on X.

The tangential space group of the uncompactified theory $\mathrm{SO}(9,1)$ is clearly broken via compactification. Since the tangential space group of three plus one

[7] One could think, that tachyonic states arise here, since $q^2 = 0$ and no bosonic oscillator excitation is allowed. However, there is no tachyonic rightmoving counterpart to tensor it with, so that heterotic string theory is finally tachyon free.

dimensional Minkowski space is by assumption SO(3,1) and that of X is SO(6) which is locally isomorphic ($\cong$) to SU(4) we arrive at the following branching

$$\mathrm{SO}(9,1) \to \mathrm{SO}(3,1) \times \mathrm{SO}(6) \cong \mathrm{SO}(3,1) \times \mathrm{SU}(4)\ . \tag{2.2.1}$$

This branching is used to decompose the spins. The ten dimensional SUSY parameter η, that transforms as a $\mathbf{16_s}$ (left–handed Majorana–Weyl spinor) decomposes in terms of (2.2.1) as $\mathbf{16_s} \to (\mathbf{2},\mathbf{4}) + (\bar{\mathbf{2}},\bar{\mathbf{4}})$ and so yields four Weyl spinors ϵ^a. These Weyl spinors provide the four SUSY parameters of $\mathcal{N}=4, D=4$ SUSY, which is more than wanted.

There are restrictions on the holonomy group coming from the enforcement of $\mathcal{N}=1, D=4$ SUSY. The (local) holonomy group[8] at a point $p \in X$ must be a subgroup of the tangential spacegroup, since parallel transportation is an isometry of the metric (it does not change lengths). The condition to have some unbroken SUSY is essentially equivalent to the condition

$$0 = \delta\psi_M = D_M\eta = D_m\epsilon^a \tag{2.2.2}$$

where ψ_M is the ten dimensional gravitino. For $M=\mu$ this means, that η should be independent of the uncompactified coordinates, which explains the last equality. With the same decomposition as before we see, that to obtain $\mathcal{N}=1, D=4$ SUSY there must be exactly one covariantly constant spinor ϵ on X. This has strong implications, i.e. it enforces X to be a complex manifold. Now, a covariant constant spinor is always invariant under parallel transport and therefore must obey $U\epsilon = \epsilon$ for some $U \in \mathrm{SU}(4)$. Let us take ϵ to have positive chirality and so transform as $\mathbf{4}$ (we can represent it as a four dimensional column vector). The subgroup that leaves exactly one component invariant is SU(3) (say, it just acts on the first three entries), i.e. under $\mathrm{SU}(4) \to \mathrm{SU}(3)\,(\times\mathrm{U}(1))$ we have $\mathbf{4} \to \mathbf{3} + \mathbf{1}$. The singlet is precisely the one unbroken SUSY parameter we were looking for. Because this is unique the condition for $\mathcal{N}=1$ SUSY in four dimensions translates into the condition that X must be a complex manifold with SU(3) holonomy. Such and similar manifolds are called Calabi–Yau manifold[9]. The U(1), which just has been omitted, acts on the unbroken SUSY parameter, transforming it by a phase, and therefore can be identified with the R–parity group of $\mathcal{N}=1, D=4$ SUSY. It is often very useful to keep this breaking pattern in mind:

$$\mathrm{SO}(9,1) \to \mathrm{SO}(3,1) \times (\mathrm{SO}(6) \cong \mathrm{SU}(4)) \to \mathrm{SO}(3,1) \times \mathrm{SU}(3)_H \times \mathrm{U}(1)_R\ , \tag{2.2.3}$$

where we indexed the holonomy group with H and the R–parity group by R.

[8] The local holonomy group is the group generated by parallell transporting tangent vectors around closed loops from a point $p \in X$ back to itself.

[9] The exact definition of a Calabi–Yau is, that it is a complex Kähler manifold with vanishing first Chern class.

We close this section by repeating the same exercise for the case, that just four dimensions are compactified and keep $\mathcal{N}=1, D=6$ in six Minkowskian dimensions (this corresponds to $\mathcal{N}=2$ in $D=4$). The outcome is summarized in the following branching:

$$\mathrm{SO}(9,1) \to \mathrm{SO}(5,1) \times \mathrm{SO}(4) \to \mathrm{SO}(5,1) \times \mathrm{SU}(2)_H \times \mathrm{SU}(2)_R\,, \qquad (2.2.4)$$

so in this case the now four dimensional compactification manifold must have $\mathrm{SU}(2)_H$ holonomy. Interestingly such a four dimensional Calabi–Yau manifold is (topologically) unique. It is called K3.

Although at first sight it might not be of interest for phenomenological purposes, we will first study this situation as a toy model, and later see, that it appears in certain subsectors of phenomenologically interesting models as well.

2.3 Geometric Orbifolds

The problem with the Calabi–Yau manifolds defined in the previous section is, that it has not yet been possible to write down metrics for them explicitly. Despite of this there is a great deal of interesting physics already following from topological informations, but this will not be further considered in this work. Fortunately, there is a special limit of these smooth manifolds, where certain points on X get singular, a so called orbifold limit. In this limit string computations get either simpler or even possible. We will comment more on this limit later, although not in detail, but first just introduce orbifolds as we will use them. In the description of geometric orbifolds as presented here we follow [21] and [50].

A geometric orbifold O is a manifold X divided by some discrete symmetry group. In our case these manifolds will always be d–tori T^d. We will think of T^d as a flat torus. Using this we can define the orbifold $O \equiv \mathbb{R}^d/S$, where S is called space group. The space group is allowed to contain rotations $\Theta \in \mathrm{SO(d)}$ and translations $u' \in \mathbb{R}^d$. We take an element $D \in S$ to act on the coordinates in the canonical way

$$Dx = (\Theta, u')x \equiv \theta x + u'\ . \qquad (2.3.1)$$

The action of D is sometimes called twist. The group generated by pure rotations is called point group P of the orbifold. The pure translations generate the lattice defining the flat torus Λ. The space group S must be a symmetry of this lattice, i.e. $S\Lambda = \Lambda$. We will further restrict to Abelian orbifolds (that is, the point group is Abelian) for which the point group is cyclic $P \in \mathbb{Z}_N$.

Orbifolds in general have fixed points, i.e. points invariant under (2.3.1). Then the space group is said to act non–freely. It is exactly at the fixed points, where orbifolds are singular and fail to be manifolds. A fixed point x_o is defined by

$$Dx_0 = (\Theta, u')x_0 \equiv \theta x_0 + u' = x_0\ . \qquad (2.3.2)$$

To analyze this condition, define the subspace I of Λ, that is invariant under the point group, and N its orthogonal complement:

$$
\begin{aligned}
I &= \{w \in \Lambda : \theta w = w\} \;, && (2.3.3)\\
N &= \{v \in \Lambda : v \cdot I = 0\} \;. && (2.3.4)
\end{aligned}
$$

Using (2.3.2), each fixed point x_0 defines an element $u' = (\mathbb{1} - \theta)x_0 \in N$. That $u' \in N$ can be easily seen by evaluating $w \cdot u'$ for $w \in I$ and using that θ is an isometry w.r.t. the scalar product. Due to the toroidal geometry, two fixed points clearly have to be identified if they differ by a lattice shift. For the corresponding elements in N that means, that they differ by $(\mathbb{1} - \theta)v$ with $v \in \Lambda$. Therefore the number of inequivalent fixed points within a chosen fundamental domain of the torus is given by the index

$$
\left| \frac{N}{(\mathbb{1} - \theta)\Lambda} \right| \;. \qquad (2.3.5)
$$

However, one has to be careful for the case that u' has components not in the directions of N. Note, that this can never be the case for the fixed points studied before. We consider an analog $\tilde{x}_0$ instead. Decompose $u' = u'^{\|} + u'^{\perp}$ into two parts, parallel and perpendicular to N. By definition $(\mathbb{1} - \theta)u'^{\|} \neq 0$, so that $\tilde{x}_0 = (\mathbb{1} - \theta)^{-1}u'^{\|} + \tilde{x}_0^{\perp}$ is well–defined for $\tilde{x}_0^{\perp} \perp N$ arbitrary. We find

$$
\begin{aligned}
\theta\tilde{x}_0 &= (\theta - \mathbb{1})(\mathbb{1} - \theta)^{-1}u'^{\|} + (\mathbb{1} - \theta)^{-1}u'^{\|} + \theta\tilde{x}_0^{\perp} && (2.3.6)\\
&= -u'^{\|} + (\mathbb{1} - \theta)^{-1}u'^{\|} + \tilde{x}_0^{\perp} = \tilde{x}_0 - u'^{\|} && (2.3.7)
\end{aligned}
$$

and using this

$$
D\tilde{x}_0 = \tilde{x}_0 - u^{\|} + u^{\|} + u^{\perp} = \tilde{x}_0 + u'^{\perp} \;. \qquad (2.3.8)
$$

Therefore, contrary to (2.3.2), eqn. (2.3.8) defines fixed planes, that extend in the compactified dimensions. We will encounter situations where they occur.

We continue with some topological properties of orbifolds. In our case of $T^d/\mathbb{Z}_N$, away from the fixed points, the orbifold is flat. Since points related by a $\mathbb{Z}_N$ rotation are identified, it follows, that the holonomy group is given by $\mathbb{Z}_N$ as well[10]. In view of section 2.2 the point group has to be a subgroup of the holonomy group as defined there (otherwise the twist would break the R–parity group and not allow for enough SUSY). Since they were $\mathrm{SU}(3)_H$ and $\mathrm{SU}(2)_H$ respectively, the embedding of the point group can be summarized via

$$
\begin{aligned}
&\mathrm{SO}(9,1) \supset \mathrm{SO}(6) \cong \mathrm{SU}(4) \supset \mathrm{SU}(3)_H \supset \mathbb{Z}_N \;,\\
&\mathrm{SO}(9,1) \supset \mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \supset \mathrm{SU}(2)_H \supset \mathbb{Z}_N \;, \qquad (2.3.9)
\end{aligned}
$$

[10]Imagine a parallel transport of a vector from a point $p \in T^d/\mathbb{Z}_N$ (not a fixed point) to $\theta^{-1}p$ (because of the identification the loop is closed). Since the torus is flat, the vector is not rotated. However, identifying corresponds to rotating by θ, so that the vector returning to its starting point appears rotated by θ, which generates the holonomy group at p.

for $d = 6$ and $d = 4$ (that is four and six uncompactified dimensions) respectively.

We close this section with a few comments on the connection between orbifolds and Calabi–Yau manifolds. The relation between them is, that orbifold singularities allow a so called blow–up, i.e. a topologically well–defined procedure to locally cut out orbifold singularities and glue in Eguchi–Hanson spheres, that possess $\mathrm{SU}(2)_H$ or $\mathrm{SU}(3)_H$ holonomy. There is one modulus for the size of theses spheres and it has been shown, that by running this size from a finite value to zero, the manifold exactly undergoes the transition from a Calabi–Yau to an orbifold. This is called orbifold limit. In view of this, the embedding chains (2.3.9) are very natural.

2.4 Embedding of Orbifolds with Wilson–lines

2.4.1 The Embedding

The next question is, how the idea of geometric orbifolds can be implemented in the construction of heterotic string theory, i.e. how an orbifold action can be defined on the different bosonic and fermionic fields. Nice introductions to this topic can be found e.g. in [51, 52, 16] or a more mathematical treatment in [27].

In (2.3.1) D acts on the internal/compactified coordinates. For the bosonic fields they appear as integration constants (center of mass coordinates) within X^μ (c.f. (2.1.1) and the following paragraph). Therefore, the straightforward generalization of (2.3.1) is given by

$$(\theta, u')X(\sigma,\tau) = \theta X(\sigma,\tau) + u' \ . \qquad (2.4.1)$$

Specializing to $d = 6$ or $d = 4$ compactified dimensions, we saw in the last section that the twist has to be embedded in $\mathrm{SU}(3)_H$ and $\mathrm{SU}(2)_H$ respectively. Therefore, we can always choose the generators of the rotations to be those of (A.2.9):

$$\theta = \exp(2\pi i \phi^i S_i) \ , \quad i = (2,)3,4 \ . \qquad (2.4.2)$$

Then, the orbifold acts on the complexified bosonic fields (transforming in the fundamental representation) like

$$(\theta, u)Z^i(z) = e^{2\pi i\phi^i} Z^i(z) + u^i \ , \qquad (2.4.3)$$

where now u is the complexified u'. The action on $Z^{\bar{\imath}}$ follows from complex conjugation of (2.4.3). Returning to the string construction, the definition (2.4.3) contains ambiguities, e.g. in decomposing Z^i in left- and rightmovers again one could easily spread the shift asymmetrically between them. We will not do so, i.e. we take

$$(\theta, u)Z^i_{L/R}(z) = e^{2\pi i\phi^i} Z^i_{L/R}(z) + \tfrac{1}{2}u^i \ . \qquad (2.4.4)$$

This implementation is called a symmetric orbifold[11].

The next question is how to define the action on the fermionic fields, but we will see that it is enforced by worldsheet SUSY. The worldsheet supercurrent $\tilde{T}_F(z)$ is given by

$$\tilde{T}_F(z) \propto \tilde{\psi}(\bar{z})\bar{\partial}X_R(\bar{z}) . \qquad (2.4.5)$$

In order to retain worldsheet supersymmetry $\tilde{T}_F(z)$ has to be invariant under the twist action. Using this and (2.4.4) enforces

$$(\theta, u)\tilde{\psi}^i(\bar{z}) = e^{2\pi i\phi^i}\tilde{\psi}^i(\bar{z}) . \qquad (2.4.6)$$

Up to now we have completely ignored the gauge d.o.f., spanned by the fermionic worldsheet fields $\lambda^I(z)$. Let us first define an action (v) on these d.o.f. in complete analogy to (2.4.2):

$$(v) = \exp(2\pi i v^I H_I) , \qquad (2.4.7)$$

where H_I now denotes the Cartan generators of $\mathrm{E}_8 \times \mathrm{E}_8$. Then by definition it acts on a state $|\, q\,\rangle$ like

$$(v)|\, q\,\rangle = e^{2\pi i v^I q_I}|\, q\,\rangle . \qquad (2.4.8)$$

Acting on the Lie algebra of $\mathrm{E}_8 \times \mathrm{E}_8$ via conjugation

$$(v)H^I(v^{-1}) = H^I , \quad (v)E^q(v^{-1}) = e^{2\pi i v^I q^I}E^q \qquad (2.4.9)$$

we see, that (2.4.7) defines nothing but a gauge transformation. However, the way of performing gauge transformations has to be compatible with the space group, i.e. the map from the space group into the group of gauge transformations must be linear (homomorphic). There are different ways to choose such a mapping. We follow [54, 44, 55] and assign[12]

$$\begin{aligned} (\theta, 0) &\rightarrow (v) , \\ (\mathbb{1}, e_i) &\rightarrow (a_i) , \end{aligned} \qquad (2.4.10)$$

where the e_i are basis vectors of the (flat) compactification torus Λ. The first map assigns a gauge transformation (v) to every $\mathbb{Z}_N$ rotation. This is sometimes called shift–embedding of the orbifold. The second map assigns a (different) gauge transformation to every non–contractible cycle of the compactification torus T^d. Such gauge transformations, which depend on the topology of the manifold, correspond to so called Wilson lines. A heterotic orbifold as defined here is therefore determined by the data $(\theta, u; \beta(v, a_i))$ with the mapping (2.4.10)

[11] This is in contrast to asymmetric orbifolds [50, 53], where from a more general point of view it is even possible to twist left- and rightmovers differently.

[12] In field theory this corresponds to an implementation of Scherk–Schwarz boundary conditions [56] by a Hosotani mechanism [57].

implicitly understood and $\beta(v, a_i)$ combining the actions of gauge shift and Wilson lines. This finishes the embedding of the orbifold in heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory, as will be used in this work.

We want to close this section by generalizing our point of view on the orbifold embedding a bit. In reviewing the steps we have performed, we note that all orbifold actions we have defined are actually contained in symmetries of $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory compactified on T^d, i.e. the complete theory, not just the geometry of space time. The gauge transformations are of course part of the gauge symmetry, whereas the transformations of the space group are all subsymmetries of the Poincare group. One can therefore turn things upside down and generalize the concept of heterotic orbifolds to say, that one is allowed to divide out any symmetry of the heterotic or another string theory (of course within consistency constraints).

For completeness we like to mention, that these symmetries are in general not related to geometric orbifolds and not always allow a geometric interpretation[13]. In the literature theses constructions are called orbifolds as well and are perfectly well defined. However, we will not follow this track further, since without geometric interpretation we would not be able to link to field theory, as we will do later. Additional possibilities are given by switching on antisymmetric background fields. Even the twist action on the gauge d.o.f. can be generalized or chosen differently, e.g. by allowing for so called continous Wilson lines. Again, we are not going to do this.

2.4.2 Embedding Ambiguities

There are some ambiguities in the definitions of twist vector, gauge shift and Wilson lines. Consider the gauge shift action (2.4.3) with $q \in \Gamma = \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8}$ for concreteness, the other cases can be treated analogous. Then, two gauge shifts (v) and (v') describe the same gauge transformation (we say, v and v' are equivalent)

$$v \cong v' \quad \Leftrightarrow \quad v - v' \in \Gamma^* \ , \tag{2.4.11}$$

where Γ^* denotes the lattice dual to Γ. This follows directly from definition (2.4.3). The $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice additionally has the special property to be self–dual, so that Γ^* can be replaced by Γ in (2.4.11) again (this can not in general be done for the rightmoving lattices).

But there are more ambiguities stemming from Weyl reflections. Let λ be a weight and α_i a simple root. Then the Weyl reflection r_i in the i–th simple root is defined by

$$r_i(\lambda) = \lambda - 2\frac{\lambda \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i \ . \tag{2.4.12}$$

[13]One example of this is provided by some asymmetric orbifolds, mentioned earlier.

This can be lifted to the group via

$$\tilde{r}_i = \exp(\frac{i\pi}{2}(E^{\alpha_i} + E^{-\alpha_i})) \ . \tag{2.4.13}$$

The action of this lift on the Lie algebra is determined by

$$\tilde{r}_i(\lambda^I H_I)\tilde{r}_i^{-1} = (r_i(\lambda))^I H_{(r_i(\lambda))^I} \ , \tag{2.4.14}$$

$$\tilde{r}_i E^\alpha \tilde{r}_i^{-1} = c_i(\alpha) E^{r_i(\alpha)} \ , \tag{2.4.15}$$

where U(1)-phases c_i stem from the ambiguity of the $[E^\alpha, E^\beta]$ commutator in the Cartan-Weyl-basis (A.3.1) and are fixed, once an explicit set of generators is choosen. What we see here is, that it does not matter which set of roots and weights we take, as long as they are related by a Weyl reflection, they define the same group theory. Let ω be another weight and let $(\lambda, \omega) = \lambda^I \omega^I$ denote the inner product. Then, it is easy to show that r_i is an isometry of the inner product, i.e.

$$(r_i(\lambda), r_i(\omega)) = (\lambda, \omega) \ . \tag{2.4.16}$$

Using this, it instantly follows from (2.4.3), that acting with a gauge shift (v) on a set of roots α and weights λ is equivalent to acting with a gauge shift $(r_i(v))$ on a set of roots $r_i(\alpha)$ and weights $r_i(\lambda)$. Therefore, again two gauge shifts are equivalent

$$v \cong v' \quad \Leftrightarrow \quad v' = r_i(v) \ , \tag{2.4.17}$$

with r_i a Weyl reflection as defined in (2.4.12).

The relations (2.4.11) and possible chains of (2.4.17) (and analogous conditions for twist vector and Wilson lines) incorporate all group theoretic equivalences within the defining orbifold data.

2.5 Hilbert Spaces of Orbifolds with Wilson–lines

In the discussion of Hilbert spaces we follow the original literature [20, 21]. Consider the bosonic worldsheet fields $Z(z, \bar{z})$ for simplicity. We start by noting that for the closed strings, a Hilbert space is defined by specifying a boundary condition w.r.t. the spacelike worldsheet direction σ. E.g. for the free heterotic string we just had single–valuedness

$$Z(z', \bar{z}') = Z(z, \bar{z}) \ , \tag{2.5.1}$$

where $z' = \exp(-2\pi i)$. However, due to the identification of points under the action of the space group S, there are the additional possibilities

$$Z(z', \bar{z}') = DZ(z, \bar{z}) \ , \tag{2.5.2}$$

with $D \in S$ non–trivial. These boundary conditions give rise to strings, which were not closed in the free theory, but only close upon a non–trivial action of the orbifold group. They are called twisted strings, in contrast to untwisted strings, where D is trivial[14]. The Hilbert spaces associated to a a boundary condition (2.5.2) with $D \in S$ (D trivial allowed) will be denoted by $\mathcal{H}_D$.

To form the orbifold, one has to project on space group invariant states in every sector. Consider a twisted string associated with D. Then, acting with (another) twist $G \in S$ on Z and using (2.5.2) we find

$$GZ(z', \bar{z}') = G\,DZ(z, \bar{z}) = GDG^{-1}\,GZ(z, \bar{z}) \; . \tag{2.5.3}$$

This means, that as long as G and D do not commute, the string is mapped to $\mathcal{H}_{GDG^{-1}}$. To form invariant states from a state in $\mathcal{H}_D$, one therefore has to project on the invariant subspace of the subgroup of S commuting with D (centralizer). Additionally, one has to sum over all states, whose Hilbert spaces can be mapped onto each other via (2.5.3). This is the case, if and only if two elements lie in the same conjugacy class.

Up to now we have not said anything about why twisted states should be present in a heterotic orbifold compactification. One reason is, that they can easily be pair–produced by untwisted strings and there is nothing to forbid this, but a much stronger argument is given by the requirement of modular invariance.

2.5.1 Modular Transformations

Consider orbifolded string theory on a worldsheet that is a two dimensional torus with Teichmüller parameter $\hat{\tau}$. This corresponds to the computation of a one–loop partition function. Then for a field $X(w)$ living on that worldsheet one can define two boundary conditions

$$X(w + 2\pi) = DX(w) \; , \quad X(w + 2\pi\hat{\tau}) = GX(w) \; , \tag{2.5.4}$$

where $D, G \in S$, which are only compatible if D and G commute. As we have seen the boundary condition along the σ direction specifies the Hilbert space, here $\mathcal{H}_D$. It can be shown, that a boundary condition along the τ direction corresponds to an insertion of an operator, here the one realizing G, into the partition function.

A consistent string theory has to be invariant under modular transformations, that is large reparametrizations of the worldsheet. For the worldsheet torus, the modular group is $\mathrm{SL}(2, \mathbb{Z})$, generated by

$$\begin{aligned} \mathcal{T} &: \; \hat{\tau} \to \hat{\tau} + 1 \; , \\ \mathcal{S} &: \; \hat{\tau} \to -\frac{1}{\hat{\tau}} \; . \end{aligned} \tag{2.5.5}$$

[14]We will see shortly, that there are different notions of what is called untwisted.

Clearly, these transformations act non–trivially on the boundary conditions (2.5.4), i.e.

$$\begin{aligned} \mathcal{T} \ &: \ D \to D \ , \\ & \quad\, G \to GD \ . \end{aligned} \tag{2.5.6}$$

$$\begin{aligned} \mathcal{S} \ &: \ D \to G \ . \\ & \quad\, G \to D^{-1} \ . \end{aligned} \tag{2.5.7}$$

We see, that different sectors of Hilbert spaces are mapped onto each other, which is something we encountered for the twisted strings before. Now, consider the special transformation $\mathcal{TST}$ and assume that G is an element of the centralizer of D in S. Then it follows from (2.5.7), that

$$\begin{aligned} \mathcal{TST} \ &: \ D \to DG \ , \\ & \quad\, G \to G \ , \end{aligned} \tag{2.5.8}$$

For D trivial, the untwisted Hilbert space is mapped to a twisted one, but since the centralizer is the whole space group, in fact to all twisted Hilbert spaces. Therefore the inclusion of twisted states is necessary to ensure modular invariance.

By cutting and gluing arguments for worldsheets it has been shown that one–loop modular invariance in string theory is sufficient to guarantee modular invariance to all orders (for details see [58, 59, 60]).

2.5.2 Toroidal Compactification

Before turning to the full space group we want to shortly consider toroidal compactifications of the free theory and interpret the torus as an orbifold (in the generalized sense). Consider compactification on the flat torus $\Lambda = \mathbb{R}^d/\{u\}$, spanned by basis vectors e_i ($\{u\}$ denotes the group generated by pure translations within the space group S). Then, a translation e_i acts as $\exp(ipe_i)$ on the Hilbert space associated with (2.5.1) with $p = (p_L + p_R)/2$ the momentum operator. Clearly, since the group of translations is Abelian, the centralizer is the whole group and every single shift corresponds to its own conjugacy class. Using this, for the untwisted states invariance under the group action enforces the momenta to lie on a lattice $2\pi\Lambda^*$ (Λ^* denotes the lattice dual to Λ) giving rise to the usual Kaluza–Klein (KK) modes n.

However, additionally there are the winding states not satisfying (2.5.1) but

$$Z(z', \bar{z}') = Z(z, \bar{z}) + u \tag{2.5.9}$$

instead ($u \in \Lambda$ and $u \neq 0$). From (2.1.1) it follows, that the winding is given by $u = 2\pi\alpha'(p_R - p_L)/2 \equiv 2\pi w$ with w the winding number. These are the twisted strings. Still the same projection as before has to be applied, but this gives no new result, the momenta have to lie on Λ^* again. Note, that because there are infinitely many different possible u's (conjugacy classes) within (2.5.9), there are as many twisted sectors.

What we have omitted up to now is, that in (2.4.10) we have associated a gauge transformation (Wilson line) (a_i) to an translation e_i, which yields an additional action of $\exp(2\pi i a_i q)$ on the gauge d.o.f.. In the presence of Wilson lines, invariance under the group action therefore also enforces the condition

$$qa_i \in \mathbb{Z} \quad \text{for all } q \in \Gamma \;, \tag{2.5.10}$$

in the untwisted sector, which has to be superimposed. These actions on the gauge d.o.f. are completely determined by the structure of the space group due to the linear map (2.4.10).

We want to further stress, that the KK– and winding modes can give rise to four dimensionally massless states by choosing specific discrete values for the radii of Λ (e.g. self–dual radii). However, we are not interested in these states but only in those descending from the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group throughout this work. Therefore, we always assume that the radii are away from such points!

2.5.3 The Untwisted Sector

Generalizing to the full space group S we first have to mention, that it is common in the literature to (re)define the term untwisted w.r.t. the point group, so to all boundary conditions $D = (\mathbb{1}, u)$ within (2.5.2). Then, even the twisted sectors of the toroidal case studied before, now belong to the untwisted sector of the orbifold. However, since for any non–trivial shift u we will get winding contributions, that we decouple by assumption, we can restrict to $D = (\mathbb{1}, 0)$ again. Therefore, the states are exactly the untwisted states of the toroidal compactified theory (those without winding), but subject to an enlarged additional projection, since the whole point group, that has been omitted before, commutes with D as well. We will work this out in detail in section 2.6. The untwisted states are sometimes called bulk states as well.

2.5.4 The Twisted Sectors

We have seen, that the structure of the space group is important to find invariant states. Moreover, for the geomteric orbifolds in section 2.3 we already expressed inequivalent elements of the space group in terms of fixed points and fixed planes. This classification directly carries over to the heterotic string, since from (2.4.1) and (2.1.1) we just have to replace the fixed point/plane coordinates of the geometric orbifold by the center of mass coordinates of the twisted string in question. This gives rise to figure 2.2, where strings of different orbifold sectors are depicted.

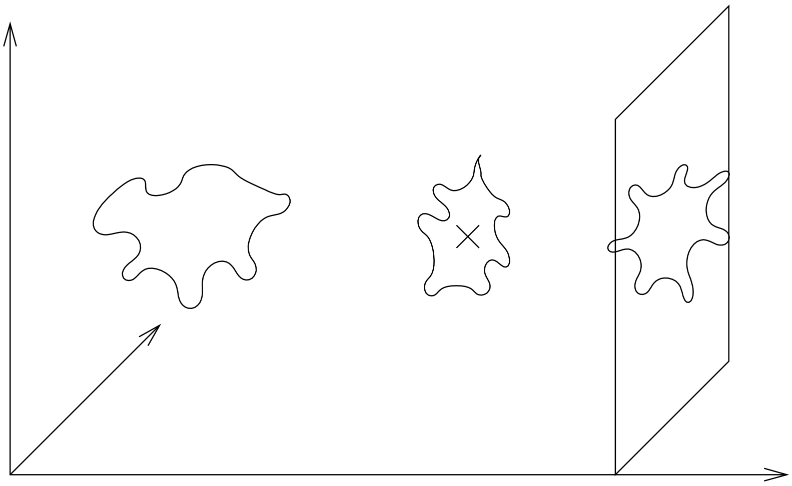

Figure 2.2: Qualitative sketch of strings of different orbifold sectors within a fictive three dimensional internal compactified spacetime. From left to right: a) untwisted strings; there is no restriction on their CM (center of mass) coordinate, they can move through the internal manifold, b) twisted strings corresponding to a fixed point; their CM is fixed to a point and cannot move to any direction, c) twisted strings corresponding to a two–dimensional fixed plane; their CM is fixed only w.r.t. one coordinate.

It is very convenient to borrow the brane language of string theories containing open strings to distinguish the different states, where here we define a brane by the span of directions in which the CM coordinate of the string can move. By k-brane we will denote branes which extend in k spatial dimensions. Please be aware, that although this language is borrowed from the D-branes of open strings, they have nearly nothing in common as physical objects!

Again, details will be worked out in section 2.6.

2.6 Spectra of Orbifolds with Wilson–lines

2.6.1 Construction Elements

In both, untwisted and twisted sectors, we have to project onto centralizer invariant subspaces, so what we in general need, are twist eigenvalues or equivalently twist phases. Becaue of this, we first have to go through all construction elements of the free heterotic string again, to collect these phases. By $(\theta, u; \beta(v, a_i)) = D$ we will from now on label the space group element D, defining the Hilbert space of the state, and by $(\hat\theta, \hat u; \hat\beta(\hat v, \hat a_i)) = G$ a space group element G appearing in the projector, that projects to D's centralizer.

We start with a bosonic worldsheet field in $\mathcal{H}_D$ again. Its mode expansion

(2.1.1) consists of the center of mass coordinates, which already has been discussed in terms of the geometric orbifold and left- and rightmoving momenta. The latter we conveniently expressed in terms of KK- and winding numbers, n and w respectively. From (2.4.4) one reads off, that $Gn = \hat{\theta}n$ and $Gw = \hat{\theta}w$. This leaves just the oscillator part of (2.1.1). They transform as (G acts only via $\hat{\theta}$ on the oscillators)

$$G\alpha^i_{r_i} = \exp(+2\pi i\hat{\phi}^i)\alpha^i_{r_i}\ , \qquad G\alpha^{\bar{\imath}}_{s_i} = \exp(-2\pi i\hat{\phi}^i)\alpha^{\bar{\imath}}_{s_i}\ , \tag{2.6.1}$$

$$G\tilde{\alpha}^i_{\tilde{r}_i} = \exp(+2\pi i\hat{\phi}^i)\tilde{\alpha}^i_{\tilde{r}_i}\ , \qquad G\tilde{\alpha}^{\bar{\imath}}_{\tilde{s}_i} = \exp(-2\pi i\hat{\phi}^i)\tilde{\alpha}^{\bar{\imath}}_{\tilde{s}_i}\ . \tag{2.6.2}$$

Now consider the rightmoving (complexified) worldsheet fermions $\tilde{\psi}^i$. Their mode expansion is given in (2.1.17)[15]. Then, from (2.4.6) one can read off the action of G on the fermionic oscillators

$$G\tilde{\psi}^i_r = \exp(+2\pi i\hat{\phi}^i)\tilde{\psi}^i_r\ , \quad G\tilde{\psi}^{\bar{\imath}}_s = \exp(-2\pi i\hat{\phi}^{\bar{\imath}})\tilde{\psi}^{\bar{\imath}}_s\ . \tag{2.6.3}$$

However, we have seen that it was very convenient to work with the lattice $\tilde{\Gamma}$ directly. Using (2.1.14) and (2.1.19) one can readily read off the transformation phase of a state $|\,\tilde{q}\,\rangle$ to be

$$G|\,\tilde{q}\,\rangle = \exp(2\pi i\tilde{q}_i\hat{\phi}^i)|\,\tilde{q}\,\rangle\ . \tag{2.6.4}$$

The leftmoving (complexified) worldsheet fermions of the gauge d.o.f. can be treated completely analogous, giving rise to

$$G|\,q\,\rangle = \exp(2\pi i q_I\hat{\beta}^I(\hat{v},\hat{a}_i))|\,q\,\rangle\ . \tag{2.6.5}$$

We can now turn to the different sectors.

2.6.2 The Untwisted Sector

As shown in subsection 2.5.3 we just consider $\mathcal{H}_D$ with $D = (\mathbb{1}, 0; 0)$. Using this, the monodromy condition (2.5.2) is nothing but single–valuedness. The solutions are those of the free heterotic string, i.e. (2.1.1) with $r_i, s_i, \tilde{r}_i, \tilde{s}_i \in \mathbb{Z}$. The same is true for the fermionic worldsheet fields, so that $q \in \Gamma = \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8}$ and $\tilde{q} \in \tilde{\Gamma}$ like in (2.1.24). The zero mode Virasoro generators determining the mass of the states are still given by (2.1.20) and (2.1.21).

Therefore, all we have to do is to ensure invariance under the full space group. Instead of generating all images under it (by including the complete projector), we will just collect all twist transformation phases seperatetly and find all possibilities to combine space group singlets (which is of course equivalent). Therefore

[15] Although the mode expansion was derived from fermionic single valuedness, by relaxing any restrictions on $\tilde{\nu}$ it is the most general one.

we define the left- and rightmoving phases under the twist $s^G_{L/R}$, which follow from the last subsection

$$\begin{aligned} s_L^G &\equiv \sum_{i=(2,)3}^{4} \hat{\phi}^i \left(\sum_{r_i<0} N_{r_i}^{\alpha^i} - \sum_{s_i<0} N_{s_i}^{\alpha^{\bar{\imath}}} \right) + \hat{\beta}^I(\hat{v},\hat{a}_i) q^I \ , \\ s_R^G &\equiv \sum_{i=(2,)3}^{4} \hat{\phi}^i \left(\sum_{\tilde{r}_i<0} N_{\tilde{r}_i}^{\tilde{\alpha}^i} - \sum_{\tilde{s}_i<0} N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{\imath}}} \right) + \hat{\phi}^i \tilde{q}^i \ . \end{aligned} \tag{2.6.6}$$

The form of the leftmoving phase is chosen such, that it incorporates the shift embedding of the orbifold (2.4.10). Then, all untwisted states have to fulfill

$$e^{2\pi i (s_L^G + s_R^G)} = 1 \tag{2.6.7}$$

for all $G \in S$. Remember, that the Wilson lines restrict the possible q's via the condition

$$q \cdot a_i \in Z \ , \tag{2.6.8}$$

due to invariance under torus shifts, just like for the toroidal compactificaton.

2.6.3 The Twisted Sectors

We have seen in the last section, that the different twisted sectors are in one-to-one correspondence to different fixed points and fixed planes, depending on the point group element. Given such a point group element, we label the corresponding fixed points/branes by an index l. Consider a Hilbert space $\mathcal{H}_D$ with $D = (\theta^k, u^{(k,l)}; kv + m^i(k,l)a_i)$. The fact, that θ^k has to be amended with a gauge shift kv follows from linearity of (2.4.10). The action of the Wilson line on the gauge d.o.f. is clearly dependent on the specific fixed point/plane in question. We absorbed this dependency in the coefficient $m^i(k,l)$. We will call the sector corresponding to such a D as (k,l)-th twisted sector[16].

From (2.5.2), (2.4.4) and (2.1.1) we can read off the conditions for the twisted worldsheet bosons. The ones for the center of mass coordinates are solved by the fixed points/planes as described before. Once the fixed point/plane l is chosen, the torus shift $u(k,l)$ is fixed, that brings the point/plane back to itself. Expanding in the torus basis $u(k,l) = m^i(k,l)e_i$ defines the coefficients $m^i(k,l)$, that appear in the Wilson line part of D via linearity of (2.4.10). We therefore identify

$$\beta(v, a_i) = kv + m^i(k,l)a_i \ . \tag{2.6.9}$$

For the internal momenta we get

$$\theta^k p_{L/R} = p_{L/R} \ , \tag{2.6.10}$$

[16]In the literature the sectors are usually only distingushed as k-th twisted sector, which is enough for orbifolds without Wilson lines.

so that they have to be zero in the directions, where θ^k acts non–trivially. This will not always be the case, since e.g. for fixed planes translations along the branes are allowed. This only leaves the oscillator part of (2.1.1), enforcing

$$\alpha^i, \alpha^{\bar{\imath}}:\ r_i, \tilde{s}_i \in \mathbb{Z} + k\phi^i \ , \quad \tilde{\alpha}^i, \tilde{\alpha}^{\bar{\imath}}:\ \tilde{r}_i, s_i \in \mathbb{Z} - k\phi^i \ . \tag{2.6.11}$$

The modes of the rightmoving fermionic worldsheet fields follow directly from subsection 2.1.2, where we already encountered non–trivial monodromies in (2.1.16)[17]. By combining the $\widetilde{\mathrm{NS}}/\widetilde{\mathrm{R}}$ boundary conditions (2.1.5) with the orbifold action (2.4.6) (to the power k) and comparing with (2.1.16), one can read off the modes from (2.1.17) to be

$$\tilde{\psi}^i:\ r \in \mathbb{Z} + \tilde{\nu} - k\phi^i \ , \quad \tilde{\psi}^{\bar{\imath}}:\ s \in \mathbb{Z} - \tilde{\nu} + k\phi^i \ . \tag{2.6.12}$$

Now analogous to 2.1.2 again one can construct a lattice on which the bosonized momenta $\tilde{q}'$ are sitting. By using the combined monodromies together with the bosonization prescription (2.1.14) and (2.1.15) it is clear that the orbifold part of the monodromy enters via an offset of $+k\phi^i$ w.r.t. the untwisted lattice without GSO modded out. The problem is to define a GSO projection for the twisted strings. Let $\tilde{\Gamma}'$ denote the lattice, where such a twisted GSO would have been imposed. Then the momenta lie on $\tilde{q}' \in \tilde{\Gamma}' + k\phi$. The answer to which GSO has to be taken comes from modular invariance, in especially from the modular transformation $\mathcal{TST}$ (2.5.8), that maps untwisted states to twisted ones. Since the orbifold action only shifts the rightmoving lattice momenta, like the oscillators generating the lattice, and shifts always commute, modular invariance requires that $\tilde{\Gamma}' = \tilde{\Gamma}$. But if the lattices have to coincide, we can choose the same basis for groundstates with worldsheet fermion numbers (2.1.22) and GSO projections (2.1.23) as for the untwisted case[18]. To summarize, we have

$$\tilde{q}' \in \tilde{\Gamma} + k\phi \ , \tag{2.6.13}$$

with $\tilde{\Gamma}$ given in (2.1.24).

For the leftmoving worldsheet fermions again everything works completely analogous, giving rise to

$$q' \in \Gamma - (kv + m^i(k,l)a_i) \ , \tag{2.6.14}$$

with $\Gamma = \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8}$ as defined in (2.1.25).

To determine the masses of the twisted states, one has to consider the zero mode Virasoro generators again. They do not coincide with the untwisted ones (2.1.20) and (2.1.21), since changing the boundary conditions of the worldsheet

[17] Please remember, that the discussion in 2.1.2 was valid for real $\nu, \tilde{\nu}$ as well.

[18] Note, that the original definition of the worldsheet fermion number (2.1.10) can not be simply applied on the twisted states $\tilde{q}'$.

fields changes the zero point energies as well. We refrain from giving the details here, but just state the result

$$\begin{aligned} L_0 &= \frac{\alpha'}{4}p_L^2 + \sum_i \left(\sum_{r_i<0}(-r_i)N_{r_i}^{\alpha^i} + \sum_{s_i<0}(-s_i)N_{s_i}^{\alpha^{\bar{\imath}}} \right) \\ &\quad + \frac{1}{2}q'^2 + \sum_I \sum_{n>0} nN_n^{\gamma^I} - 1 + \frac{1}{2}\sum_i ||k\phi^i||(1-||k\phi^i||) \ , \quad (2.6.15) \\ \tilde{L}_0 &= \frac{\alpha'}{4}p_R^2 + \sum_i \left(\sum_{\tilde{r}_i<0}(-\tilde{r}_i)N_{\tilde{r}_i}^{\tilde{\alpha}^i} + \sum_{\tilde{s}_i<0}(-\tilde{s}_i)N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{\imath}}} \right) \\ &\quad + \frac{1}{2}\tilde{q}'^2 + \sum_i \sum_{n>0} nN_n^{\tilde{\gamma}^i} - \frac{1}{2} + \frac{1}{2}\sum_i ||k\phi^i||(1-||k\phi^i||) \ , \quad (2.6.16) \end{aligned}$$

where $||\phi^i||$ denotes ϕ^i integrally shifted into the set $[0,1)$.

An element G in the centralizer of D in S acting on such a state, gives rise to phases

$$\begin{aligned} s_L^G &\equiv \sum_{i=(2,)3}^{4} \hat{\phi}^i \left(\sum_{r_i<0} N_{r_i}^{\alpha^i} - \sum_{s_i<0} N_{s_i}^{\alpha^{\bar{\imath}}} \right) + \hat{\beta}^I(\hat{v},\hat{a}_i)q'^I \ , \\ s_R^G &\equiv \sum_{i=(2,)3}^{4} \hat{\phi}^i \left(\sum_{\tilde{r}_i<0} N_{\tilde{r}_i}^{\tilde{\alpha}^i} - \sum_{\tilde{s}_i<0} N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{\imath}}} \right) + \hat{\phi}^i\tilde{q}'^i \ . \end{aligned} \quad (2.6.17)$$

Collecting the phases, invariance under the centralizer enforces

$$e^{2\pi i(s_L^G+s_R^G)} = 1 \ . \quad (2.6.18)$$

2.7 Constraints on Heterotic Orbifolds

Up to now, we have neglected most of the conditions that constrain the defining orbifold data: twist θ or equivalently twist vector ϕ, gauge shift v and Wilson lines a_i. For a discussion of target space constraint see e.g. [44, 50], concerning the worldsheet constraints we use and extend the approaches of [16, 45].

2.7.1 Target Space Constraints

We start by considering the twist vector ϕ and remember, that one of our original motivations to study orbifolds, was to realize $\mathcal{N}=1$ SUSY in four or six uncompactified dimensions. The 16 supersymmetries of the $D=10, \mathcal{N}=1$ SUSY algebra of the free heterotic theory are given by a single left–handed Majorana–Weyl spinor $Q_{\mathbf{s}}$ (c.f. appendix B). In the spin basis of the SO(8) little group it is given by

$$Q_{\mathbf{s}} = |\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}\rangle \quad (2.7.1)$$

with #(−) even. The orbifold twist acts on the supercharge as usual

$$Q_{\mathbf{s}} \to \exp(2\pi i s_i \phi^i) Q_{\mathbf{s}} \ , \tag{2.7.2}$$

so that only those supercharges survive for which $s_i \phi^i \in \mathbb{Z}$. It is easy to verify, that the condition[19]

$$\sum_i \phi^i = 0 \mod 2 \tag{2.7.3}$$

gives the right amount of unbroken SUSY, four for $d = 6$ and eight for $d = 4$ compactified dimensions.

This is not the only restriction on the twist vector. There is a much stronger one, coming from the fact, that θ should be a symmetry of the compactification torus, and therefore act as a lattice automorphism on the torus lattice Λ. This restricts not only θ, but Λ as well. All possibilities have been classified for $d = 4$ and $d = 6$ in the literature. We will not list them here but refer to [61, 62, 44, 63, 64] where several aspects and methods of classifications are discussed.

The gauge transformations (shift and Wilson lines) are restricted through linearity of the mapping (2.4.10) that associates space group transformations to them. What we are going to exploit is the simple fact, that a linear map always maps one identity to the other. So by producing identities within the space group and using the linear mapping we get restrictions on v and a_i. Let $P = \mathbb{Z}_N$, i.e. $\theta^N = \mathbb{1}$. Then, the equation

$$(\sum_{k=1}^{M} \theta^{kN/M}) e_i = 0 \tag{2.7.4}$$

holds with $N = M$ for all i. For orbifolds with N non–prime, so called $\mathbb{Z}_M$ subsectors appear ($M \neq 1$). Then there exists some i, such that (2.7.4) is fulfilled for an $M \neq N$, that divides N. Note, that this is exactly the case when there are fixed planes (not only fixed points). The space group multiplication law is given by

$$(\theta, u) \cdot (\omega, t) = (\theta\omega, u + \theta t) \ . \tag{2.7.5}$$

- Consider first the following mapping

  $$(\mathbb{1}, 0) = (\theta^N, 0) = (\theta, 0)^N \to (Nv) \ .$$

  in the group of gauge transformations. In order to (Nv) describe a trivial gauge transformation (no transformation at all), it follows from (2.4.8) that Nv has to be an element of the lattice dual to Γ. However, Γ is self–dual so that the gauge shift is restricted by

  $$Nv \in \Gamma = \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8} \ . \tag{2.7.6}$$

[19] The condition $\sum_i \phi^i = 1 \mod 2$ would have worked equally well, therefore the exact form of (2.7.3) is a convention w.r.t. which SUSY spinors we want to survive, not about the amount of unbroken SUSY itself.

- Next consider

$$
\begin{aligned}
(\mathbb{1},0) &= (\theta^N,(\sum_{k=1}^{M}\theta^{kN/M})e_i) = [(\theta^{N/M},0)\cdot(\mathbb{1},e_i)]^M \\
&\to (M[(N/M)v+a_i]) = (Nv+Ma_i)\ .
\end{aligned}
$$

With $M=N$ this holds for all i. Since from the first result we know that (Nv) acts trivially, it then follows that

$$
Na_i \in \Gamma = \Gamma_{\mathrm{E}_8}\times\Gamma_{\mathrm{E}_8} \quad \text{for all } i. \tag{2.7.7}
$$

However, in the presence of a $\mathbb{Z}_M$ subsector in, say, the j-th direction, the stronger condition

$$
Ma_j \in \Gamma = \Gamma_{\mathrm{E}_8}\times\Gamma_{\mathrm{E}_8} \quad \text{for } \mathbb{Z}_M \text{ along } e_j \tag{2.7.8}
$$

has to hold.

- Another constraint on the Wilson lines arises, whenever two basis vectors are related via a linear transformation, $e_j = M_{ji}e_i$ with M_{ji} a $d\times d$ dimensional integral[20] matrix. Then

$$
(\mathbb{1},0) = (\mathbb{1},e_j - M_{ji}e_i) \to (a_j - M_{ji}a_i)\ ,
$$

giving rise to the condition

$$
e_j = M_{ji}e_i \quad \Rightarrow \quad a_j - M_{ji}a_i \in \Gamma = \Gamma_{\mathrm{E}_8}\times\Gamma_{\mathrm{E}_8}\ . \tag{2.7.9}
$$

Equations (2.7.6), (2.7.7), (2.7.8) and (2.7.9) incorporate all constraints following from linearity of the map (2.4.10).

2.7.2 Worldsheet Constraints

We first concentrate on modular transformations again. Consider the (k,l)-th twisted sector (allowing for the untwisted sector as well) corresponding to the space group element D again. Assume there is an M such that $D^M = \mathbb{1}$. We assume $M=N$, which is always an option. (Sometimes there is an $M<N$ for a certain group element D. We will comment on this case later). Then the modular transformation $\mathcal{T}^N$ (2.5.5) maps the twisted Hilbert space back to itself. This is clear, since $\mathbb{1}$ is always in the centralizer of D in S. On the other hand $\mathcal{T}^N$ maps the Teichmüller parameter $\hat{\tau}$ of the worldsheet torus to $\hat{\tau}+N$, so that the partition function of every twisted sector has to be invariant under

[20]Integrality is a necessary condition for the compactification torus to possess a cyclic symmetry, so that it must be given.

this transformation. Since the partition function is given by a trace over $q^{L_0}\bar{q}^{\tilde{L}_0}$, where $q = \exp(2\pi i\hat{\tau})$, the condition

$$L_0 - \tilde{L}_0 = 0 \mod 1/N \tag{2.7.10}$$

has to hold as an operator identity for every (k,l)-th twisted sector seperately. Equation (2.7.10) is called level matching condition. It has been shown in [21] for heterotic symmetric orbifolds, like considered here, that level matching already guarantees invariance under all modular transformations.

On the other hand $L_0 - \tilde{L}_0$ is the generator for worldsheet translations $\sigma \to \sigma + 2\pi$. Since we are only considering closed strings, invariance under this requires the seemingly stronger condition

$$L_0 - \tilde{L}_0 = 0 \mod 1 \tag{2.7.11}$$

to hold. In [45] the conditions (2.7.10) and (2.7.11) have therefore been distinguished as weak and strong level matching condition. We will adopt this language but show later, that both conditions lead to the same constraints for the models considered in this work. We will start with (2.7.11).

We now work out the consequences of (strong) level matching on the different sectors, starting with the untwisted one again. Using the Virasoro generators (2.1.20) and (2.1.21), that apply to the untwisted sector, we get the condition

$$L_0 - \tilde{L}_0 = \frac{\alpha'}{4}(p_L^2 - p_R^2) + \frac{1}{2}(q^2 - \tilde{q}^2) + \frac{1}{2} = 0 \mod 1 \ , \tag{2.7.12}$$

because all mode numbers are integer as follows from (2.1.2). From the same equation we see, that $p_L = p_R$ in the non–compact directions. For the internal directions it follows from the discussion of KK– and winding numbers, n and w respectively, that $\alpha'(p_L^2 - p_R^2)/4 = -w \cdot n$ (c.f. subsection 2.5.2). Since w and n are integral (2.7.12) boils down to the condition

$$q^2 - \tilde{q}^2 = 1 \mod 2 \ , \tag{2.7.13}$$

thus constraining the lattices Γ and $\tilde{\Gamma}$. Using (2.1.25) and (2.1.24), we see that the untwisted level matching condition is fulfilled and does not give new constraints.

Now consider the (k,l)-th twisted sector. From (2.6.15) and (2.6.16) one gets

$$\begin{aligned} L_0 - \tilde{L}_0 &= \frac{\alpha'}{4}(p_L^2 - p_R^2) - s_L^D - s_R^D + \frac{1}{2}(q'^2 - \tilde{q}'^2) + \\ &\quad + (kv + m^i(k,l)a_i)q' + (k\phi)\tilde{q}' = 0 \mod 1 \ , \end{aligned} \tag{2.7.14}$$

with $s^D_{L/R}$ as defined in (2.6.17). That the Virasoro eigenvalues can be recursed to the twist phases (up to integers) is due to the structure of the mode numbers

(2.6.11). Again the term proportional to $p_L^2 - p_R^2$ gives only integer contributions, analogous to the untwisted sector with the only difference, that in directions transversal to the fixed planes both momenta have to be zero. Next, the sum $s_L^D + s_R^D$ has to be integer, as follows from the projection (2.6.18): Note that D is in its own centralizer, since every element commutes with itself. Rewriting $q' = q - (kv + m^i(k,l)a_i)$ and $\tilde{q}' = q + \phi$ with $q \in \Gamma$ and $\tilde{q} \in \Gamma'$ as follows from (2.6.14) and (2.6.13), condition (2.7.14) simplifies to

$$L_0 - \tilde{L}_0 = \frac{1}{2}(q^2 - \tilde{q}^2) + \frac{1}{2}[(k\phi)^2 - (kv + m^i(k,l)a_i)^2] = 0 \mod 1 \tag{2.7.15}$$

and with level matching of the untwisted sector finally to

$$(k\phi)^2 - (kv + m^i(k,l)a_i)^2 = 0 \mod 2 \quad \text{for all } k, l. \tag{2.7.16}$$

These are the final conditions constraining gauge shift and/or Wilson lines.

We close this section with a comparison of (2.7.16) with its weak level matching counterpart.

$$(k\phi)^2 - (kv + m^i(k,l)a_i)^2 = 0 \mod 2/N \quad \text{for all } k, l. \tag{2.7.17}$$

The key observation (c.f. [45]) is, that unlike the weak level matching condition, the strong version (2.7.16) is not invariant under $\mathrm{E}_8 \times \mathrm{E}_8$ lattice shifts of gauge shift v and Wilson lines a_i. Following (2.4.11) this invariance should be given, since the orbifold actions in both models are identical. Let

$$LM \equiv (k\phi)^2 - (kv + m^i(k,l)a_i)^2 \,. \tag{2.7.18}$$

According to (2.7.6) and (2.7.7) we rewrite $v = s_v/N$ and $a_i = s_{a_i}/N$ with $s_{v/a_i} \in \Gamma$. Then under the transformations $v \to v + w$ and $a_i \to a_i + w$ with $w \in \Gamma$, LM changes according to

$$\begin{aligned}
\delta_v LM &= -k^2(w \cdot w) - 2k^2(w \cdot v) - 2km^i(k,l)(w \cdot a_i) \\
&= -\underbrace{[k^2(w \cdot s_v) + km^i(k,l)(w \cdot s_{a_i})]}_{\in \mathbb{Z}} \frac{2}{N} \mod 2 \\
\delta_{a_i} LM &= -(m^i(k,l))^2(w \cdot w) - 2km^i(k,l)(w \cdot v) - 2m^i(k,l)m^j(k,l)(w \cdot a_j) \\
&= -\underbrace{[km^i(k,l)(w \cdot s_v) + m^i(k,l)m^j(k,l)(w \cdot s_{a_j})]}_{\in \mathbb{Z}} \frac{2}{N} \mod 2
\end{aligned} \tag{2.7.19}$$

Here we used that since Γ is even, the square of every element is even, and the product of two different elements is an integer and further, that k and $m^i(k,l)$ are integers. Therefore we see, that enforcing invariance of the strong level matching condition under lattice shfits of v and a_i naturally leads to the weak form of the level matching condition. On the other hand the question arises, wether it is possible to allow for all orbifold data determined by the weak version only. The answer is yes, because (by integrality of Γ), it is always possible to find such a w that by tuning the underbraced terms in (2.7.19), $v+w$ and/or a_i+w satisfy the strong level matching conditions[21]

However, we still have to make sure, that a shift in v and/or a_i does not change the spectrum of the orbifold in our construction. In reviewing our construction one realizes, that the untwisted sector is unchanged, but there are changes in the twisted sectors: First consider (2.6.13). There q' is defined modulo lattice shifts, so that an additional lattice shift does not matter. The only restriction on q' comes from the mass level of the states we are going to determine, but this is unchanged for both situations. Therefore, the only place where the change matters is in the twisted projection phase s_L^D (2.6.17), in especially the term $\beta(v,a_i)q'$. Now, there is a nice trick to repair the phases, i.e. we add an additional phase to the projection condition, that exactly offsets the mismatch of weak and strong level matching conditions. Let $s^G_{\delta\mathrm{LM}}$ denote this phase. Then it is straightforward to show, that

$$s^G_{\delta\mathrm{LM}} = \frac{k}{2}[(\hat{\beta}(\hat{v},\hat{a}_i))^2 - \hat{\phi}^2] \tag{2.7.20}$$

does the job. Using this, the twisted projection (2.6.18) is changed to

$$e^{2\pi i(s_L^G+s_R^G+s^G_{\delta\mathrm{LM}})} = 1 \ . \tag{2.7.21}$$

There are two reasons, why this effort was worthwhile: Firstly, the adopted result now coincides with the results from a partition function approach, where modular invariance and therefore as we have seen invariance under lattice shifts is manifest. Secondly, it is much easier to find gauge shift and Wilson lines that define a valid orbifold. Therefore we will work with the weak version of the level matching condition (2.7.17) and the enhanced projection (2.7.21).

2.8 Examples

In this section we apply the general machinary developed in the previous sections to specific examples. There will be two steps of specialization: Firstly, only fixing the twist vector ϕ, which already determines the target space symmetries (and SUSYs) of states, and secondly fixing gauge shift v and Wilson lines a_i, which determines the gauge symmetries of states.

The examples will be worked out with decreasing information on details. Important differences or novelties will be stressed however.

[21] There is one possible loophole here, because the integrality argument of Γ breaks down exactly for $v=a_i=0$, which is an allowed orbifold for $\mathbb{Z}_3$ as we will see later. But then one can apply the same reasoning as before to $\phi\to\phi+\chi$ with $\chi\in\tilde{\Gamma}^*$.

2.8.1 $T^4/\mathbb{Z}_2$

Preliminaries

Orbifolds of this kind have been classified e.g. in [65]. We take $T^4 = T^2 \times T^2$ and choose the compactification two–tori to be quadratic with common radii R (This is not enforced by any crystallographic restrictions here, but will prove convenient later). We realize the $\mathbb{Z}_2$ symmetry by a twist vector[22] $\phi = (1/2, -1/2)$, acting on the worldsheet field components $i = 3, 4$.

Beside the untwisted, there is only one, the first, twisted sector for $k = 1$. In this sector there are only fixed points, that in brane terminology appear as 5-branes, therefore supporting six dimensional states. In the language of section 2.3 we find

$$I = \{0\} \ , \quad N = \Lambda \ , \quad \left| \frac{N}{(\mathbb{1} - \theta)\Lambda} \right| = 2^4 = 16 \ , \tag{2.8.1}$$

so that there 16 inequivalent 5-branes, labeled by $l = 1, \ldots, 16$. The situation is depicted in figure 2.3. In fact, drawing such figures is mostly the easiest possibility to solve the fixed point/plane conditions and read off the relevant compactification lattice shifts u, needed to bring a point back to itself.

The space group element corresponding to fixed point l is given by $D = (\theta, u(l))$. Using the space group multiplication law (2.7.5) it follows, that the centralizer $C(D)$ of D in S contains only the identity and D itself (and of course D^{-1}, which does not give new information). Therefore we need invariance under D. We choose one representative within the fundamental domain of the torus and have to superimpose all states, that are images of this representative under $S - C(D)$. For states corresponding to a given fixed point within the fundamental domain, there is no such mapping to a state corresponding to another fixed point, so in this sense the fixed points are unique.

To construct the spectrum we first note, that the spacetime SO(8) is broken down to $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_H$ as explicitly shown in appendix A.4.3. However, from the discussion at the end of section 2.3 we know, that the $\mathrm{SU}(2)_H$ holonomy group is broken to $\mathbb{Z}_2$ in our orbifold limit. Therefore, a non–trivial representation under the "would–be" holonomy $\mathrm{SU}(2)_H$ gives just rise to multiplicities. Since the spectrum will possess $D = 6, \mathcal{N} = 1$ SUSY, corresponding supermultipletts are listed in appendix B.2.

The Untwisted Sector

We now construct the untwisted sector, starting with the rightmovers. In subsection 2.6.2 we learned, that we just need to classify the states $|\,\tilde{q}\,\rangle$ with $\tilde{q} \in \tilde{\Gamma}$ and $\tilde{q}^2 = 1$ according to their twist eigenvalues s_R^G given in (2.6.6). We label

[22] Up to equivalent twist vectors, this is the only choice consistent with $D = 6, \mathcal{N} = 1$ SUSY.

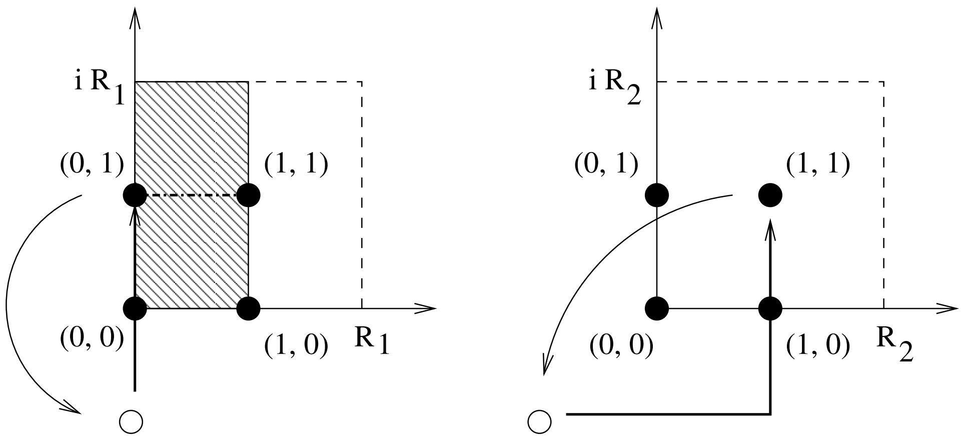

Figure 2.3: The orbifold $(T^2 \times T^2)/\mathbb{Z}_2$: A fundamental domain of $T^2 \times T^2$ is framed by the dashed boxes. Every T^2 is chosen to be spanned by the vectors, R and iR. There are 16 fixed points, which are here labeled by pairs of numbers in each T^2 and depicted by filled circles. That the points are fixed can be seen e.g. for $l = ((0,1),(1,1)')$ whose image under θ is depicted by empty circles. The lattice shifts needed to bring the point back to itself (within the fundamental domain), is decomposed into the torus basis vectors and depicted by straight lines. They determine possible Wilson line contributions. The shaded region in the first torus times the whole fundamental domain of the second torus shows a possible choice of fundamental domain for the orbifold.

the eigenvalues in powers of $\alpha = \exp(2\pi i/N) = -1$, spacetime/SUSY representations are worked out in appendix A.4.3. The result is given in table 2.1.

Sector	State(s)	Twist eigenvalue	$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R$ rep
$\widetilde{\mathrm{NS}}$	$\vert\, \underline{\pm 1, 0}, 0, 0\,\rangle$	α^0	$(\mathbf{2},\, \mathbf{2};\, \mathbf{1})$
$\widetilde{\mathrm{NS}}$	$\vert\, 0, 0, \underline{\pm 1, 0}\,\rangle$	α^1	$2\ (\mathbf{1},\, \mathbf{1};\, \mathbf{2})$
$\widetilde{\mathrm{R}}$	$\vert\, \pm(\frac{1}{2}, \frac{1}{2}), \pm(\frac{1}{2}, \frac{1}{2})\,\rangle$	α^0	$(\mathbf{2},\, \mathbf{1};\, \mathbf{2})$
$\widetilde{\mathrm{R}}$	$\vert\, \underline{+\frac{1}{2}, -\frac{1}{2}}, \underline{+\frac{1}{2}, -\frac{1}{2}}\,\rangle$	α^1	$2\ (\mathbf{1},\, \mathbf{2};\, \mathbf{1})$

Table 2.1: Twist eigenvalues $\exp(2\pi i \tilde{q}_i \phi^i)$ of the rightmovers and $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ representations for $T^4/\mathbb{Z}_2$.

For the leftmovers the situation is more involved. Like for the free heterotic string, from (2.1.20) there are two different ways to arrive at massless states. Those with $q^2 = 0$ but bosonic oscillator excitations, and states with $q^2 = 2$

and without bosonic oscillator excitations ($q \in \Gamma$). The states descending from the first case are sometimes called universal in the literature, because they never depend on gauge shift and/or Wilson lines, but only on N. The twist eigenvalues are computed from (2.6.6) and given in table 2.2.

State(s)	Twist eigenvalue	$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R$ rep	Gauge rep
$\alpha^{\mu}_{-1}\,\|\,0\,\rangle$	α^0	$(\mathbf{2},\,\mathbf{2};\,\mathbf{1})$	$\|\,\mathbf{1}\,\rangle$
$\alpha^{3/\bar{4}}_{-1}\,\|\,0\,\rangle$	α^1	$2\,(\mathbf{1},\,\mathbf{1};\,\mathbf{1})$	$\|\,\mathbf{1}\,\rangle$
$\alpha^{\bar{3}/4}_{-1}\,\|\,0\,\rangle$	α^1	$2\,(\mathbf{1},\,\mathbf{1};\,\mathbf{1})$	$\|\,\mathbf{1}\,\rangle$

Table 2.2: Twist eigenvalues of the leftmovers in the universal sector ($q^2 = 0$), $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ and gauge representations for $T^4/\mathbb{Z}_2$.

The states with $q^2 = 2$ all transform as spacetime singlet. Their twist eigenvalues are given by $\exp(2\pi i p\cdot v)$ and therefore depend heavily on the gauge shift. For a fixed twist eigenvalue $c/N = c/2$ for $c = 0, \cdots N-1$ we face the usual situation of Cartan symmetry breaking (c.f. appendix A.4), i.e. we will find representations of the gauge group unbroken by v and all a_i (remember, that $q \cdot a_i \in \mathbb{Z}$ has to be superimposed). In table 2.3 we listed all possible eigenvalues and introduced notation for the representations, that will finally emerge from the breaking.

State(s)	Twist eigenvalue	$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R$ rep	Gauge rep
$q \cdot v = 0 \mod 1$	α^0	$(\mathbf{1},\,\mathbf{1};\,\mathbf{1})$	$\|\,\mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]}\,\rangle$
$q \cdot v = 1/2 \mod 1$	α^1	$(\mathbf{1},\,\mathbf{1};\,\mathbf{1})$	$\|\,\mathbf{R}^{(\mathbf{1})}_{[\mathbf{v}][\mathbf{a_i}]}\,\rangle$

Table 2.3: Twist eigenvalues of the leftmovers with $q^2 = 2$, $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ representations and notation for gauge representations for $T^4/\mathbb{Z}_2$.

The notation may seem overkill, but will be very useful in what follows. For general N we define

$$\begin{aligned} \mathbf{R}^{(\mathbf{c})}_{[\mathbf{v}][\mathbf{a_i}]} &\cong \{q \in \Gamma \mid q^2 = 2,\ q \cdot v = c/N \bmod 1,\ q \cdot a_i = 0 \bmod 1\}\ , \\ \mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]} &= \mathbf{R}^{(\mathbf{0})}_{[\mathbf{v}][\mathbf{a_i}]}\ , \end{aligned} \tag{2.8.2}$$

and denote the unbroken gauge group (that is completely determined by $\mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]}$) by $G_{[v][a_i]}$. The representations $\mathbf{R}^{(\mathbf{c})}_{[\mathbf{v}][\mathbf{a_i}]}$ and $\mathbf{R}^{(\mathbf{N-c})}_{[\mathbf{v}][\mathbf{a_i}]} = \mathbf{R}^{(-\mathbf{c})}_{[\mathbf{v}][\mathbf{a_i}]}$ are always complex conjugate to each other, as is explained in appendix A.4.1. For $N = 2$ this leads to the conclusion, that $R^{(1)}_{[v][a_i]}$ is always real, as it is complex conjugated to itself.

Tensoring the leftmoving states from tables 2.2 and 2.3 with the rightmoving ones from table 2.1 in such a way that the product of twist eigenvalues is always α^0, one arrives at the massless untwisted spectrum displayed in table 2.5.

The Twisted Sectors

We only have to deal with $k = 1$ (l arbitrary) and start with the rightmovers. Due to the twist the zero–point energy is shifted from $-1/2$ to $-1/4$, as can be read off from (2.6.16). But since all oscillator modes are at least half–integral, as can be read off from (2.6.11) and (2.6.12), the only possibility to find massless states, is from the lattice contributions. Masslessness therefore imposes the condition

$$\frac{1}{2}(\tilde{q}')^2 = \frac{1}{2}(\tilde{q} + \phi)^2 = \frac{1}{4}\,, \quad \tilde{q} \in \tilde{\Gamma}\,, \tag{2.8.3}$$

which from (2.1.24) and $\phi = (0, 0, 1/2, -1/2)$ implies

$$\tilde{q'} \in \{|\,0, 0, \pm(\tfrac{1}{2}, \tfrac{1}{2})\,\rangle, |\,\pm(\tfrac{1}{2}, -\tfrac{1}{2}), 0, 0\,\rangle\}\,. \tag{2.8.4}$$

In appendix A.4.3 the representations are worked out for the highest weights, resulting in table 2.4.

Sector	State(s)	$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R$ rep
$\widetilde{\mathrm{NS}}$	$\|\,0, 0, \pm(\frac{1}{2}, \frac{1}{2})\,\rangle$	$(\mathbf{1},\, \mathbf{1};\, \mathbf{2})$
$\widetilde{\mathrm{R}}$	$\|\,\pm(\frac{1}{2}, -\frac{1}{2}), 0, 0\,\rangle$	$(\mathbf{1},\, \mathbf{2};\, \mathbf{1})$

Table 2.4: Twisted rightmovers and $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ representations for $T^4/\mathbb{Z}_2$.

For the leftmovers again the situation is more involved. Due to (2.6.15) the zero–point energy is shifted from -1 to $-3/4$ and since the (four) internal leftmoving bosonic oscillators have mode numbers $1/2 \mod 1$, which follows from (2.6.11), bosonic excitations are allowed. Let $\mathbf{S}_{[\beta(1,l)]}$ and $\mathbf{D}_{[\beta(1,l)}$ denote the gauge representations emerging from states without and with bosonic excitations. Accordingly, there are two masslessness conditions defining those representations

$$\begin{aligned} \mathbf{S}_{[\beta(1,l)]} &\cong \frac{1}{2}(q')^2 = \frac{1}{2}[q - (v + m^i(1,l)a_i)]^2 = \frac{3}{4}\,, \quad q \in \Gamma\,, \\ (4\times)\ \mathbf{D}_{[\beta(1,l)} &\cong \frac{1}{2}(q')^2 = \frac{1}{2}[q - (v + m^i(1,l)a_i)]^2 = \frac{1}{4}\,, \quad q \in \Gamma\,. \end{aligned} \tag{2.8.5}$$

Note, that the latter states come with a multiplicity of 4, corresponding to the four internal dimensions.

It is interesting to consider the twist eigenvalues under D (which is the only independent non–trivial element of the centralizer of D in S). Clearly the rightmoving states are invariant. For the leftmoving states one can use the masslessness conditions and the fact that Γ is even to show that the phase is given by

$$q' \cdot \beta(1,l) = -\frac{3}{4} - \frac{1}{2}(\beta(1,l))^2 \mod 1 \tag{2.8.6}$$

for both groups of states (the bosonic oscillators transform with α^1). Together with the phase correction $s^G_{\delta\mathrm{LM}}$ introduced in (2.7.20) and the explicit form of ϕ it is easy to see, that the twisted states are invariant as a consequence of level matching. Although this result is fine for orbifolds with prime N, it should be handled with care, since there are subtleties arising for orbifolds with non–prime N.

Tensoring together left– and rightmovers gives rise to the massless twisted spectrum displayed in table 2.5.

Untwisted Sector	
$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R \times G_{[v][a_i]}$ rep	Multiplet
$(\mathbf{3},\mathbf{3};\mathbf{1}\,\vert\,\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{1}\,\vert\,\mathbf{1}) + (\mathbf{3},\mathbf{2};\mathbf{2}\,\vert\,\mathbf{1})$	SUGRA
$(\mathbf{1},\mathbf{3};\mathbf{1}\,\vert\,\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1}\,\vert\,\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{2}\,\vert\,\mathbf{1})$	tensor
$(\mathbf{2},\mathbf{2};\mathbf{1}\,\vert\,\mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]}) + (\mathbf{2},\mathbf{1};\mathbf{2}\,\vert\,\mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]})$	gauge
$4\ [\ 2(\mathbf{1},\mathbf{1};\mathbf{2}\,\vert\,\mathbf{1}) + 2(\mathbf{1},\mathbf{2};\mathbf{1}\,\vert\,\mathbf{1})\ ]$	4 neutral hyper
$2(\mathbf{1},\mathbf{1};\mathbf{2}\,\vert\,\mathbf{R}^{(\mathbf{1})}_{[\mathbf{v}][\mathbf{a_i}]}) + 2(\mathbf{1},\mathbf{2};\mathbf{1}\,\vert\,\mathbf{R}^{(\mathbf{1})}_{[\mathbf{v}][\mathbf{a_i}]})$	charged hyper
(1, l)–Twisted Sector	
$\mathrm{SU}(2)^2 \times \mathrm{SU}(2)_R \times G_{[v][a_i]}$ rep	Multiplet
$(\mathbf{1},\mathbf{1};\mathbf{2}\,\vert\,\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]}) + (\mathbf{1},\mathbf{2};\mathbf{1}\,\vert\,\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]})$	charged half–hyper
$2\ \left[\ 2(\mathbf{1},\mathbf{1};\mathbf{2}\,\vert\,\mathbf{D}_{[\beta(1,l)})} + 2(\mathbf{1},\mathbf{2};\mathbf{1}\,\vert\,\mathbf{D}_{[\beta(1,l)})})\ \right]$	2 charged hyper

Table 2.5: Massless spectrum for $T^4/\mathbb{Z}_2$. The last term within each row of representations always corresponds to the fermionic multiplet content. SUGRA, tensor and the four neutral hyper multiplets are independent of gauge shift and Wilson lines.

2.8.2 $T^6/\mathbb{Z}_3$

The important and obvious difference w.r.t. $T^4/\mathbb{Z}_2$ is, that $T^6/\mathbb{Z}_3$ leads to a four dimensional theory. Since N is still prime, there is not so much difference in the construction of spectra, as we will see. Orbifolds of this type have been discussed extensively in the literature (see e.g. [66] and references therein).

Preliminaries

We take $T^6 = T^2 \times T^2 \times T^2$ and choose the compactification two–tori to be SU(3) root lattices with non–critical radii R_1, R_2 and R_3 (Unlike before, this choice is here uniquely determined by the crystallographic restriction coming from $\mathbb{Z}_3$ to be a symmetry of the torus). Within the j–th T^2 we choose basis vectors R_j and $R_j \exp(2\pi i/3)$ and realize the $\mathbb{Z}_2$ symmetry by a twist vector[23] $\phi = (1/3, 1/3, -2/3)$, acting on the worldsheet field components $i = 2, 3, 4$.

There are now two twisted sectors, $k = 1$ and $k = 2$, but each of them only containing fixed points again. In brane terminology they appear as 3-branes, therefore supporting four dimensional states. In the language of section 2.3 we find

$$I = \{0\}\ , \quad N = \Lambda\ , \quad \left|\frac{N}{(\mathbb{1}-\theta)\Lambda}\right| = 3^3 = 27\ , \tag{2.8.7}$$

so that there 27 inequivalent 3-branes in each sector, labeled by $l = 1, \ldots, 27$. The situation is depicted in figure 2.4 for $k = 1$. Note, that the picture for $k = 2$ is just a copy.

The space group element corresponding to fixed point (k, l) is given by $D = (\theta^k, u(k, l))$. Again, the centralizer of D in S contains only the identity, D itself and D^{-1} for $k = 1, 2$ and the fixed points are unique.

The spacetime SO(8) is broken down to $\mathrm{SU}(2) \times \mathrm{SU}(3)_H \times \mathrm{U}(1)_R$ as explicitly shown in appendix A.4.4. However, from the discussion at the end of section 2.3 we know, that the $\mathrm{SU}(3)_H$ holonomy group is broken to $\mathbb{Z}_3$ and a non–trivial representation under it gives just rise to multiplicities again. Since the spectrum will possess $D = 4, \mathcal{N} = 1$ SUSY, corresponding supermultipletts are listed in appendix B.3. The multiplets are classified w.r.t. their chiralities, i.e. SO(2) quantum numbers and not w.r.t. SU(2) as it occurs from the breaking. However, from the definitions of the chirality operator, one can make sure, that the SU(2) weights cooincide with the chirality.

The Untwisted Sector

The untwisted sector works parallel to the $T^4\mathbb{Z}_2$ case. Twist eigenvalues are labeled in powers of $\alpha = \exp(2\pi i/3)$. The chiralities can be read off from the

[23] Up to equivalent twist vectors, this is the only choice consistent with $D = 4, \mathcal{N} = 1$ SUSY.

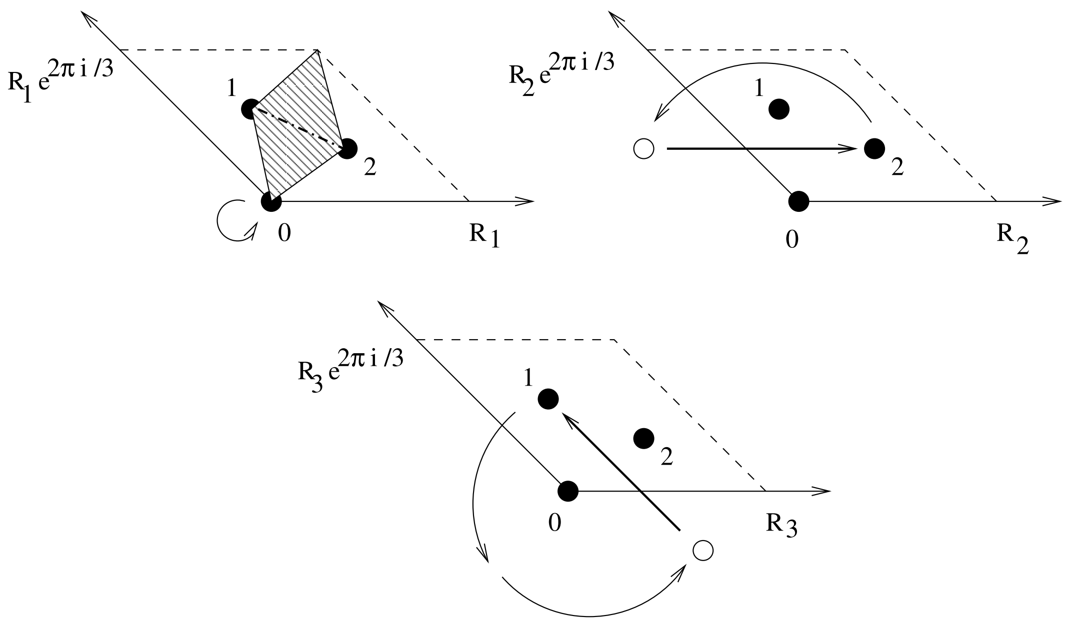

Figure 2.4: The orbifold $(T^2 \times T^2 \times T^2)/\mathbb{Z}_3$: A fundamental domain of $T^2 \times T^2 \times T^2$ is framed by the dashed boxes. Every T^2 is chosen to be spanned by the vectors, R_j and $R_j \exp(2\pi i/3)$. There are 27 fixed points, which are here labeled from 0 to 2 in each T^2 and depicted by filled circles. That the points are fixed can be seen e.g. for $l = (0, 2, 1)$ whose image under θ is depicted by empty circles. Note, that we chose $\theta^3 = \exp(2\pi i 2/3)$. The lattice shifts needed to bring the point back to itself (within the fundamental domain), is decomposed into the torus basis vectors and depicted by straight lines. They determine possible Wilson line contributions. The shaded region in the first torus times the whole fundamental domain of the second and third torus shows a possible choice of fundamental domain for the orbifold.

first SU(2) weight, thus specifying the multiplets, that are listed in B.3. The result for the rightmovers is given in table 2.6.

The results for the leftmovers are summarized in tables 2.7 and 2.8, where the notation for the gauge representations is taken from (2.8.2).

Tensoring the leftmoving states from tables 2.7 and 2.8 with the rightmoving ones from table 2.6 in such a way that the product of twist eigenvalues is always α^0, one arrives at the massless untwisted spectrum displayed in table 2.9.

The Twisted Sectors

We start with $k = 1$ (l arbitrary) and the rightmovers. Due to the twist the zero–point energy is shifted from $-1/2$ to $-1/6$, and there are no oscillator excitations leading to massless states. Again, the only possibility to find massless states, is

Sector	State(s)	Twist eigenvalue	SO(2) rep (chirality)
$\widetilde{\mathrm{NS}}$	$\lvert 1,0,0,0\rangle$	α^0	$(+1)$
$\widetilde{\mathrm{NS}}$	$\lvert 0,\underline{1,0},0\rangle, \lvert 0,0,0,-1\rangle$	α^1	(0)
$\widetilde{\mathrm{NS}}$	$\lvert 0,\underline{-1,0},0\rangle, \lvert 0,0,0,1\rangle$	α^2	(0)
$\widetilde{\mathrm{NS}}$	$\lvert -1,0,0,0\rangle$	α^0	(-1)
$\widetilde{\mathrm{R}}$	$\lvert \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle$	α^0	$(+1/2)$
$\widetilde{\mathrm{R}}$	$\lvert \frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\rangle$	α^1	$(+1/2)$
$\widetilde{\mathrm{R}}$	$\lvert -\frac{1}{2},\underline{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\rangle$	α^2	$(-1/2)$
$\widetilde{\mathrm{R}}$	$\lvert -\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle$	α^0	$(-1/2)$

Table 2.6: Twist eigenvalues $\exp(2\pi i\tilde{q}_i\phi^i)$ of the rightmovers and SO(2) representations for $T^6/\mathbb{Z}_3$.

State(s)	Twist eigenvalue	SO(2) rep	Gauge rep
$\alpha^{\mu}_{-1}\lvert 0\rangle$	α^0	(± 1)	$\lvert \mathbf{1}\rangle$
$\alpha^{2/3/\bar{4}}_{-1}\lvert 0\rangle$	α^1	(0)	$\lvert \mathbf{1}\rangle$
$\alpha^{\bar{2}/\bar{3}/4}_{-1}\lvert 0\rangle$	α^2	(0)	$\lvert \mathbf{1}\rangle$

Table 2.7: Twist eigenvalues of the leftmovers in the universal sector ($q^2=0$), SO(2) and gauge representations for $T^6/\mathbb{Z}_3$.

State(s)	Twist eigenvalue	SO(2) rep	Gauge rep
$q\cdot v = 0 \mod 1$	α^0	(0)	$\lvert \mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]}\rangle$
$q\cdot v = 1/3 \mod 1$	α^1	(0)	$\lvert \mathbf{R}^{(\mathbf{1})}_{[\mathbf{v}][\mathbf{a_i}]}\rangle$
$q\cdot v = 2/3 \mod 1$	α^2	(0)	$\lvert \mathbf{R}^{(\mathbf{2})}_{[\mathbf{v}][\mathbf{a_i}]}\rangle$

Table 2.8: Twist eigenvalues of the leftmovers with $q^2=2$, SO(2) and gauge representations for $T^6/\mathbb{Z}_3$.

from the lattice contributions. Masslessness imposes the condition

$$\frac{1}{2}(\tilde{q}')^2 = \frac{1}{2}(\tilde{q}+\phi)^2 = \frac{1}{6}\,, \quad \tilde{q}\in\tilde{\Gamma}\,, \tag{2.8.8}$$

which from (2.1.24) and $\phi = (0, 1/3, 1/3, -2/3)$ implies

$$\tilde{q}' \in \{|\,0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\,\rangle, |\,\tfrac{1}{2}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}\,\rangle\}\ , \tag{2.8.9}$$

with chiralities (0) and (+1/2) respectively.

For the leftmovers the zero–point energy is shifted from -1 to $-2/3$ and internal leftmoving bosonic excitations are allowed. The bosonic oscillators $\alpha^{2/3/4}_{-2/3}$ shift the mass–squared by 2/3 and $\alpha^{\bar{2}/\bar{3}/\bar{4}}_{-1/3}$ by 1/3.

Let $\mathbf{S}_{[\beta(1,l)]}$ and $\mathbf{T}_{[\beta(1,l)]}$ denote the gauge representations emerging from states without and with one bosonic excitation $\alpha^{\bar{\imath}}_{-1/3}$. There is still the possiblity of excitations $\alpha^{i}_{-2/3}$ and $\alpha^{(\bar{\imath}}_{-1/3}\alpha^{\bar{\jmath})}_{-1/3}$ (indices symmetrized). We denote the corresponding representation by $\mathbf{N}$. But then the states must have $q' = (0^{16})$, which implies $\beta(1,l) \in \Gamma$ and therefore $\mathbf{N} = \mathbf{1}$. This is only possible for one model, as we will see later. There are three masslessness conditions defining those representations

$$\begin{aligned}
\mathbf{S}_{[\beta(1,l)]} &\cong \frac{1}{2}(q')^2 = \frac{1}{2}[q - (v + m^i(1,l)a_i)]^2 = \frac{2}{3}\ , \quad q \in \Gamma\ , \\
(3\times)\ \mathbf{T}_{[\beta(1,l)} &\cong \frac{1}{2}(q')^2 = \frac{1}{2}[q - (v + m^i(1,l)a_i)]^2 = \frac{1}{3}\ , \quad q \in \Gamma\ , \\
(9\times)\ \mathbf{N} &\Leftrightarrow (v + m^i(1,l)a_i) \in \Gamma\ .
\end{aligned} \tag{2.8.10}$$

The multiplicity of 3 and 9 are due to the bosonic oscillator structure.

The discussion of twist eigenvalues works completely along the lines of $T^4/\mathbb{Z}_2$. All combinations of states are centralizer invariant.

From the group theoretical discussion of twisted states in appendix A.4.1 one can see, that the second twisted sector $k = 2 = N - 1$ contributes a set of states complex conjugate to those of the first twisted sector. Therefore we just state the result.

Tensoring together left– and rightmovers gives rise to the massless twisted spectrum displayed in table 2.9. The SO(2) weights are additive.

2.8.3 $T^6/\mathbb{Z}_4$

For the untwisted sector and the first and third twisted sector (which again combine to fit complete multiplets) everything works in complete analogy to the $T^6/\mathbb{Z}_3$ case. Therefore we just state the result. The fixed point structure of the first (third analogous) twisted sector is depicted in figure 2.5.

Exactly like $T^6/\mathbb{Z}_3$, $T^6/\mathbb{Z}_4$ leads to a four dimensional theory. The novelty here is, that N is non–prime, which will lead to fixed planes in the second twisted sectors.

This type of orbifolds has as well been discussed extensively in the literature, see e.g. [44, 67, 68] and references therein. For a very good mathematical discussion containing the local structure see [69].

Untwisted Sector	
$\mathrm{SO}(2) \times G_{[v][a_i]}$ rep	Multiplet
$((-2 + -3/2) \,\vert\, \mathbf{1}) + ((3/2 + 2) \,\vert\, \mathbf{1})$	SUGRA
$((-1/2 + 0) \,\vert\, \mathbf{1}) + ((0 + 1/2) \,\vert\, \mathbf{1})$	chiral (dilaton/axion)
$((-1 + -1/2) \,\vert\, \mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]}) + ((1/2 + 1) \,\vert\, \mathbf{Ad}_{[\mathbf{v}][\mathbf{a_i}]})$	gauge
$9\ [\ ((-1/2 + 0) \,\vert\, \mathbf{1}) + ((0 + 1/2) \,\vert\, \mathbf{1})\ ]$	9 neutral chiral
$3\ \left[\ ((-1/2 + 0) \,\vert\, \mathbf{R}^{(1)}_{[\mathbf{v}][\mathbf{a_i}]}) + ((0 + 1/2) \,\vert\, \mathbf{R}^{(2)}_{[\mathbf{v}][\mathbf{a_i}]})\ \right]$	3 charged chiral in $\mathbf{R}^{(2)}_{[\mathbf{v}][\mathbf{a_i}]}$
(1, l)–Twisted + (2, l)–Twisted Sector	
$\mathrm{SO}(2) \times G_{[v][a_i]}$ rep	Multiplet
$((-1/2 + 0) \,\vert\, \bar{\mathbf{S}}_{[\beta(\mathbf{1},\mathbf{l})]}) + ((0 + 1/2) \,\vert\, \mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]})$	charged chiral in $\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]}$
$3\ [\ ((-1/2 + 0) \,\vert\, \bar{\mathbf{T}}_{[\beta(\mathbf{1},\mathbf{l})]}) + ((0 + 1/2) \,\vert\, \mathbf{T}_{[\beta(\mathbf{1},\mathbf{l})]})\ ]$	3 charged chiral in $\mathbf{T}_{[\beta(\mathbf{1},\mathbf{l})]}$
$9\ [\ ((-1/2 + 0) \,\vert\, \mathbf{N}) + ((0 + 1/2) \,\vert\, \mathbf{1})\ ]$	9 neutral chiral iff $\beta(1, l) \in \Gamma$

Table 2.9: Massless spectrum for $T^6/\mathbb{Z}_3$. A charged chiral multiplet is said to transform as $\mathbf{R}$ if and only if its positive chirality components transform as $\mathbf{R}$. Then its negative chirality components transform as $\bar{\mathbf{R}}$. Note, that $\mathbf{R}^{(1)}_{[\mathbf{v}][\mathbf{a_i}]} = \bar{\mathbf{R}}^{(2)}_{[\mathbf{v}][\mathbf{a_i}]}$ and $\mathbf{N} = \mathbf{1}$.

Preliminaries

Again, we take $T^6 = T^2 \times T^2 \times T^2$ and choose the compactification lattice to be quadratic with non–critical radii R_1, R_2 and R_3 (This is one of two possible choices determined by the crystallographic restriction coming from $\mathbb{Z}_4$ to be a symmetry of the torus). Within the j–th T^2 we choose basis vectors R_j and iR_j and realize the $\mathbb{Z}_4$ symmetry by a twist vector $\phi = (-1/2, 1/4, 1/4)$, acting on the worldsheet field components $i = 2, 3, 4$.

Again we do not yet specify gauge shift and Wilson lines but already comment on the following subtelty: The boundary conditions of the second twisted sector are of order $M = 2 < 4$ and therefore we have to be careful with level matching, as has been indicated at the beginning of subsection 2.7.2. By reviewing the derivation of the level matching conditions it is easy to see that we get an additional level matching condition

$$(2\phi)^2 - (2v + m^i(2, l)a_i)^2 = 0 \mod 2/M = 0 \mod 1\ , \qquad (2.8.11)$$

which in fact is stronger than the one following from the mod $1/4$ studied before. The appearance of this kind of extra level matching conditions stemming from subsectors of orbifolds is largely overlooked in the standard literature. It has to be fulfilled in order to define valid models.

There are three twisted sectors, of which the first and third only contain fixed points, thus supporting four dimensional states. In the language of section 2.3 we find

$$I = \{0\} \,, \quad N = \Lambda \,, \quad \left| \frac{N}{(\mathbb{1} - \theta)\Lambda} \right| = 4 \cdot 2^2 = 16 \,, \tag{2.8.12}$$

so that there 16 inequivalent 3-branes in sectors $k = 1$ and $k = 3$. The situation is depicted in figure 2.5 for $k = 1$. The picture for $k = 3$ is just a copy.

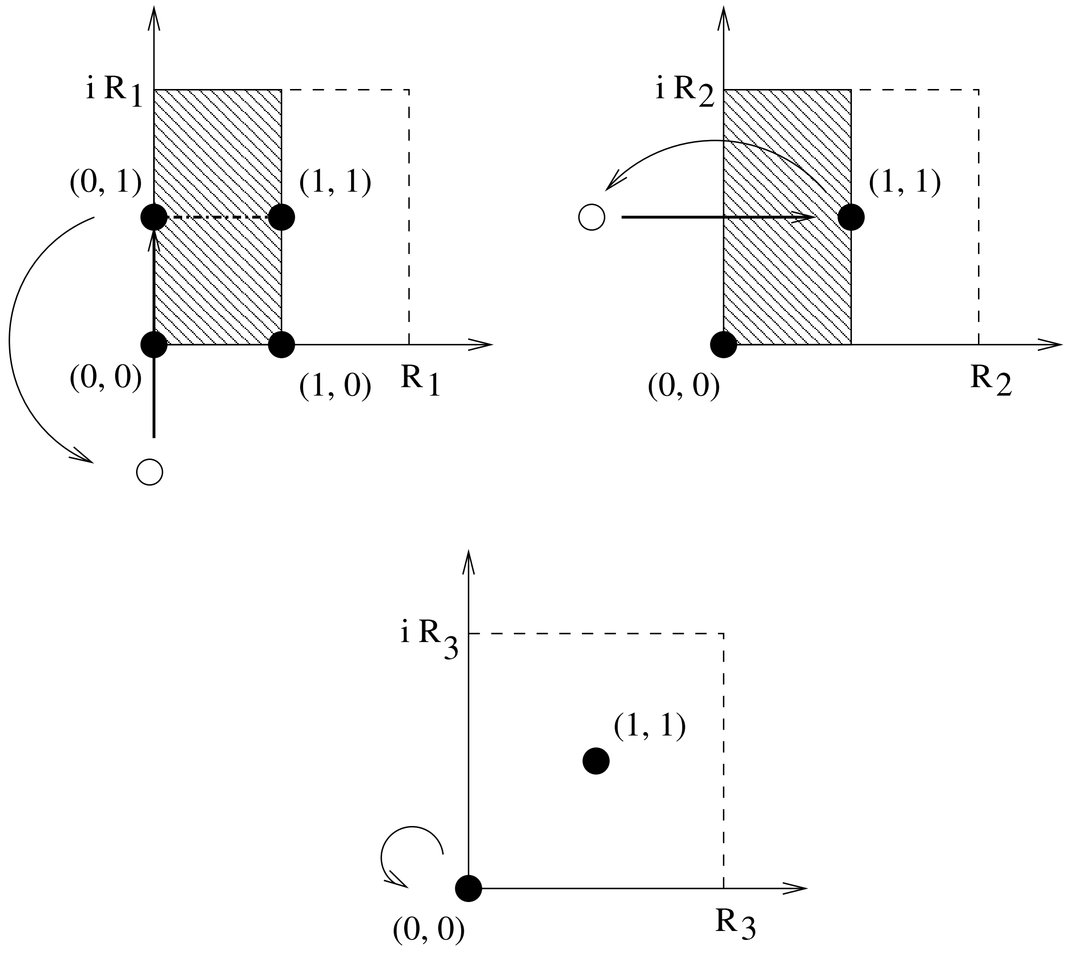

Figure 2.5: Fixed points of the orbifold $(T^2 \times T^2 \times T^2)/\mathbb{Z}_4$ appearing for $k = 1$ and $k = 3$.

Now consider the second twisted sectors $k = 2$. The effective twist vector is given by $2\phi = (-1, 1/2, 1/2)$. The twisted states seem therefore free to propagate within the first T^2. The fixed planes would appear as 5–branes, supporting six dimensional states, but this is only partially true, as we will see. The situation is depicted in figure 2.6.

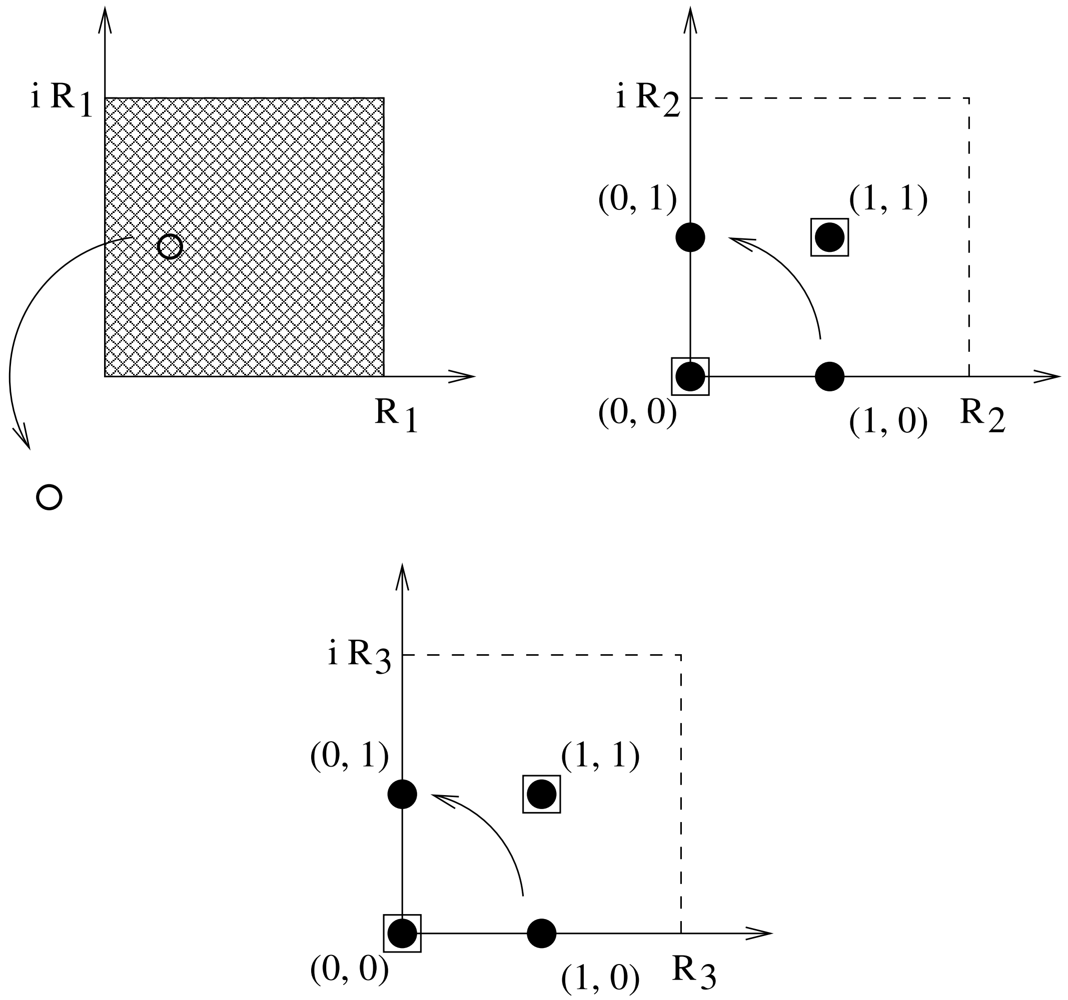

Figure 2.6: Fixed planes of the orbifold $(T^2 \times T^2 \times T^2)/\mathbb{Z}_4$ appearing for $k = 2$.

Since the space group for this example gives rise to interesting physics, we will discuss it in some detail. Consider the first twisted sector and figure 2.5 again: We label a fixed point by an ordered triple of pairs of numbers $(p, q) = ((p_1, q_1), (p_2, q_2), (p_3, q_3))$. In general $p_i, q_i \in \{0, 1\}$, but for the first twisted sector only the points with $p_2 = q_2$ and $p_3 = q_3$ are fixed. Therefore the space group elements corresponding to all possible fixed points of the first twisted sector are given by

$$(\theta, p_1 e_1 + q_1 e_2 + p_2(e_3 + e_4) + p_3(e_5 + e_6)) \ . \tag{2.8.13}$$

Using the multiplication law (2.7.5) $(\theta, u(1, (p, q)))$ and $(\theta, u(1, (p', q')))$ only commute if $(p, q) = (p', q')$. There are no additional projections arising from higher order twists, so nothing new here.

However, consider the second twisted sector and figure 2.6: The fixed planes correspond to the space group elements

$$(\theta^2, r_2 e_3 + s_2 e_4 + r_3 e_5 + s_3 e_6)) \ , \tag{2.8.14}$$

with $r_2, s_2, r_3, s_3 \in \{0, 1\}$. Restricting only to the second and third T^2, the situation is a copy of the first twisted sector of $T^4/\mathbb{Z}_2$. Given an element $D' = (\theta^2, u(2,(r,s)))$ and restricting the space group on elements with point transformations θ^2, the centralizer would only consist of identity, D' itself and D'^{-1} again. But it is easy to show, that the space group elements $(\theta, u(1,(p,q)))$ of (2.8.13) commute with D' if and only if

$$p_2 = r_2 = s_2 \ , \quad p_3 = r_3 = s_3 \tag{2.8.15}$$

is fulfilled. This has strong implications.

Assume, that there is no solution for (2.8.15), i.e. $r_2 \neq s_2$ and/or $r_3 \neq s_3$. There are exactly 12 possiblities. Then, the states are superpositions of all images of space group transformations not in the centralizer. However, there are fixed planes, that are not independent of each other, i.e. they lie in the same conjugacy class (c.f. discussion in section 2.5) and we have to take care, that we do not overcount. E.g. a state corresponding to the fixed plane $((z^2, \bar{z}^2), (1,0), (1,0))$ in figure 2.6 is mapped to a state corresponding to $((-z^2, -\bar{z}^2), (0,1), (0,1))$ via $(\theta, 0)$, so that they are not independent of each other. Repeating this for all such fixed planes, one finds exactly six independent pairs $((z^2, \bar{z}^2), (p_2, q_2), (p_3, q_3))$ and $((-z^2, -\bar{z}^2), (q_2, p_2), (q_3, p_3))$ (note, that q's and p's are exchanged), that are related by a space group transformation.

Now assume that there is a solution of (2.8.15) for some p_2 and p_3, i.e. $r_2 = s_2$ and $r_3 = s_3$. Then the fixed planes are in one–to–one correspondence w.r.t. the twisted sectors (there is no space group mapping between them), and the states have to be invariant under the additional projector

$$\frac{1}{4}\left[(\mathbb{1}, 0) + (\theta, u(1,(r,s))) + (\theta^2, u(2,(r,s))) + (\theta^3, u(3,(r,s)))\right] \ . \tag{2.8.16}$$

By construction there is no non–trivial action of this projector on the second and third tori, because the fixed points of the first twisted sector are contained within those of the second. Therefore, effectively the projector only acts as

$$\frac{1}{2}\left[(\mathbb{1}, 0) + (\theta, u(1,(r,s)))\right] \ , \tag{2.8.17}$$

since the first and third, as well as the second and last term within (2.8.16) act identical on the first T^2. However, still there is a non–trivial action on the first torus. For the first T^2 there was no restriction on the center–of–mass position before. Now, there is an additional projection for states sitting at certain points within the first T^2. The order of the projection is clearly 2 and therefore, the first torus is orbifolded to $T^2/\mathbb{Z}_2$. The fixed points are exactly those of the first twisted sector.

Summarizing, the states of the second twisted sector fall into two categories, that are classified w.r.t. their corresponding fixed planes. Firstly, the 12 fixed planes not satisfying (2.8.15) give rise to six independent pairs of fixed planes

$$((z^2,\bar{z}^2),(p_2,q_2),(p_3,q_3)) \quad \textit{and} \quad ((-z^2,-\bar{z}^2),(q_2,p_2),(q_3,p_3)) \ . \tag{2.8.18}$$

The states are free to propagate on the first T^2. These give rise to the kind of states expected. Secondly, the four fixed planes satisfying (2.8.15) give rise to four independent orbifolds $T^2/\mathbb{Z}_2$. Due to the $\mathbb{Z}_2$ projection the six dimensional states at the fixed points of the first twisted sector therefore contribute effectively four dimensional states. The states must be invariant under the additional $\mathbb{Z}_2$.

The spacetime SO(8) is broken down to $\mathrm{SU}(2)\times\mathrm{SU}(3)_H\times\mathrm{U}(1)_R$ as explicitly shown in appendix A.4.4. However, from the discussion at the end of section 2.3 we know, that the $\mathrm{SU}(3)_H$ holonomy group is broken to $\mathbb{Z}_3$ and a non–trivial representation under it gives just rise to multiplicities again. Since the spectrum will possess $D=4,\mathcal{N}=1$ SUSY, corresponding supermultipletts are listed in appendix B.3. The multiplets are classified w.r.t. their chiralities, i.e. SO(2) quantum numbers and not w.r.t. SU(2) as it occurs from the breaking. However, from the definitions of the chirality operator, one can make sure, that the SU(2) weights cooincide with the chirality.

The Untwisted Sector

The untwisted sector works completely parallel to those of $T^6\mathbb{Z}_3$ (including classification of little group representation). Therefore we just state the result in table 2.10. The notation of the gauge representations again follows the general defintion (2.8.2) with $N=4$.

The Twisted Sectors

The first and third twisted sector work analogous to $T^6/\mathbb{Z}_3$ as well. There is only one possibility to get massless states in each sector, the corresponding gauge representations $\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]}$ and $\bar{\mathbf{S}}_{[\beta(\mathbf{1},\mathbf{l})]}$ follow from the masslessness condition

$$\mathbf{S}_{[\beta(1,l)]} \cong \frac{1}{2}(q')^2 = \frac{1}{2}[q-(v+m^i(2,l)a_i)]^2 = \frac{11}{16} \ , \quad q\in\Gamma \ . \tag{2.8.19}$$

The corresponding condition in the third twisted sector delivers the missing $\bar{\mathbf{S}}_{[\beta(\mathbf{1},\mathbf{l})]}$ to fill a chiral multiplet at each fixed point.

The gauge representations in the second twisted sector do not dependent on the complicated space group structure and are determined by the usual procedure. Masslessness defines

$$\hat{\mathbf{T}}_{[\beta(2,l)]} \cong \frac{1}{2}(q')^2 = \frac{1}{2}[q-(2v+m^i(2,l)a_i)]^2 = \frac{11}{16} \ , \quad q\in\Gamma \ . \tag{2.8.20}$$

The states corresponding to the six pairwise combined fixed planes living on T^2 on the first torus, transform under this representation. Clearly, for the states corresponding to the four $T^2/\mathbb{Z}_2$'s half of the states are projected out. From group theory we know that $\hat{\mathbf{T}}_{[\beta(2,l)]}$ must be real. But then we can always write it as

$$\hat{\mathbf{T}}_{[\beta(2,l)]} = \mathbf{T}_{[\beta(2,l)]} + \bar{\mathbf{T}}_{[\beta(2,l)]} \ . \tag{2.8.21}$$

We fix the ambiguity in this splitting in such a way that the states assigned to $\mathbf{T}_{[\beta(2,l)]}$ survive the $\mathbb{Z}_2$ projection (2.8.17) and those assigned to $\bar{\mathbf{T}}_{[\beta(2,l)]}$ are projected out. By considering the explicit twist eigenvalues this translates into the condition

$$\mathbf{T}_{[\beta(2,l)]} \ \cong \ [\beta(1,l)^2 - \phi^2] + q'\beta(1,l) = 1/4 \mod 1 \ , \tag{2.8.22}$$

that then is satisfied for all states in $\mathbf{T}_{[\beta(2,l)]}$. The first term descends from $s^G_{\delta\mathrm{LM}}$ and renders the condition invariant under lattice shifts of β.

To conclude, in the second twisted sector, we get a chiral multiplet in gauge representation $\mathbf{T}_{[\beta(2,l)]} + \bar{\mathbf{T}}_{[\beta(2,l)]}$, for every independent pair of fixed planes. The states are superpositions and propagate on the first T^2. Therefore we will call these fixed planes T^2 fixed planes or T^2 5–branes. Since there are six different sectors, we let l run from 1 to 6 for these states.

Additionally, we get a chiral multiplet in gauge representation $\mathbf{T}_{[\beta(2,l)]}$ for every $T^2/\mathbb{Z}_2$ fixed plane. Due to the projection the states are confined to the $T^2\mathbb{Z}_2$ fixed points within the first T^2. Since there are four different sectors, we let l run from 7 to 10 for these states.

The result is summarized in table 2.10.

2.9 Bulk and Brane Spectra

In subsection 2.8.3 we have already seen, that the brane language is very convenient to describe, where certain states are situated or confined to. This confinement was always due to projections. In this section we generalize this picture to brane spectra or, as they have been called in [46, 47, 48], fixed point/plane equivalent models. We will show, that this concept is particularly useful in the presence of Wilson lines. Before doing that we describe the very important case of orbifolds without Wilson lines.

2.9.1 Orbifolds without Wilson lines

In this subsection we concentrate on orbifolds without Wilson lines. Therefore we have $\beta(v, a_i) = \beta(v) = kv, a_i = (0^8 \,|\, 0^8)$. (We will just drop subscripts $[a_i]$, where they appear.) Then, in a (k, l)-th twisted sector of given k, the gauge shift does not depend on l anymore and the states are degenerate w.r.t. l. The

Untwisted Sector	
$\mathrm{SO}(2) \times G_{[v][a_i]}$ rep	Multiplet
$((-2 + -3/2)\,\vert\,\mathbf{1}) + ((3/2 + 2)\,\vert\,\mathbf{1})$	SUGRA
$((-1/2 + 0)\,\vert\,\mathbf{1}) + ((0 + 1/2)\,\vert\,\mathbf{1})$	chiral (dilaton/axion)
$((-1 + -1/2)\,\vert\,\mathbf{Ad_{[v][a_i]}}) + ((1/2 + 1)\,\vert\,\mathbf{Ad_{[v][a_i]}})$	gauge
$6\ [\ ((-1/2 + 0)\,\vert\,\mathbf{1}) + ((0 + 1/2)\,\vert\,\mathbf{1})\ ]$	6 neutral chiral
$2\ \left[\ ((-1/2 + 0)\,\vert\,\mathbf{R^{(1)}_{[v][a_i]}}) + ((0 + 1/2)\,\vert\,\mathbf{R^{(3)}_{[v][a_i]}})\ \right]$	2 charged chiral in $\mathbf{R^{(3)}_{[v][a_i]}}$
$((-1/2 + 0)\,\vert\,\mathbf{R^{(2)}_{[v][a_i]}}) + ((0 + 1/2)\,\vert\,\mathbf{R^{(2)}_{[v][a_i]}})$	charged chiral in $\mathbf{R^{(2)}_{[v][a_i]}}$
(1, l)–Twisted + (3, l)–Twisted Sector	
$\mathrm{SO}(2) \times G_{[v][a_i]}$ rep	Multiplet
$((-1/2 + 0)\,\vert\,\bar{\mathbf{S}}_{[\beta(\mathbf{1},\mathbf{l})]}) + ((0 + 1/2)\,\vert\,\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]})$	charged chiral in $\mathbf{S}_{[\beta(\mathbf{1},\mathbf{l})]}$
(2, l=1, …, 6)–Twisted Sectors (T^2 5–branes)	
$\mathrm{SO}(2) \times G_{[v][a_i]}$ rep	Multiplet
$((-1/2 + 0)\,\vert\,\hat{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]}) + ((0 + 1/2)\,\vert\,\hat{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]})$	charged chiral in $\hat{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]}$
(2, l=7, …, 10)–Twisted Sectors ($T^2/\mathbb{Z}_2$ 3–branes)	
$((-1/2 + 0)\,\vert\,\bar{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]}) + ((0 + 1/2)\,\vert\,\mathbf{T}_{[\beta(\mathbf{2},\mathbf{l})]})$	charged chiral in $\mathbf{T}_{[\beta(\mathbf{2},\mathbf{l})]}$

Table 2.10: Massless spectrum for $T^6/\mathbb{Z}_4$. Note, that $\mathbf{R^{(1)}_{[v][a_i]}} = \bar{\mathbf{R}}^{(3)}_{\mathbf{[v][a_i]}}$ and $\hat{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]} = \mathbf{T}_{[\beta(\mathbf{2},\mathbf{l})]} + \bar{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]}$. Furthermore $\mathbf{R^{(2)}_{[v][a_i]}}$, $\mathbf{S}_{[\beta(\mathbf{2},\mathbf{l})]}$ and $\hat{\mathbf{T}}_{[\beta(\mathbf{2},\mathbf{l})]}$ are real.

constraints due to level matching (2.7.17) get particularly simple, since the case $k = 1$ already includes the constraints coming from all other twisted sectors. For the examples considered in this work, all inequivalent gauge shifts have been worked out in appendix A.3.2 for a single E_8. For $E_8 \times E_8$ two of such shifts have to be combined in such a way, that the combination weak level matching condition (2.7.17) is fulfilled. However, since for the untwisted sector both groups of internal gauge d.o.f. do not interfere with each other (i.e. if a state is non-trivially charged under one E_8 it must be neutral under the other), unbroken gauge group and untwisted matter do not depend on the specific combinations and are therefore displayed in seperate tables. For the twisted states this is not true (they might be charged under both E_8's simultaneously[24]) and do depend on the level matching invariant combination of gauge shifts. We just state all physically inequivalent[25] possiblities together with the twisted representations for the different examples.

The breaking of the E_8's is again just an application of Cartan symmetry breaking (implying rank conservation), which has been introduced in appendix A.4.1. Since the procedure is completely analogous to the SO(8) breakings of the rightmovers, that have been considered in appendices A.4.3 and A.4.4 in detail, we just state the results here.

$T^4/\mathbb{Z}_2$

Gauge groups and untwisted matter correspond to $\mathbf{Ad}_{[\mathbf{v}]}$ and $\mathbf{R}^{(\mathbf{1})}_{[\mathbf{v}]}$ and are determined via (2.8.2). All possible results are listed in table 2.11, where the model numbers refer to table A.1.

Number	Gauge Shift $2v$	Gauge group $\mathbf{Ad}_{[v]}$	Untwisted Matter $(1\times)$ $\mathbf{R}^{(1)}_{[v]}$
0	(00000000)	E_8	nothing
1	(11000000)	$E_7 \times SU(2)$	$(\mathbf{56}, \mathbf{2})$
2	(20000000)	SO(16)	$\mathbf{128}_c$

Table 2.11: The resulting gauge groups and six dimensional untwisted matter representations are given for representatives of the possible $\mathbb{Z}_2$ gauge shifts. The multiplicity for the untwisted matter in the second row indicates the number of hyper multiplets (not half–hyper multiplets), that appear in the full orbifold spectrum.

All modular invariant gauge shift combinations together with the corresponding twisted states are listed in table 2.12. The representations are determined from (2.8.5).

Since these are the first specific examples for heterotic $E_8 \times E_8$ orbifold models, we will state one model in full detail to explain, how the tables should be read. From table 2.12 we see that e.g. the combination of gauge shifts (1;0) fulfills all consistency conditions and therefore defines a model. From 2.11 we identify the full gauge shift to be given by (1/2)(11000000—00000000). From these two tables and table 2.5 we can now read off the full massless spectrum. We find the universal SUGRA and tensor multiplets of $D = 6, \mathcal{N} = 1$ SUSY, together with

[24] This very interesting feature of orbifolds has been tackled in [70].

[25] We do not distunguish models that are identical up to an exchange of both E_8 factors.

Number	Gauge Group $G_{[v]}$	Twisted Matter $(\frac{1}{2}\times)$ $\mathbf{S}_{[v]}$	$(2\times)$ $\mathbf{D}_{[v]}$
$(1;0)$	$\mathrm{E}_7\times\mathrm{SU}(2)\ \times\ \mathrm{E}_8$	$(\mathbf{56},\mathbf{1};\mathbf{1})$	$(\mathbf{1},\mathbf{2};\mathbf{1})$
$(1;2)$	$\mathrm{E}_7\times\mathrm{SU}(2)\ \times\ \mathrm{SO}(16)$	$(\mathbf{1},\mathbf{2};\mathbf{16}_v)$	

Table 2.12: There are two modular invariant combinations of $\mathbb{Z}_2$ gauge shifts. The number specifies the shift combination and refers to the first column of table 2.11. The resulting gauge group and the twisted matter at a single fixed point is given. The multiplicites in the second row indicate, how many hyper multiplets transform in this representations. By (1/2) times a hyper multiplet, we denote a half–hyper multiplet.

four neutral hyper multiplets. There is a gauge multiplet and the adjoint representations of 2.11 determines the gauge group to be $\mathrm{E}_7\times\mathrm{SU}(2)\times\mathrm{E}_8$. Additionally there is one charged hyper multiplet transforming as $(\mathbf{56},\mathbf{2};\mathbf{1})+(\mathbf{1};\text{nothing})$, which is equal to $(\mathbf{56},\mathbf{2};\mathbf{1})$. This completes the untwisted sector. For the twisted sectors we learned that due to the absence of Wilson lines all twisted spectra are identical giving rise to a multiplicity of 16 (the number of fixed points). Therefore from table 2.12 we get 16 half–hyper multiplets transforming as $(\mathbf{56},\mathbf{1};\mathbf{1})$ and $16\times 2=32$ hyper multiplets transforming as $(\mathbf{1},\mathbf{2};\mathbf{1})$ under the unbroken gauge group. This completes the model.

$T^6/\mathbb{Z}_3$

Gauge groups and untwisted matter (positive chirality part) correspond to $\mathbf{Ad}_{[\mathbf{v}]}$ and $\mathbf{R}^{(\mathbf{2})}_{[\mathbf{v}]}$ and are determined via (2.8.2). All possible results are listed in table 2.13, where the model numbers refer to table A.2. In these models we explicitly have to deal with U(1) charges within the context of Cartan symmetry breaking. Some details about the U(1)s are exposed in appendix A.4.2 and the example of $\mathrm{E}_8\to\mathrm{E}_7\times\mathrm{U}(1)$ has been worked out for the untwisted states in appendix A.4.5 explicitly.

All modular invariant gauge shift combinations together with the corresponding twisted states are listed in table 2.14. The representations are determined from (2.8.10).

$T^6/\mathbb{Z}_4$

Gauge groups and untwisted matter (positive chirality part) correspond to $\mathbf{Ad}_{[\mathbf{v}]}$, $\mathbf{R}^{(\mathbf{3})}_{[\mathbf{v}]}$ and $\mathbf{R}^{(\mathbf{2})}_{[\mathbf{v}]}$ and are determined via (2.8.2). All possible results are listed in table 2.15, where the model numbers refer to table A.3.

All modular invariant gauge shift combinations together with the correspond-

Number	Gauge Shift $3v$	Gauge group $\mathbf{Ad}_{[v]}$	Untwisted Matter $(3\times)\ \mathbf{R}^{(2)}_{[v]}$
0	(00000000)	E_8	nothing
1	(11000000)	$E_7 \times U(1)$	$(\mathbf{56})_1 + (\mathbf{1})_{-2}$
2	(20000000)	$SO(14) \times U(1)$	$(\mathbf{64_c})_1 + (\mathbf{14_v})_{-2}$
3	(21100000)	$E_6 \times SU(3)$	$(\overline{\mathbf{27}}, \overline{\mathbf{3}})$
4	(21111000)	$SU(9)$	$(\overline{\mathbf{84}})$

Table 2.13: The resulting gauge groups and four dimensional untwisted matter representations are given for representatives of the possible $\mathbb{Z}_3$ gauge shifts. The multiplicity for the untwisted matter in the second row indicates the number of chiral multiplets appearing in the full spectrum of the orbifold.

No.	Gauge Group $G_{[v]}$	Twisted Matter $(1\times)\ \mathbf{S}_{[v]}$	$(3\times)\ \mathbf{T}_{[v]}$	$(9\times)\ \mathbf{N}$
$(0;0)$	$E_8 \times E_8$	nothing	$(\mathbf{1};\mathbf{1})$	$(\mathbf{1};\mathbf{1})$
$(3;0)$	$E_6 \times SU(3) \times E_8$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})$	$(\mathbf{1},\mathbf{3};\mathbf{1})$	nothing
$(3;3)$	$E_6 \times SU(3) \times E_6 \times SU(3)$	$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	nothing	nothing
$(1;2)$	$E_7 \times U(1) \times SO(14) \times U(1)$	$(\mathbf{1};\mathbf{14}_v)_{2/3,-2/3}$ $+(\mathbf{1};\mathbf{1})_{-4/3,4/3}$	$(\mathbf{1};\mathbf{1})_{2/3,4/3}$	nothing
$(4;2)$	$SU(9) \times SO(14) \times U(1)$	$(\mathbf{9};\mathbf{1})_{4/3}$	nothing	nothing

Table 2.14: There are five modular invariant combinations of $\mathbb{Z}_3$ gauge shifts. The number specifies the shift combination and refers to the first column of table 2.13. The resulting gauge group and the twisted matter at a single fixed point is given. The multiplicites in the second row indicate, how many chiral multiplets transform in this representations.

ing twisted states are listed in table 2.16. The representations are determined from (2.8.19), (2.8.21) and (2.8.20).

2.9.2 From a Global to a Local Point of View

We have already seen, that the twisted states are constrained to live on branes of lower dimensionality w.r.t. the bulk. Therefore, we have a good understanding of where twisted states are living. However, our understanding is not so clear for the untwisted states, whose CM coordinates are not constrained due to worldsheet monodromies. Up to now there has been no assignment of certain untwisted states to certain positions in the orbifold. Our understanding is therefore only

No.	Gauge shift	Gauge group	Untwisted matter	
	$4v$	$\mathbf{Ad}_{[v]}$	$(2\times)\ \mathbf{R}^{(3)}_{[\mathbf{v}]}$	$(1\times)\ \mathbf{R}^{(2)}_{[\mathbf{v}]}$
0	(00000000)	E_8	nothing	nothing
1	(11000000)	$\mathrm{E}_7 \times \mathrm{U}(1)$	$(\mathbf{56})_1$	$(\mathbf{1})_2 + (\mathbf{1})_{-2}$
2	(20000000)	$\mathrm{SO}(14) \times \mathrm{U}(1)$	$(\mathbf{64}_s)_1$	$(\mathbf{14}_v)_2 + (\mathbf{14}_v)_{-2}$
3	(21100000)	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$(\overline{\mathbf{27}}, \mathbf{2})_1$ $+(\mathbf{1}, \mathbf{2})_{-3}$	$(\overline{\mathbf{27}}, \mathbf{1})_{-2}$ $+(\mathbf{27}, \mathbf{1})_2$
4	(22000000)	$\mathrm{E}_7 \times \mathrm{SU}(2)$	nothing	$(\mathbf{56}, \mathbf{2})$
5	(31000000)	$\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$(\mathbf{32}_s, \mathbf{1})_{-1}$ $+(\mathbf{12}_v, \mathbf{2})_1$	$(\mathbf{32}_c, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_2$ $+(\mathbf{1}, \mathbf{1})_{-2}$
6	(22200000)	$\mathrm{SO}(10) \times \mathrm{SU}(4)$	$(\mathbf{16}_c, \overline{\mathbf{4}})$	$(\mathbf{10}_v, \mathbf{6})$
7	(31111100)	$\mathrm{SU}(8) \times \mathrm{SU}(2)$	$(\overline{\mathbf{28}}, \mathbf{2})$	$(\mathbf{70}, \mathbf{1})$
8	(40000000)	$\mathrm{SO}(16)$	nothing	$\mathbf{128}_c$
9	(1111111-1)	$\mathrm{SU}(8) \times \mathrm{U}(1)$	$(\overline{\mathbf{56}})_1$ $+(\mathbf{8})_{-3}$	$(\mathbf{28})_2$ $(\overline{\mathbf{28}})_{-2}$

Table 2.15: The resulting gauge groups and four dimensional untwisted matter representations are given for representatives of the possible $\mathbb{Z}_4$ gauge shifts. The multiplicity for the untwisted matter in the second row indicates the number of chiral multiplets appearing in the full spectrum of the orbifold.

No.	Gauge group $G_{[v]}$	Single twisted matter $(1\times)\ \mathbf{S}_{[v]}$	Double twisted matter $(1\times)\ \mathbf{T}_{[2v]}\ (+\overline{\mathbf{T}}_{[2v]})$
(3; 0)	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)\times$ E_8	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_{-1/2}+2(\mathbf{1},\mathbf{2};\mathbf{1})_{-3/2}$ $+5(\mathbf{1},\mathbf{1};\mathbf{1})_{3/2}$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_{1}+(\mathbf{1},\mathbf{1};\mathbf{1})_{-3}$ $+2(\mathbf{1},\mathbf{2};\mathbf{1})_{0}$
(3; 4)	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)\times$ $\mathrm{E}_7 \times \mathrm{SU}(2)$	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})_{-3/2}+2(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{3/2}$	$(\mathbf{27},\mathbf{1};\mathbf{1},\mathbf{1})_{-1}$ $+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{3}$ $+2(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{0}$
(3; 9)	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)\times$ $\mathrm{SU}(8) \times \mathrm{U}(1)$	$(\mathbf{1},\mathbf{1};\mathbf{8})_{3/2,-1}+(\mathbf{1},\mathbf{2};\mathbf{1})_{-3/2,2}$ $+2(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{1})_{2}$	$(\mathbf{1},\mathbf{2};\overline{\mathbf{8}})_{0,-1}$
(3; 8)	$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)\times$ $\mathrm{SO}(16)$	$(\mathbf{1},\mathbf{1};\mathbf{16}_v)_{3/2}$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_{1}+(\mathbf{1},\mathbf{1};\mathbf{1})_{-3}$ $+2(\mathbf{1},\mathbf{2};\mathbf{1})_{0}$
(7; 0)	$\mathrm{SU}(8) \times \mathrm{SU}(2)\times$ $\times\mathrm{E}_8$	$(\overline{\mathbf{8}},\mathbf{2};\mathbf{1})+2(\mathbf{8},\mathbf{1};\mathbf{1})$	$(\mathbf{28},\mathbf{1};\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1})$
(7; 4)	$\mathrm{SU}(8) \times \mathrm{SU}(2)\times$ $\mathrm{E}_7 \times \mathrm{SU}(2)$	$(\mathbf{8},\mathbf{1};\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{28}},\mathbf{1};\mathbf{1},\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$
(7; 9)	$\mathrm{SU}(8) \times \mathrm{SU}(2)\times$ $\mathrm{SU}(8) \times \mathrm{U}(1)$	$(\mathbf{8},\mathbf{1};\mathbf{1})_{2}$	$(\mathbf{1},\mathbf{2};\mathbf{8})_{1}$
(7; 8)	$\mathrm{SU}(8) \times \mathrm{SU}(2)\times$ $\mathrm{SO}(16)$	nothing	$(\mathbf{28},\mathbf{1};\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1})$
(2; 1)	$\mathrm{SO}(14) \times \mathrm{U}(1)\times$ $\mathrm{E}_7 \times \mathrm{U}(1)$	$(\mathbf{14}_v;\mathbf{1})_{-1,1/2}+(\mathbf{1};\mathbf{1})_{1,-3/2}$ $+5(\mathbf{1};\mathbf{1})_{1,1/2}$	$(\mathbf{14}_v;\mathbf{1})_{0,1}+(\mathbf{1};\mathbf{1})_{2,-1}$ $+(\mathbf{1};\mathbf{1})_{-2,-1}$
(2; 5)	$\mathrm{SO}(14) \times \mathrm{U}(1)\times$ $\mathrm{SO}(12)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	$(\mathbf{1};\mathbf{12}_v,\mathbf{1})_{1,1/2}+2(\mathbf{1};\mathbf{1},\mathbf{2})_{1,-1/2}$	$(\mathbf{14}_v;\mathbf{1},\mathbf{1})_{0,1}+(\mathbf{1};\mathbf{1},\mathbf{1})_{2,-1}$ $+(\mathbf{1};\mathbf{1},\mathbf{1})_{-2,-1}$
(6; 1)	$\mathrm{SO}(10) \times \mathrm{SU}(4)\times$ $\mathrm{E}_7 \times \mathrm{U}(1)$	$(\mathbf{16}_s,\mathbf{1};\mathbf{1})_{1/2}+2(\mathbf{1},\mathbf{4};\mathbf{1})_{1/2}$	$(\mathbf{10}_v,\mathbf{1};\mathbf{1})_{-1}+(\mathbf{1},\mathbf{6};\mathbf{1})_{1}$
(6; 5)	$\mathrm{SO}(10) \times \mathrm{SU}(4)\times$ $\mathrm{SO}(12)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	$(\mathbf{1},\mathbf{4};\mathbf{1},\mathbf{2})_{-1/2}$	$(\mathbf{10}_v,\mathbf{1};\mathbf{1},\mathbf{1})_{-1}$ $+(\mathbf{1},\mathbf{6};\mathbf{1},\mathbf{1})_{1}$

Table 2.16: There are twelve modular invariant combinations of $\mathbb{Z}_4$ gauge shifts. The number specifies the shift combination and refers to the first column of table 2.15. The resulting gauge group and the twisted matter at a single fixed point is given. For the double twisted matter a chiral multiplet $\mathbf{T}_{[2v]}$ is supported at every $T^2/\mathbb{Z}_2$ brane, whereas a multiplet $\mathbf{T}_{[2v]} + \overline{\mathbf{T}}_{[2v]}$ is supported at every T^2 brane. The multiplicites in the second row indicate, how many chiral multiplets transform in this representations.

global. Accordingly we call the spectra we have computed before as global spectra. To develop a local picture we will take a target space point of view. Therefore we reconsider the untwisted projection condition.

From target space perspective a field clearly depends on the uncompactified target space coordinates x^μ as well as on the internal coordinates z^i and $z^{\bar{\imath}}$. Consider a field $\phi(x^\mu, z^i)$, which can be of any particle type allowed in the spectrum and let $\mathcal{P}$ denote the projector, projecting on space group singlets. Clearly

$$\mathcal{P}\phi(x^\mu, z^i) = \phi(z^\mu, z^i) \tag{2.9.1}$$

in the untwisted sector. However, we have to distunguish between internal points that correspond to a brane, and those where the orbifold acts freely.

Assume, that $(z^i, \bar{z}^{\bar{\imath}})$ is situated in the bulk, i.e. away from any branes. Then ϕ must be a superposition of a state with all its space group images. Therefore all $\mathcal{P}$ does, is permuting these images and map them onto each other. It follows that (2.9.1) effectively is no projection. This is a first very important result: As long as we look at an orbifolded field at an internal position away from all branes, it looks as if it has not been orbifolded at all, It appears like a usual massless ten dimensional field of free heterotic string theory. This in particular implies, that all charged multiplets fill complete representations of the full $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group.

Now assume, that $(z^i_{fix}, \bar{z}^{\bar{\imath}}_{fix})$ describe the coordinates of a (k, l)–brane, i.e. we are looking at the field ϕ at the internal point where the (k, l)–twisted strings are living. We will first assume that the branes correspond to fixed points, but relax this assumption later. Then the geometric point is fixed under the space group element $(\theta^k, u(k, l))$ and now (2.9.1) acts as a real projector on the gauge d.o.f.. The gauge transformation associated to the space group element under consideration is given by $(\beta(k, l))$ with $\beta(k, l) = kv + m^i(k, l)a_i$. To find out the consequences we decompose the complete projector (2.6.7) in new phases. Let s_{part} contain all phases descending from those quantum numbers, that define the space time and SUSY transformation properties. These are i.e. all rightmoving phases and those from leftmoving bosonic oscillators. Further denote by s_{gauge} only the phases that come from the leftmoving gauge d.o.f.. Then (2.6.7) can be rewritten as

$$e^{2\pi i(s_{\text{part}} + s_{\text{gauge}})} = 1 \ . \tag{2.9.2}$$

Now from the argumentation above it is clear, that the gauge projection is only due to $q \cdot \beta(k, l)$, where $q \in \Gamma$. So the natural question to ask is, what the gauge group looks like. To answer this question we remember, that for the gauge multiplet we found $s_{\text{part}} = 0 \mod 1$ (this is valid for four and for six uncompactified dimensinos). But then the projection determining the adjoint representation as it appears at the brane in question (and therefore the relevant gauge group) is just given by

$$q \cdot V = q \cdot (v + m^i(k, l)a_i) = 0 \mod 1 \ , \tag{2.9.3}$$

without any other restrictions. By comparison with (2.8.2) we are now in the position to reinterpret this result in the following way: The gauge group as it appears on a (k,l)–brane is exactly that of an orbifold model without Wilson lines and gauge shift $\beta(k,l) = kv + m^i(k,l)a_i$. The same reasoning can be repeated for all other types of particles stemming from the untwisted sector (including the neutral ones) and holds in general. Refering to (2.8.2) we write for the representations

$$\mathbf{R}^{(\mathbf{c})}_{[\mathbf{kv}][\mathbf{a_i}]}\Big|_{(k,l)} = \mathbf{R}^{(\mathbf{c})}_{[\mathbf{kv}+\mathbf{m^i}(\mathbf{k},\mathbf{l})\mathbf{a_i}][\mathbf{0}]}\,, \tag{2.9.4}$$

where we understand $|_{(k,l)}$ like "as it appears on the (k,l)–brane". We will refer to the gauge group and matter representations as they appear on the (k,l)–brane as local gauge group and local matter representations.

Now reconsider the twisted states at the (k,l)–brane. It is easy to see from the general construction prescriptions given in subsection 2.6.3 or from the direct computation of spectra for the different examples, that even all (k,l)–twisted states only depend on the specific combination $\beta(k,l) = kv + m^i(k,l)a_i$, i.e. they do not feel a difference in gauge shift and Wilson lines at the brane as well. Therefore again it is clear, that they fill complete representations under the local gauge group.

Note, that by now we have completely disentangled the untwisted as well as twisted local spectra of different branes from each other. Locally at the (k,l)–brane they only depend on the specific combination $\beta(k,l) = kv + m^i(k,l)a_i$. The spectra can easily be inferred from the spectra of orbifolds without Wilson lines as introduced in subsection 2.9.1 by using $\beta(k,l)$ as gauge shift.

Up to now, we have assumed that the (k,l)–brane corresponds to a fixed point of the geometric orbifold. We want to relax this assumption and allow for fixed planes as well. We restrict the discussion to the $\mathbb{Z}_2$ subsectors of the $T^6/\mathbb{Z}_4$ orbifold of subsection 2.8.3 for simplicity (However, a similar reasoning should be possible for all kind of valid heterotic orbifold compactifications with Wilson lines that include subsectors.).

From worldsheet monodromies the strings are only confined to live on 5–branes. The relevant twist vector is given by $2\phi = (-1, 1/2, 1/2)$. By adding the SO(8) root $(0; 1, 0, -1)$ to it (which leaves the action of the orbifold unchanged), we arrive at an effective $2\phi = (0, 1/2, -1/2)$. But this is nothing but the twist vector of the $T^4/\mathbb{Z}_2$ orbifold discussed in subsection 2.8.1 and so the states give rise to multiplets of $D = 6, \mathcal{N} = 1$ SUSY exactly like there. However, on these six dimensional states we still have to deal with the $\mathbb{Z}_2$ projector (2.8.17), but the reasoning now is parallel to that for the untwisted states: Away from the $T^2/\mathbb{Z}_2$ fixed points the $\mathbb{Z}_2$ projector acts freely, so that there effectively is no projection. The states still appear six dimensional. The local gauge group is determined via $\mathbf{Ad}_{[\mathbf{2v}+\mathbf{m^i}(\mathbf{2},\mathbf{l})][\mathbf{0}]}$, again with all Wilson lines turned off. Similarly the representations for the charged hyper multiplets can be worked out. On the

$T^2/\mathbb{Z}_2$–fixed points however there is an effective projection again, so that the states appear four dimensional. It is now crucial to rethink where we are in internal space, i.e. exactly on the fixed points of the first (and third) twisted sector (note, that we started with a fixed plane of the second twisted sector). But such a fixed point is charakterized by an space group element $(\theta^k, u(k,l))$ with $k \neq 2$ and so the corresponding projection only depends on $kv + m^i(k,l)a_i$. This coincides with the disentanglement of brane spectra from the first and third twisted sectors and so the picture closes.

In this picture it is very natural to associate the level matching conditions directly to the branes, since they just ensure that the brane spectra are well–defined.

Summarizing the main conclusions again, we are facing the following situation: Away from the branes, i.e. in the bulk, the states are the ten dimensional ones of free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory. Depending on the space group of the orbifold (k,l)–branes occur, that either support four or six dimensional states[26]. The gauge representations on that branes only depend on the gauge shift and Wilson line combination $\beta(k,l) = kv + m^i(k,l)a_i$ and can be extracted from an orbifold model without Wilson line and gauge shift $\beta(k,l)$. Each brane comes with its own level matching condition, that renders the spectrum on that brane consistent. The states on the branes fall into two classes: Firstly the twisted ones, whose CM is restricted on the brane and which therefore are explicitly of the dimensionality of the brane. Secondly the untwisted ones that are in fact ten dimensional, but at the brane decompose into multiplets of brane dimensional $\mathcal{N} = 1$ SUSY and local gauge group. Thirdly, in models with subsectors a mixture occurs, because there are 4–branes contained in six–branes. The six dimensional states undergo projections at the loci of the 4–branes and decompose in gauge representations of the 4–brane in question.

We close this section with a comment on phenomenological implications: Discrete Wilson lines as studied in this work have been and still are widely used to construct SM gauge groups at zero–mode level. It has to be questioned wether their use for phenomenology makes any sense: Usually it is assumed that by some (unknown) dynamic mechanism the compactification radii are fixed at a small non–vanishing value, so that a new scale is established in the theory. As we have learned from the local perspective in such a scenario the unknown mechanism has to also guarantee that our universe is internally located a 3–brane. But the local gauge group and spectrum at this 3–brane is not distinguishable from the spectrum of a 3–brane that is part of a model without Wilson lines. Therefore a zero–mode gauge group that is the cut of all local gauge groups looses its meaning for the description of our world.

In this spirit from a local perspective discrete Wilson–lines should be seen as a

[26]This holds true for all possible orbifolds of type $T^4/\mathbb{Z}_N$ or $T^6/\mathbb{Z}_N$ although only examples are considered in this work.

tool to incorporate different brane spectra in one heterotic orbifold in a consistent way at the same time. This interpretation nicely reflects their topological origin.

2.9.3 A Journey through an Orbifold

To get some intuition for the concept of brane spectra as developed in the last subsection we will describe a small journey through the interior of the specific $T^6/\mathbb{Z}_4$ orbifold model

$$v = \tfrac{1}{4}(22200000|31000000)\ , \quad a_2 = \tfrac{1}{4}(22000000|11111133)\ , \qquad (2.9.5)$$

and all other Wilson lines vanish. The level matching conditions are fulfilled at each brane, including the extra one (2.8.11) responsible for the 5–branes. We are not going to state all different brane spectra, but only those we call by on our trip. Consider the path depicted in figure 2.7. Starting point is $(\bullet\bullet\bullet)$. Clearly

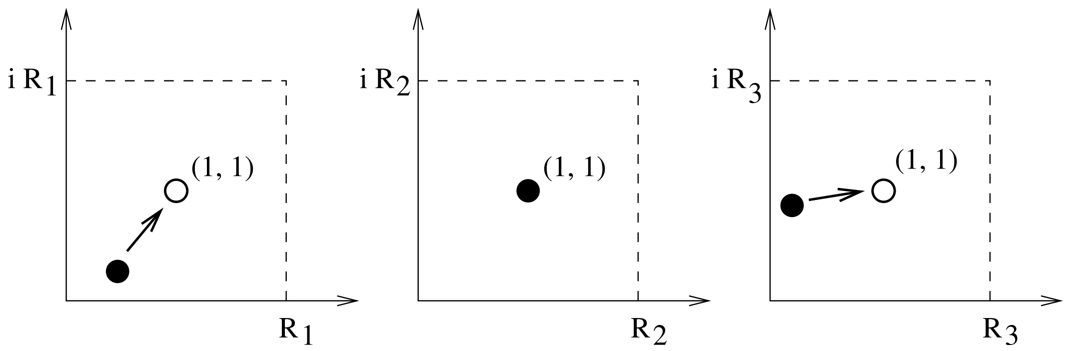

Figure 2.7: Journey through the interior of $T^6/\mathbb{Z}_4$: Starting point is $(\bullet\bullet\bullet)$. From there we travel on the third T^2 to reach the point $(\bullet\bullet\circ)$. Finally, we walk the first T^2 to end our journey at $(\circ\bullet\circ)$.

on this point the space group acts freely, we are on no brane and therefore we experience our world as ten dimensional. Beside $D = 10, \mathcal{N} = 1$ SUGRA we only see an $\mathrm{E}_8 \times \mathrm{E}_8$ super Yang-Mills multiplet.

Since this is not the world we would like to stay in, we move along the third torus in direction of $(\bullet\bullet\circ)$. We approch the $T^2/\mathbb{Z}_2$ brane. As soon as we reach $(\bullet\bullet\circ)$ our world suddenly seems six dimensional. Parts of the untwisted ten dimensional states of the bulk theory are projected out to leave a SUGRA, a tensor and four neutral hyper multiplets of $D = 6, \mathcal{N} = 1$ SUSY. Additionally the ten dimensional gauge multiplet decomposes into a six dimensional gauge multiplet, spanning the local gauge group, and a charged hyper multiplet. Furthermore, we can see twisted strings that are bound to the 5–brane. They contribute a half–hyper multiplet and two charged hyper multiplets. To state it briefly, we see a spectrum with the same structure as given in table 2.5.

But what is the local gauge group and what are the right representations? To answer this question we note, that the space group element $(\theta^2, e_3+e_4+e_5+e_6)$ leaves $(\bullet, \bullet, \circ)$ invariant and therefore corresponds to the 5–brane we are sitting at. The associated gauge shift is therefore given by $2v+a_2+a_2+0+0 = 2v+2a_2$. By manipulating this local shift vector like described in appendix A.3.5 and using table 2.11 (or recomputing everything) one can read off the local gauge group to be SO(16) $\times$ E$_7$ $\times$ SU(2). From the same table and table 2.12 one reads off the missing representations. The untwisted charged hyper multiplets that survive the projection transform as $(\mathbf{128_c};\mathbf{1},\mathbf{1}) + (\mathbf{1};\mathbf{56},\mathbf{2})$. The twisted half–hyper multiplet transforms as $(\mathbf{16_v};\mathbf{1},\mathbf{2})$, whereas for the two charged hyper multiplets the gauge representation is empty, so that finally they are simply absent in this specific model. That is it.

Six dimensions are better than ten, but worse than four, so we continue our journey on the first torus in direction of $(\circ\bullet\circ)$. Note that as long as we do not move on the second and third torus, we are still on the 5–brane and the world looks to us exactly as on $(\bullet\bullet\circ)$, namely six dimensional. As soon as we reach $(\circ\bullet\circ)$ the world seems to change again and now appears four dimensional to us. We arrive at a 3–brane. Even parts of the untwisted six dimensional states are projected out by the additional $\mathbb{Z}_2$ action: The six dimensional SUGRA, tensor and the four neutral hyper multiplets split into four dimensional $\mathcal{N}=\infty$ SUGRA, a chiral dilaton/axion and six neutral chiral multiplets. Additionally the six dimensional gauge multiplet decomposes into a four dimensional gauge multiplet and a charged chiral multiplet, and the untwisted charged hyper into two four dimensional charged chiral multiplets. Now to the six dimensional twisted states. They as well are subject to the extra $\mathbb{Z}_2$ projection and give rise to a charged chiral multiplet (corresponding to $\mathbf{T}_{[\beta(\mathbf{2},\mathbf{1})]}$ of table 2.10). Finally, purely four dimensional twisted states appear and contribute a charged chiral multiplet. Again, the structure of the spectrum is just like in table 2.10.

The representations are found like in the six dimensional case. The 3–brane we are at is invariant under the space group element $(\theta, e_1+e_2+e_3+e_5)$, yielding the associated gauge shift $v+a_2$. It is then easy to see, that the local gauge group is given by SO(14)$\times$U(1)$\times$SO(12)$\times$SU(2)$\times$U(1). All appearing representations can then be extracted from tables (2.15) and (2.16).

Our journey is finished. We just would like to mention, that the path was chosen in such a way, that the notion of our world changed from ten to six and then to four dimensions. This is not necessarily the case. If e.g. we would have approached the end point $(\circ\bullet\circ)$ from the second torus only, we would not have experienced an explicit six dimensional image on our way. The breaking would be ten to four directly. However, by construction we know, that for the case of $T^6/\mathbb{Z}_4$ as considered here every appearing 3–brane with associated local shift V is embedded in a 5–brane with associated local shift $2V$. Therefore the two step breaking as observed on our journey must always be possible. In fact it is.

For all models without Wilson lines (and only they appear as brane spectra) the branching rules have been listed in appendix A.5.

2.9.4 From a Local to a Global Point of View

With the local knowledge we just developed it is very interesting to reinterpret the global spectra as described until section 2.8. We start by comparing the local gauge group at the (k,l)–brane with the global gauge group and therefore the definitions (2.8.2) of the corresponding adjoint representations $\mathbf{Ad_{[v][a_i]}}$ and $\mathbf{Ad_{[kv+m^i(k,l)a_i][0]}}$, respectively. Clearly, the projection for the local gauge group is weaker. From

$$q\cdot v = 0 \mod 1\,,\; q\cdot a_i = 0 \mod 1 \;\Rightarrow\; q\cdot(kv+m^i(k,l)a_i) = 0 \mod 1 \quad (2.9.6)$$

we see, that the global gauge group is always a subgroup of the local one. Note, that this is true for every local gauge group. Even more, for the examples studied in this work for every k one can find an l such that $m^i(k,l)$ reaches all integers from 0 to $N-1$. But then the global gauge group can be thought of as the cut of all local ones (the superpositions of projections for all (k,l)–branes is identical to the global projection). In summary we have

$$G_{[v][a_i]} \subset G_{[kv+m^i(k,l)a_i][0]}\ \forall k,l\ , \quad \cap_{k,l} G_{[kv+m^i(k,l)a_i][0]} = G_{[v][a_i]}\ . \quad (2.9.7)$$

We first work out the consequences for the twisted states that appear on the fixed points, since their treatment is easiest. As seen before, they fill irreducible representations of the unbroken local gauge group. These irreducible representations are now in general reducible as representations of the unbroken global gauge group. Therefore from the global picture they appear decomposed. However, the sum of dimensions of the representations is unchanged because there is no difference in projection w.r.t. global and local point of view (the centralizers are trivial).

This is not true for the untwisted states at the branes. In analogy to (2.9.6) we have for the untwisted matter representations as defined in (2.8.2)

$$q\cdot v = c/N \mod 1\,,\; q\cdot a_i = 0 \mod 1 \;\Rightarrow\; q\cdot(kv+m^i(k,l)a_i) = c/N \mod 1\ . \quad (2.9.8)$$

and therefore[27]

$$\mathbf{R^{(c)}_{[v][a_i]}} \subset \mathbf{R^{(c)}_{[v+m^i(1,l)a_i][0]}}\ \forall l\ , \quad \cap_l \mathbf{R^{(c)}_{[v+m^i(1,l)a_i][0]}} = \mathbf{R^{(c)}_{[v][a_i]}}\ . \quad (2.9.9)$$

[27]Firstly, for this comparison of representations to make sense of course all of them have to be decomposed w.r.t. the global gauge group $G_{[v][a_i]}$. Secondly, analogous statements can be made for $k \neq 1$, however we do not need this for our examples and so we restrict to keep the notation at least quite compact.

We see, that in general there are bigger representations in the local picture than in the global one.

How can this be consistent? The answer is the following: One of our starting points was to work out massless states. Consider e.g. a fermion. Then, in dimensional reducing a Dirac equation for a massless ten dimensional particle, the lower dimensional mass is given by the internal derivative of the field. Therefore in order to describe a massless particle again, the modes of the field in question have to be constant over the whole internal manifold. However, constancy especially implies that the mode has to be present everywhere on the manifold. And this is certainly not the case for the local matter representations. Consider e.g. the local matter representations at the (k, l)–brane. From the Virasoro generators we know, that it is perfectly massless. However, in general parts of this representation are not compatible with the space group action, i.e. the gauge projection, on another different brane (of equal dimensionality). The common part of all local representations that is compatible with all projections on all different branes is exactly the global representation. We arrive at the very important conclusion: The global spectra as presented up to section 2.8 describe only the modes of the fields that are constant and present everywhere on the whole orbifold (including all branes). This is equivalent to the zero mode limit $R_i \to 0$ that sends all internal radii to zero. Then the orbifold collapses to a point and all spacetime and gauge projections are superimposed, yielding the global spectrum again. Therefore we use the terms global and zero mode spectra synonymical. It should be clear, that for the description of local phenomena, the pure zero mode picture is not enough.

Chapter 3

Anomalies

In this chapter we show how local anomalies can be computed on orbifolds and how they cancel. The computations performed in this section are solely on field theoric grounds. Following [43] we start with two brief introductions, one to low energy field theory in section 3.1 and one to anomalies in section 3.2. We move on to the computation of anomalies on orbifolds in section 3.3. There we first derive the zero–mode anomalies for all examples discussed in this thesis. Then we present how anomalies localize and compute the localized anomalies explicitly. These computations are central issues of this thesis. They have first been done in [33] for five dimensional field theory orbifolds, then in [46] for the special case of $T^6/\mathbb{Z}_3$ and using general results obtained in [40] for the other two examples presented here in [48]. We show that the localized anomalies always cancel up to terms that factorize in a special way. Finally in section 3.4 we show that these factorized local anomalies can always be canceled by a local version of the so called Green–Schwarz mechanism. Therefore we give a short introduction to the orginial Green–Schwarz mechanism and discuss it globally and locally. The original version has been proposed in [71], versions for zero–mode anomalies on orbifolds in [72, 73] and the local version in [47, 48].

3.1 Low Energy Effective Field Theory

In this section we follow [74, 43]. We first want to define the action of a low energy effective field theory. In fact there are in principle two approximations to be taken as will become apparent in the following, firstly a string to field theory limit and secondly a low energy limit. However, we will see that it is necessary to do both at the same time.

Consider ten dimensions for concreteness, although the discussion is valid in more generality. Let ϕ_0 and ϕ_h collectively denote all (ten dimensional) massless and massive fields respectively. Then there should be a corresponding field theory action $S(\phi_0, \phi_h)$. Unfortunately up to now such an action is in general unknown.

However, the (ten dimensional) masses of states are of order of the Planck scale $M = 10^{19} GeV$ and so perturbatively[1] decouple in the low energy regime of order $E << M$, as their participation in low energy interactions is supressed. Therefore, a useful low energy effective theory should primarily depend on the (ten dimensional) massless states. In fact such an action is related to $S(\phi_0, \phi_h)$ in the following way

$$e^{iS_{eff}(\phi_0)} \sim \int D\phi_h \, e^{iS(\phi_0,\phi_h)} \ . \tag{3.1.1}$$

Note, that in the full path integral, where all fields, massive and massless ones are integrated out, this is just a reordering of contributions. Therefore in the effective path integral the massive d.o.f. are removed from the quantum theory in a proper way.

This still does not mean that one is able to write down the low energy effective action S_{eff} exactly. However, one can expand in the number of derivatives, since each derivative goes along with a supression by a power of E/M. Therefore already the first few terms give a very precise approximation and most important, they are completely determined by symmetries, i.e. gauge invariance and SUSY. The low energy effective action is therefore practicially defined from low energy considerations.

With help of these symmetries it has been shown, that the low energy effective action for the free heterotic string ($D = 10, \mathcal{N} = 1$) can be cast in the form [75, 76, 77]

$$S_{het} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left[R + 4\partial_M \phi \partial^M \phi - \frac{1}{2}|H_3|^2 - \frac{\kappa^2}{g^2} \mathrm{tr}(|iF_2|^2) \right] \ , \tag{3.1.2}$$

where we have restricted to the bosonic part for simplicity. κ and g are (ten dimensional) gravitational and gauge coupling constants, respectively, G is the spacetime metric, R the corresponding Ricci scalar and ϕ is the dilaton. We will use the symbol tr for traces over the fundamental and the symbol Tr for traces over the adjoint representation. It can be shown that $\kappa^2/g^2 = \alpha'/4$. Additionally $iF_2 = d(iA_1) + (iA_1)^2$ (wedge product implicit) denotes the field strength of the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge field A_1. They are understood as Lie algebra valued two and one forms respectively. We take the generators of the gauge group to be hermitean. The norm–squared is defined for a q–form C_q via $|C_q|^2 = (1/q!) C_{M_1 \dots M_q} C^{M_1 \dots M_q}$. Finally H_3 corresponds to the field strenght of the $\widetilde{\mathrm{NS}}$ two–form field B_2 but already includes a correction term

$$H_3 = dB_2 - \frac{\kappa^2}{g^2} \omega_{YM\,3} \ , \quad \omega_{YM\,3} = \mathrm{tr}\left(A_1 dA_1 - \frac{2i}{3} A_1^3 \right) \ , \tag{3.1.3}$$

[1] Note that there are effects where this is not true. Anomalies e.g. provide a counter example as will be shown below.

that has to be included due to local SUSY, when $D = 10, \mathcal{N} = 1$ SUGRA and $E_8 \times E_8$ Yang–Mills theories are coupled. The three form $\omega_{YM\,3}$ is called Yang–Mills or gauge Chern–Simons form. Note that really all massless (bosonic) fields of free heterotic $E_8 \times E_8$ string theory as computed in subsection 2.1.3 are contained in (3.1.2) therfore allowing an interpretation as low energy effective action.

Clearly (3.1.2) is invariant under a gauge transformation $\delta_{\Lambda_1} B_2 = d\Lambda_1$ with Λ_1 an infinitesimal one form. However, under a Yang–Mills gauge transformation $\delta_{\Lambda_0} A_1 = d\Lambda_0 - i[A_1, \Lambda]$ we have $\delta_{\Lambda_0}\omega_{YM\,3} = d\mathrm{tr}(\Lambda_0 dA_1) \equiv d\omega^1_{YM\,2}$ for Λ_0 a zero form and therefore

$$\delta_{\Lambda_0} B_2 = \frac{\kappa^2}{g^2}\omega^1_{YM\,2} \,, \tag{3.1.4}$$

so that Yang–Mills gauge invariance is restored.

One problem in (3.1.2) arises via the fact, that the dilaton ϕ appears as a modulus, i.e. different background values for ϕ correspond to different string vacua, e.g. the low energy gauge couplings depend on it. The potentials appearing for the dilaton mostly possess a runaway property in the sense that they drive the vacuum expectation value to infinity and therefore the low energy couplings to zero. Therefore there should be a (presumably non–perturbative) mechanism, that stabilizes the dilaton. There has been some effort to find such mechanisms (c.f. e.g. [78] and references therein). However, in going to lower dimensions there are other moduli, that are not fixed by dynamical mechanisms either: E.g. why should a $\mathbb{Z}_N$ orbifold be realized for a definite N, what determines the compactification radii and so on. Questions like this turn out to be notoriously hard to anwer, and since there are no convincing concepts to tackle them as well, we will not go into detail and assume that all such moduli are fixed.

3.2 A Brief Introduction to Anomalies

Anomalies are quantum effects that break symmetries present in the classical field theory. They were first found in certain Feynman diagrams. For the case of four dimensions and gauge symmetry the structure of such Feynman diagrams is depicted in figure 3.1. The three V–A currents attached to the vertices are

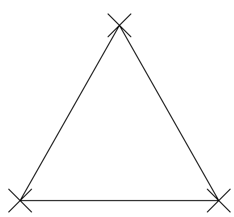

Figure 3.1: Anomalous triangle diragram appearing in four dimensions. There are V–A currents attached to each vertex that are classically conserved.

conserved in the classical field theory. On the other hand the contributions from the one–loop triangle diagram are divergent and therefore enforce the introduction of a regulator. In turns out that in parity–violating amplitudes (and only there as we will see later) this regulator is not compatible with conservation of all attached currents. Therefore a classical symmetry is broken on the quantum level and the diagram is called anomalous.

We want to generalize the concept of anomalies not only for a gaugino but for gravitinos as well and to arbitrary even[2] dimensions $D = 2k$. For the gravitational part we again write spin connection ω and Ricci tensor R as $\mathrm{SO}(1,9)$ Lie algebra valued 1 and 2 forms and use their analogy to gauge potential and field strength, respectively[3]. Let

$$\not{D} = \not{\partial} + (i\not{A} + \not{\omega})\frac{1+\tilde{\gamma}}{2} \tag{3.2.1}$$

be the Dirac operator and $\tilde{\gamma}$ the chirality operator in $2k$ dimensions. Because the Dirac operator is linear in Dirac matrices states of a given chirality are mapped to the other. Considering a left–handed fermion χ we then introduce a non–interacting right–handed fermion ξ, so that the Hilbert space corresponding to the Dirac fermion $\psi = \chi + \xi$ is mapped to itself under $\not{D}$. The action

$$S(\psi, A, G) = -\frac{1}{2}\int d^{2k}x\,\sqrt{-G}\;\bar{\psi}\,\not{D}\,\psi\;. \tag{3.2.2}$$

is classically invariant under the gauge transformation $g : (\psi, A) \to ({}^g\psi, {}^gA)$, i.e. $S({}^g\psi, {}^gA, G) = S(\psi, A, G)$.

Now consider an effective action $\Gamma(A_M, G_{MN})$ in terms of gauge field and gravition with all other fields integrated out. Then an anomaly occurs if and only if

$$e^{i\Gamma(A,G)} = \int d\psi d\bar{\psi}\; e^{iS(\psi,A,G)} \neq e^{i\Gamma({}^gA,G)} = e^{i\Gamma(A,G)}\, e^{iG(g)}\;. \tag{3.2.3}$$

$G(g)$ depends on the gauge transformation and is called anomaly, since for $G(g) \neq 0$ the symmetry is spoiled.

From now on we restrict to infinitesimal gauge transformations Λ only. Then the anomaly is given as

$$G(\Lambda) = \delta_\Lambda\Gamma = \overline{Tr}\,[\Lambda\tilde{\gamma}]\;\;, \tag{3.2.4}$$

[2]In odd dimensions there is no notion of chirality and the kind of anomalies considered here can not occur.

[3]Unlike the generators of the gauge symmetries that we took to be hermitean, we take those of the local Lorentz symmetries to be antihermitean. Therefore we usually write powers of R, but iF.

where in the last equation the trace $\overline{Tr}$ runs over space time and gauge indices as well as (over)countable states (the full Hilbert space). Therefore regularization has to be performed.

Consider an (infinitesimal) gauge transformation $\Lambda = [\Lambda_1, \Lambda_2]$. Then the anomaly has to fulfill

$$\delta_{\Lambda_1} G(\Lambda_2) - \delta_{\Lambda_2} G(\Lambda_1) = G(\Lambda) \,, \tag{3.2.5}$$

as follows from (3.2.3) by applying two gauge transformations g_1 and g_2 and taking the linear approximation. This very important equation is known as Wess–Zumino consistency condition [79] and is so strong, that it fixes the complete structure of anomalies[4]. Therefore instead of computing Feynman diagrams we just search for a general solution to (3.2.5).

To do so, we again proceed along the lines of [43, 80]: By construction $G(\Lambda)$ is always given as an integral over the full spacetime, i.e. it can be written as $\Gamma(\Lambda) = \int I_D^1$ with I_D^1 a D form, that is clearly only defined up to an exact form. Starting point for the construction of a solution is a gauge invariant $D+2$ form[5] I_{D+2}. To find the most general form of I_{D+2} we first consider a $2m$ form $\mathrm{tr}(iF)^m$. Due to $\delta_\Lambda iF = [iF, \Lambda]$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, which is valid for forms of even degree, $\mathrm{tr}(iF)^m$ is gauge invariant. In complete analogy a $2m$ form $\mathrm{tr}(R^m)$ is invariant under an (infinitesimal) local Lorentz transformation Θ. Since for $\mathrm{SO}(1,9)$ the form R is represented by an antisymmetric matrix $\mathrm{tr}(R^m)$ is only non–vanishing for m even. Using $\mathrm{tr}(iF)^m$ and $\mathrm{tr}R^m$ as building blocks for a polynomial then defines the most general form of I_{D+2}, that is invariant under gauge and local Lorentz transformations. Therefore I_{D+2} is usually called anomaly polynomial.

Locally every term $tr(iF)^m$ can be written as outer derivative of a $(2m-1)$ form

$$\mathrm{tr}(iF)^m = d\omega_{2m-1} \,, \tag{3.2.6}$$

where

$$\omega_{YM\,2m-1} = m \int_0^1 dt\, t^{m-1}\, \mathrm{tr}(iA(d(iA_+t(iA)^2)^{m-1}) \tag{3.2.7}$$

is a general form of these so called Chern–Simons forms[6]. Replacing A by the spin connection in (3.2.7) defines the Lorentz Chern–Simons forms, that will be denoted by $\omega_{L\,2m-1}$.

[4]To pe precise the anomaly is fixed up to a multiplicative factor. But this factor can be fixed by comparison with an explicit computation of an anomalous diagram.

[5]I_{D+2} has to be taken formally, because in D–dimensional spacetime it should be actually zero. However, this procedure is mathematically well defined (see e.g. [81] and references therein).

[6]We encountered the special case of a gauge Chern–Simons 3 form $\omega_{YM\,3}$ already in the previous subsection.

Since $\mathrm{tr}(iF)^m$ and trR^m are exact it follows that $d\mathrm{tr}(iF)^m = d\mathrm{tr}R^m = 0$ and that any polynomial made out of them is as well exact. This implies that I_{D+2} is exact as well, i.e. $I_{D+2} = dI_{D+1}$ with some $(D+1)$ form I_{D+1}. Now unlike I_{D+2}, the form I_{D+1} is not in general invariant under gauge and/or local Lorentz transformations. However, due to invariance of I_{D+2} it always transforms as a total derivative $\delta I_{D+1} = dI^1_D$. The superscript in I^1_D indicates, that the expression is linear in Λ and/or Θ and defined up to a closed form. However its integral over spacetime is unique and nothing but the anomaly

$$G = \int I^1_D \, . \tag{3.2.8}$$

In summary starting with the gauge invariant I_{D+2} we have a chain of formulas determining the anomaly via

$$\delta I_{D+2} = 0 \; , I_{D+2} = dI_{D+1} \; , \quad \delta I_{D+1} = dI^1_D \, . \tag{3.2.9}$$

This chain is known as the descend equations [80]. Note that except for I_{D+2} the forms appearing in this chain are always defined locally.

We still have to show, that the Wess–Zumino consistency conditions (3.2.5) are satisfied. Let M be the spacetime manifold and suppose it is represented as a boundary $M = \partial N$ of some other manifold N (which can always be achieved). Then we have

$$G = \int_M I^1_D = \int_N dI^1_D = \delta \int_N I_{D+1} \, . \tag{3.2.10}$$

Then, with help of $[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_\Lambda$ (3.2.5) is implied.

Now we know that the anomaly polynomial I_{D+2} is a linear combination of terms with structure $(\mathrm{tr}(iF)^m)^k(\mathrm{tr}R^n)^l$ with $2(km + ln) = D + 2$ and how it is related to the anomaly via the descend equations. However, we have not yet determined the coefficients. In view of anomalous Feynman diagrams they depend on the gauge/Lorentz quantum numbers of the particles running in the loop. It has been shown that they can be determined from topological invariants defined in $D+2$ dimensions. The relation involves lots of interesting mathematics like index theorems, but is rather lengthy and will therefore not be presented here. In the context of physical applications they were first discussed in [82, 83]. The result can be summarized for chiral spin 1/2 fields in representation $\mathbf{r}$ as $\hat{I}_{1/2\,\mathbf{r}}|_{D+2}$, chiral spin 3/2 fields $\hat{I}_{3/2}|_{D+2}$ and (anti–)self–dual $D/2$ rank tensor $\hat{I}_A|_{D+2}$. For the explicit expressions see appendix A.6. By $|_{D+2}$ we denote that the $D+2$ form part of the expression has to be extracted. In this restricted expression terms are distinguished from each other in the following way: those without any R dependence are called pure gauge anomalies, those without any iF dependence gravitational anomalies and all others mixed anomalies, because in the corresponding anomalous Feynman diagrams either gauge fields or gravitons are attached to the vertices. In d uncompactified dimensions we distinguish between contributions from Abelian and non–Abelian gauge fields as follows: Contributions from an anomalous diagram with only (non–)Abelian gauge fields attached will be called pure (non–)Abelian anomalies, those with Abelian as well as non–Abelian gauge fields attached mixed Abelian/non–Abelian anomalies and those with (non–)Abelian gauge fields and spin connections attached mixed (non–)Abelian/gravitational anomalies.

3.3 Anomalies on Orbifolds

In this section apply the concept of anomalies on orbifolds. We start with the consideration of global anomalies as they arise from the zero–mode perspective and compute these anomalies for all examples considered in this work explicitly. Next we show that localization effects occur on orbifolds. They are responsible for the confinement of anomalies on the branes. We close this section with a discussion of local anomalies on orbifolds.

To avoid confusion that is sometimes present in the literature we want to stress that we are always considering anomalies due to (local) gauge and local Lorentz symmetries throughout the whole chapter. In the literature the notion of Abelian anomalies is sometimes connected to the chiral U(1) symmetry, that is only a global classical symmetry. We are not considering this (global) chiral symmetry.

3.3.1 Global Anomalies

We are now prepared to compute the global anomalies arising in orbifold compactifications. We want to emphasize that the terminology global here refers to the zero–mode perspective on orbifolds as considered in subsection 2.9.4. In the same spirit we will consider local anomalies from a local perspective.

We start with the computation of anomalies for the free heterotic $E_8 \times E_8$ orbifold[7] and then move over to the other examples considered in section 2.8.

We only compute global zero–mode anomalies for orbifolds without Wilson lines for reasons that will be clarified later.

The Free Heterotic String

The spectrum of the free heterotic $E_8 \times E_8$ string was derived in section 2.1.3. It lives in ten dimensions so that the relevant anomaly polynomial is a 12 form. The chiral fermions contained in the theory that potentially can lead to anomalies are the gravitino, the dilatino and the gauginos transforming in the adjoint

[7] We consider it as an orbifold that is not orbifolded here for reasons that will become apparent later.

representation **Ad** of $E_8 \times E_8$. Gravitino and gaugino are left–handed, the dilatino right–handed. Then using the definitions and results of appendix A.6 the anomaly polynomial reads

$$\hat{I}_{12} = \frac{1}{2}(\hat{I}_{3/2}(R) - \hat{I}_{1/2}(R) + \hat{I}_{1/2,\mathbf{Ad}}(F,R)) \ , \tag{3.3.1}$$

where the extra factor of 1/2 stems from the fact that the chiral fermions are additionally Majorana. It is straightforward but tedious to show that $\hat{I}_{12}$ can be written as

$$\begin{aligned} \hat{I}_{12} \ = \ & \frac{1}{1440}\left(-\mathrm{Tr}(iF)^6 + \frac{1}{48}\mathrm{Tr}(iF)^2\mathrm{Tr}(iF)^4 - \frac{1}{14400}[\mathrm{Tr}(iF)^2]^3\right) \\ & +(\dim(\mathbf{Ad}) - 496)\left(\frac{\mathrm{tr}(R)^6}{725760} + \frac{\mathrm{tr}R^4\mathrm{tr}R^2}{552960} + \frac{[\mathrm{tr}R^2]^3}{1327104}\right) \\ & + \left(\mathrm{tr}R^2 - \frac{1}{30}\mathrm{Tr}(iF)^2\right) X_8 \ , \end{aligned} \tag{3.3.2}$$

where

$$X_8 = \frac{1}{768}\left(\mathrm{tr}R^4 + \frac{[\mathrm{tr}R^2]^2}{4} - \frac{\mathrm{Tr}(iF)^2\mathrm{tr}R^2}{30} + \frac{\mathrm{Tr}(iF)^4}{3} - \frac{[\mathrm{Tr}(iF)^2]^2}{900}\right) \ . \tag{3.3.3}$$

In both expression tr denotes the traces over the fundamental representation of $SO(1,9)$ and Tr the trace over the adjoint representation of the gauge group. For E_8 we define $\mathrm{tr}(iF)^2 = \frac{1}{30}\mathrm{Tr}(iF)^2$.

Equation (3.3.2) is structured as follows: The first line contains only pure gauge anomalies, whereas the second and third lines factor in a special way. One apparent observation is that the second line contains the non–vanishing pure gravitational anomaly $\mathrm{tr}R^6$, therefore enforcing $\dim(\mathbf{Ad}) = 496$. This is fulfilled for $E_8 \times E_8$ and so the whole second line vanishes identically. Even more, using the identities (A.6.21) and (A.6.27) the first line vanishes as well. However, the third line does not. Fortunately it will cancel via the so called Green–Schwarz mechanism but we want to leave this open for a while. For now we want to just summarize that the anomaly of free heterotic $E_8 \times E_8$ string theory does not yet vanish. The corresponding anomaly polynomial factorizes into the form

$$\hat{I}_{12} = \left(\mathrm{tr}R^2 - \mathrm{tr}(iF)^2\right) X_8 \ . \tag{3.3.4}$$

This finishes our computation of ten dimensional anomalies.

However, at this stage we want to make some side remarks: Already the conditions that the second and third line of (3.3.2) have to vanish are so strong that there are in total only four gauge groups leading to the factorized form (3.3.4): $E_8 \times E_8$, $\mathrm{Spin}(32)/\mathbb{Z}_2$, $E_8 \times U(1)^{248}$ and $U(1)^{248} \times E_8$. The second in this list is realized as heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ string theory, whereas string realizations for the other two are not known. Tracing back the construction of the free heterotic string in subsection 2.1.3 the gauge group arose from the GSO projections that were ad hoc there. We could now justify our choice of GSOs as essential to yield anomaly cancellation in ten dimensions and therefore close our argumentation. In fact this argumentation is often used because historically this was how $E_8 \times E_8$ as possible gauge group was encountered and so this string theory was found. However, we do not want to use anomaly cancellation to define the GSOs. It is certainly a hint but there are other ways to justify the GSO projections as mentioned in 2.1.3. As we will see string theory and its compactifications are very powerful in constructing anomaly free low energy effective field theories. The above argumentation would weaken this predictive power.

$T^4/\mathbb{Z}_2$

The computation of anomalies for this class of models can be found e.g. in [84]. The global zero–mode spectra of all heterotic $T^4/\mathbb{Z}_2$ orbifolds without Wilson lines were derived in subsections 2.8.1 and 2.9.1. The multiplet structure is given in table 2.5, possible gauge groups and representations in tables 2.11 and 2.12. From the strict zero–mode persepective the theory is completely six dimensional so that the relevant anomaly polynomial is a 8 form. The chiral fermions that potentially can lead to anomalies are the gravitino, the dilatino, the gauginos transforming in the adjoint representation **Ad** of the gauge group and matter fermions. Gravitino and gaugino are left–handed, dilatino and matter fermions right–handed. Additionally there are anomaly contributions from the self–dual tensor of the tensor multiplet and from the anti–self–dual tensor of the SUGRA multiplet. Let s_i denote the number of hyper multiplets transforming in representation $\mathbf{r_i}$ and further assume that the gauge group is semi–simple[8]. Then the anomaly polynomial reads

$$\hat{I}_8 = \hat{I}_{3/2}(R) - \hat{I}_{1/2}(R) + \hat{I}_{1/2,\,\mathbf{Ad}}(F,R)) - \sum_i s_i \hat{I}_{1/2,\,\mathbf{r_i}}(F,R) \ , \tag{3.3.5}$$

where the contributions from the self–dual and anti–self–dual tensors are not included because they cancel each other (see e.g. [43]). Let $s = \sum_i s_i$ be the total number of matter hyper multiplets (including the neutral ones). Using the definitions from subsection A.6 it is then again straightforward to show that $\hat{I}_8$

[8] This is true for all six dimensional spectra considered in this work but not in general. For a complete treatment we refer to [84].

can be written as

$$\begin{aligned}\hat{I}_8 &= \frac{1}{5760}(244+\dim(\mathbf{Ad})-s)\mathrm{tr}R^4\\ &+\frac{1}{4608}(-44+\dim(\mathbf{Ad})-s)[\mathrm{tr}R^2]^2\\ &-\frac{1}{96}\mathrm{tr}R^2\left(\mathrm{Tr}(iF)^2-\sum_i s_i\mathrm{tr}_{\mathbf{r_i}}(iF)^2\right)\\ &+\frac{1}{24}\left(\mathrm{Tr}(iF)^4-\sum_i s_i\mathrm{tr}_{\mathbf{r_i}}(iF)^4\right)\ .\end{aligned} \tag{3.3.6}$$

We see that there is an irreducible gravitational anomaly unless

$$s-\dim(\mathbf{Ad})=244\ . \tag{3.3.7}$$

It can be checked for both possbible models of table 2.12 that this condition is fulfilled.

Since this is the first time we extract an anomaly from the given tables we demonstrate the procedure for the model with gauge group $\mathrm{E}_7\times\mathrm{SU}(2)\times\mathrm{SO}(16)$. Clearly $\dim(\mathbf{Ad})=133+3+120=256$. The number of charged hyper multiplets can be extracted from tables 2.11 and 2.12 again. The multiplicities in the second lines of the table captions can be used. We see that from the untwisted sector we get $56\cdot 2+128=240$ hyper multiplets not to forget the four additional neutral hypers from table 2.5 making a total of 244. There are 16 twisted sectors contributing $16\cdot(\frac{1}{2}\cdot 2\cdot 16)=256$ hypers. Therefore we have in total $s-\dim(\mathbf{Ad})=244+256-256=244$ which proves the condition.

This relation can now be used to fix the value in the round brackets of the second line of (3.3.6) to (-288). At this point one can show using the trace decompositions (A.6.21) and identities (A.6.28) that the anomaly polynomial can be cast in the form

$$\hat{I}_8=\frac{1}{16}\left(\mathrm{tr}R^2-\sum_{G_i}c_i\,\mathrm{tr}(iF)^2_{G_i}\right)X_4 \tag{3.3.8}$$

where

$$X_4=-\frac{1}{16}\left(\mathrm{tr}R^2-\sum_{G_i}d_i\,\mathrm{tr}(iF)^2_{G_i}\right)\ . \tag{3.3.9}$$

The sums run over the simple gauge group factors G_i. The constants c_i and d_i are contained in tables 3.2 and 3.1 for both models. The essential observation here is that unlike the d_i and therefore X_4 the coefficients c_i only depend on the gauge groups and not on the matter content of the theory. In fact in [84] it has been shown in more generality by enforcing factorization that

$$c_i=Q_2(G_i)/g(G_i) \tag{3.3.10}$$

Model	$d_i(G_i)$			
	E_7	SU(2)	E_8	SO(16)
$E_7 \times SU(2) \times E_8$	1	12	–1/5	–
$E_7 \times SU(2) \times SO(16)$	–1/3	28	–	2

Table 3.1: The factorization coefficients d_i in dependence of the appearing gauge group factors G_i relevant for (3.3.8) are given.

where the quadratic index $Q_2(G_i)$ is implicitly defined by

$$\mathrm{Tr}(iF)^2_{G_i} = Q_2(G_i)\mathrm{tr}(iF)^2_{G_i} \tag{3.3.11}$$

and $g(G_i)$ denotes the dual Coxeter number of the gauge group factor G_i. This will be very important later. However, again for now like in the ten dimensional case we are left with an anomaly polynomial that factorizes.

$T^6/\mathbb{Z}_3$

The global zero–mode spectra of all heterotic $T^6/\mathbb{Z}_3$ orbifolds without Wilson lines were derived in subsections 2.8.2 and 2.9.1. The multiplet structure is given in table 2.9, possible gauge groups and representations in tables 2.13 and 2.14. From the strict zero–mode persepective the theory is completely four dimensional so that the relevant anomaly polynomial is a 6 form. The chiral fermions that potentially can lead to anomalies are the gauginos transforming in the adjoint representation **Ad** of the gauge group and matter fermions. However, the gauginos do not contribute because they are never in a complex representation. We consider a theory with s_i chiral multiplets transforming in representation $\mathbf{r_i}$. Then the anomaly polynomial reads

$$\hat{I}_6 = \sum_i s_i \hat{I}_{1/2,\,\mathbf{r_i}}(F,R)\ . \tag{3.3.12}$$

Note that there are no contributions from $\hat{I}_{3/2}$ since they are only present in dimensions $D = 4k + 2$ with k integer. Using the definitions from subsection A.6 it is then again straightforward to show that $\hat{I}_6$ can be written as

$$\hat{I}_6 = \sum_i s_i \left(\frac{1}{48}\mathrm{tr}R^2 \mathrm{tr}_{\mathbf{r_i}}(iF) + \frac{1}{6}\mathrm{tr}_{\mathbf{r_i}}(iF)^3 \right)\ . \tag{3.3.13}$$

It is now useful to distinguish between non–Abelian anomalies and others. From the spectra we see that maximally one U(1) factor can arise per E_8. We denote the field strengths of all non–Abelian field strengths collectively by $i\hat{F}$ and the

Abelian ones by iF again. Then (3.3.13) decomposes according to (A.6.21) to (note that $\mathrm{tr}(i\hat{F}) = 0$ in the non–Abelian case)

$$\hat{I}_6 = \sum_i s_i \left(\frac{1}{48}\mathrm{tr}R^2\mathrm{tr}_{\mathbf{r_i}}(iF) + \frac{1}{6}\mathrm{tr}_{\mathbf{r_i}}(iF)^3 + \frac{1}{2}\mathrm{tr}_{\mathbf{r_i}}(i\hat{F})^2\mathrm{tr}_{\mathbf{r_i}}(iF) + \frac{1}{6}\mathrm{tr}_{\mathbf{r_i}}(i\hat{F})^3 \right) , \tag{3.3.14}$$

where only the last term corresponds to a possible non–Abelian anomaly. We compute them first. Non–Abelian anomalies in four dimensions can only occur for gauge group factors $G_i = \mathrm{SU(N)}$ with $N \geq 3$ otherwise $\mathrm{tr}(iF)^3 = 0$. It is easy to directly read off from tables 2.13 and 2.14 that all non–Abelian anomalies sum up to zero, using that the fixed point multiplicity of the twisted matter is 27 and further $\mathbf{84} = \binom{\mathbf{9}}{\mathbf{3}}$ and therefore $\mathrm{tr}_{\overline{\mathbf{84}}}(iF)^3_{\mathrm{SU(9)}} = -\mathrm{tr}_{\mathbf{84}}(iF)^3_{\mathrm{SU(9)}} = -9\,\mathrm{tr}(iF)^3_{\mathrm{SU(9)}}$ with help of (A.6.29).

Knowing that all non–Abelian anomalies sum up to zero the anomaly polynomial reduces to

$$\hat{I}_6 = \sum_i s_i \left(\frac{1}{48}\mathrm{tr}R^2\mathrm{tr}_{\mathbf{r_i}}(iF) + \frac{1}{6}\mathrm{tr}_{\mathbf{r_i}}(iF)^3 + \frac{1}{2}\mathrm{tr}_{\mathbf{r_i}}(i\hat{F})^2\mathrm{tr}_{\mathbf{r_i}}(iF) \right) , \tag{3.3.15}$$

and can only be non–vanishing for models containing at least one U(1) factor. There is a lot of interesting physics connected to anomalous U(1)'s that we are not able to discuss in this thesis. We refer the reader to [85, 86, 87, 88, 89] instead.

It has been shown in [41], that the remaining anomaly can again always be written in a factorized form

$$\hat{I}_6 = \frac{1}{48}\left(\mathrm{tr}R^2 - \sum_i c_i \mathrm{tr}(iF)^2_{G_i} \right) X_2 \tag{3.3.16}$$

with

$$X_2 = \sum_n (iF_{(n)})\mathrm{tr}(q_n) \; . \tag{3.3.17}$$

Note that we made contact with the U(1) generators q_n as defined in appendix A.4.2 by replacing the non–Abelian "would–be" generator T by q_n. Analogously $F_{(n)}$ denotes the field strength of the n–th U(1) factor appearing in the gauge group. It was further shown in [41, 90] that such a factorization is only possible because of the so called universality condition

$$\frac{1}{6}\frac{1}{k_{q_n}}\sum_i s_i\mathrm{tr}_{\mathbf{r_i}}(q_n^3) = \frac{1}{2}\sum_i s_i \sum_{G_i \neq \mathrm{U(1)}} Q_n(\mathbf{r_i})Q_2(\mathbf{r_i}) = \frac{1}{48}\sum_i s_i\mathrm{tr}_{\mathbf{r_i}}(q_n) \; . \tag{3.3.18}$$

Note that unlike in the case of non–Abelian generators the normalizations for the Abelian ones is not fixed by some Lie algebra structure. This can be accounted for by inclusion of the so called level $k_{q_n} = 2q_n \cdot q_n$ of the U(1). The universality condition (3.3.18) states that pure Abelian, mixed Abelian/non–Abelian and mixed Abelian/gravitational anomalies are not independent of each other. If one cancels all do. The universality condition (3.3.18) has been checked for every model appearing in this work explicitly and turns out to be fulfilled. The coefficients c_i appearing in (3.3.16) again only depend on the gauge group factors and coincide with those defined in the six dimensional context whenever the gauge group factors are identical. The values for all possible gauge group factors are contained in table 3.2. Due to universality it is sufficient to compute

Coeff.	Gauge group factors G_i					
	E_8	E_7	E_6	SO(N)	SU(N)	U(1)
c_i	1	1/6	1/3	1	2	k_q

Table 3.2: All possible gauge group dependend factorization coefficients c_i are given.

$\sum_i s_i \mathrm{tr}_{\mathbf{r_i}}(q_n)$ to decide wether a U(1) is anomalous or not. The result of this computation for the different $T^6/\mathbb{Z}_3$ models that contain U(1)'s is given in table 3.3. The relevant U(1) charges are taken from tables 2.13 and 2.14.

Model	U(1) generators	$\sum_i s_i \mathrm{tr}_{\mathbf{r_i}}(q_n)$
$E_7 \times U(1)_{(1)} \times$	$q_{(1)} = (110^6\|0^8)$	$27 \cdot 16$
$SO(14) \times U(1)_{(2)}$	$q_{(2)} = (0^8\|10^7)$	0
$SU(9) \times SO(14) \times U(1)_{(1)}$	$q_{(1)} = (0^8\|10^7)$	$27 \cdot 28$

Table 3.3: Sum of U(1) charges for all $T^6/\mathbb{Z}_3$ models that contain U(1)'s. The U(1) generators q_n are given explicitly (see discussion in section A.4.2).

$T^6/\mathbb{Z}_4$

The global zero–mode spectra of all heterotic $T^6/\mathbb{Z}_4$ orbifolds without Wilson lines were derived in subsections 2.8.3 and 2.9.1. The multiplet structure is given in table 2.10, possible gauge groups and representations in tables 2.15 and 2.16. From the strict zero–mode persepective the theory is completely four dimensional so that the relevant anomaly polynomial is a 6 form.

Since the whole discussion works exactly like for $T^6/\mathbb{Z}_3$ we do not repeat it here. Again it turns out that all four dimensional non–Abelian anomalies sum up to zero. The universality condition (3.3.18) is fulfilled for every model containing U(1)s and then the corresponding Abelian anomaly polynomial factorizes like

(3.3.16). Again the coefficients c_i appearing there only depend on the gauge group factors (not on the matter field content) and are consistent with those already obtained previously. They are listed in table 3.2 again. The results for the sums of U(1) charges are given in table 3.4. For the model with gauge group $\mathrm{SO}(14) \times \mathrm{U}(1) \times \mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ one can rotate the generators of the two U(1)'s into a basis such that only one of them remains anomalous. In fact it has been shown that in heterotic $T^6/\mathbb{Z}_N$ orbifolds always at most one U(1) is anomalous at the zero–mode level (see [85, 90] and references therein).

Model	U(1) generators	$\sum_i s_i \mathrm{tr}_{\mathbf{r_i}}(q_n)$
$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(1)} \times \mathrm{E}_8$	$q_{(1)} = -\frac{1}{6}(2110^5 \vert 0^8)$	0
$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(1)} \times$ $\mathrm{E}_7 \times \mathrm{SU}(2)$	$q_{(1)} = -\frac{1}{6}(2110^5 \vert 0^8)$	0
$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(1)} \times$ $\mathrm{SU}(8) \times \mathrm{U}(1)_{(2)}$	$q_{(1)} = -\frac{1}{6}(2110^5 \vert 0^8)$ $q_{(2)} = -\frac{1}{8}(0^8 \vert 1^7-1)$	$16 \cdot 18$ 0
$\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(1)} \times \mathrm{SO}(16)$	$q_{(1)} = -\frac{1}{6}(2110^5 \vert 0^8)$	$16 \cdot 36$
$\mathrm{SU}(8) \times \mathrm{SU}(2) \times \mathrm{SU}(8) \times \mathrm{U}(1)_{(1)}$	$q_{(1)} = -\frac{1}{8}(0^8 \vert 1^7-1)$	$16 \cdot 24$
$\mathrm{SO}(14) \times \mathrm{U}(1)_{(1)} \times$ $\mathrm{E}_7 \times \mathrm{U}(1)_{(2)}$	$q_{(1)} = -\frac{1}{2}(10^7 \vert 0^8)$ $q_{(2)} = -\frac{1}{2}(0^8 \vert 110^6)$	0 $16 \cdot 18$
$\mathrm{SO}(14) \times \mathrm{U}(1)_{(1)} \times$ $\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(2)}$	$q_{(1)} = -\frac{1}{2}(10^7 \vert 0^8)$ $q_{(2)} = \frac{1}{2}(0^8 \vert 1-10^6)$	$16 \cdot 24$ $16 \cdot 6$
$\mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{E}_7 \times \mathrm{U}(1)_{(1)}$	$q_{(1)} = -\frac{1}{2}(0^8 \vert 110^6)$	$16 \cdot 18$
$\mathrm{SO}(10) \times \mathrm{SU}(4) \times$ $\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{(1)}$	$q_{(1)} = \frac{1}{2}(0^8 \vert 1-10^6)$	$16 \cdot (-6)$

Table 3.4: Sum of U(1) charges for all $T^6/\mathbb{Z}_4$ models that contain U(1)'s. The U(1) generators q_n are given explicitly (see discussion in section A.4.2).

Summary

In summary we have seen that whenever anomalies appeared in this subsection they either vanish or the corresponding anomaly polynomial factorizes in the form

$$\hat{I}_{D+2} = \left(\mathrm{tr} R^2 - \sum_i c_i \mathrm{tr}(iF)^2_{G_i} \right) X_{D-2} \tag{3.3.19}$$

with X_{D-2} a model dependend $D-2$ form[9]. The coefficients c_i are unique[10] and never depend on the chiral matter content nor on quantities like e.g. the number of compactified dimensions. They only depend on the gauge group factors and are given in table 3.2.

The result is in fact quite interesting because the anomaly polynomial always seems to factorize. In fact it has been shown in [41] for zero–mode spectra of orbifolds without Wilson lines that this is no accident but a deep consequence of modular invariance of the free string theory and its orbifold compactifications. However, the coefficients c_i were determined by a conspiration of trace identities and seemingly field dependend anomaly contributions. Therefore it is still surprising that they only depend on the gauge group factors. We will return to this issue in subsection 3.4.2.

3.3.2 Localization of Anomalies on Orbifolds

Up to now we have completely ignored the local structure of the orbifold as has been described in subsection 2.9.2, where we have seen that the local spectrum depends on the position within internal space, i.e. on wether we sit on a brane or are in the bulk. We start with discussing the untwisted states, where we have seen that the space group projector acts non–trivially on them. Note that in this subsection we are again just doing field theory.

The computation as presented here is based on [46] and the more general paper [40]. Similar computations starting from six dimensional field theories have been performed in [38, 39, 91].

We remember that the anomaly in D dimensions is given by

$$G^D(\lambda) = \overline{Tr}[\lambda \tilde{\gamma}_D] \tag{3.3.20}$$

with λ collectively denoting an infinitesimal gauge or local Lorentz transformation and $\tilde{\gamma}_D$ the chirality operator in D dimensions. The trace symbolized by $\overline{Tr}$ runs over spacetime and gauge indices as well as over the full Hilbert space of D dimensional states. Analogously we now want to compute

$$G(\lambda) = \underline{Tr}[\lambda \tilde{\gamma}_D] \tag{3.3.21}$$

on the orbifold, where $\underline{Tr}$ now again runs over spacetime and gauge indices but only over the full orbifold Hilbert space. Note that this time the Hilbert space only contains those states that are invariant under the full space group. To guarantee this, it is very useful to include the complete $\mathbb{Z}_N$ space group projector

$$\mathcal{P} = \frac{1}{N} \sum_{k=0}^{N-1} \Theta^k \tag{3.3.22}$$

[9]Note, that this is even true for the ten dimensional case (3.3.4) with $c_{\mathrm{E}_8} = 1$ in agreement with table 3.2.

[10]To be precise this statement is only true up to a trivial and here irrelevant exchange of the c_i's and d_i's in six dimensions of course.

into the operator instead of working with invariant states. To regularize the expression we use Fujikawa's method [92, 93, 94] and introduce a heat kernel regulator

$$\begin{aligned} G(\lambda) &= \overline{\mathrm{Tr}}\left[\mathcal{P}\,\lambda\,\tilde{\gamma}_D\,e^{(\not{D}/M)^2}\right] \\ &= \sum_i \langle\,\psi_i\,|\,k,q\,\rangle\langle\,k,q\,|\,\mathcal{P}\,\lambda\,\tilde{\gamma}_D\,e^{(\not{D}/M)^2}\,|\,\psi_i\,\rangle \end{aligned} \tag{3.3.23}$$

and let $M \to \infty$ once the expression is well–defined. Here we again consider d uncompactified and $D-d$ compactified dimensions with coordinates x and z and momenta k and q respectively. $|\,\psi_i\,\rangle$ represents a particle running in the loop and $\langle\,k,q\,|\,\psi_i\,\rangle = \psi_i(k,q)$. In the second line we left the trace over spacetime and gauge indices implicit.

We now have to take care which particles run in the loop. By only considering zero–modes of course we are never able to get informations on localization since they are by definition constant over the whole internal manifold. So over which states do we have to sum? In the string construction we only discussed states with vanishing mass^2, but we have seen that there are two different sources for masses. Setting all KK– and winding–modes to zero, there are different integer spaced mass^2–levels in units of $4/\alpha'$. We will argue later that only those on the 0–th level (that have been discussed already) can contribute to anomalies. On the other hand there are the masses induced by the KK– and winding–modes. Their scale is governed by the compactification radii R_i. Since we are doing field theory, all winding modes have to be zero, so that we are left with the usual KK tower on top of each state on mass–level 0. We have to take them into account. As has been shown in subsections 2.5.2 and 2.5.3 their momenta q have to lie on the lattice $2\pi\Lambda^*$ with Λ the compactification lattice and are therefore discrete.

Then we can take the plane wave basis $\langle\,x,z\,|\,k,q\,\rangle = \exp(ik\cdot x + iq\cdot z)$ to expand

$$\langle\,k,q\,|\,\psi_i\,\rangle = \int d^dx\,\frac{1}{N}\int d^{D-d}z\,\langle\,k,q\,|\,x,z\,\rangle\langle\,x,z\,|\,\psi_i\,\rangle\,, \tag{3.3.24}$$

where the factor $1/N$ takes into account that the volume of the orbifold is $1/N$ compared to that of the torus (note that we are not working with orbifold invariant states). Then, the expression (3.3.23) can be rewritten as

$$\begin{aligned} G(\lambda) = {} & \frac{1}{N^2}\int d^dx\int d^{D-d}z\int d^dx'\int d^{D-d}z'\sum_i\langle\,\psi_i\,|\,x',z'\,\rangle\,\times \\ & \times\langle\,x',z'\,|\,k,q\,\rangle\,\mathcal{P}\,\lambda\,\tilde{\gamma}_D\,e^{(\not{D}/M)^2}\langle\,k,q\,|\,x,z\,\rangle\langle\,x,z\,|\,\psi_i\,\rangle\,. \end{aligned} \tag{3.3.25}$$

We now want to make use of the completeness relation

$$\sum_i\langle\,x,z\,|\,\psi_i\,\rangle\langle\,\psi_i\,|\,x',z'\,\rangle = \delta^d(x'-x)\,\delta^{D-d}(z'-z-\Lambda) \tag{3.3.26}$$

with Λ the compactification torus. However, in order to use it we have to pull the expression $\langle x, z \,|\, \psi_i \rangle$ through $\mathcal{P}$. But we know how Θ^k acts on z, i.e. like $z \to \theta^k z$. Note, that it does not act on z'. We arrive at

$$\begin{aligned} G(\lambda) &= \frac{1}{N^2} \sum_{k=0}^{N-1} \int d^d x \int d^{D-d} z \, \delta^d(x - x') \, \delta^{D-d}(z' - \theta^k z - \Lambda) \times \\ &\quad \times \underline{tr} \left[\Theta^k \, \lambda \, \tilde{\gamma}_D \, e^{(\not{D}/M)^2} \right] , \end{aligned} \tag{3.3.27}$$

where $z' \to z$ and $x' \to x$. By $\underline{tr}$ we now made the trace over spacetime and gauge indices explicit again. Note that it still runs over ten dimensional spacetime and gauge indices. We can further simplify this expression by using

$$\delta(f(z)) = \sum_i \frac{1}{|f'(z_i)|} \delta(z - z_i) , \tag{3.3.28}$$

where i now labels all positions z_i with $f(z_i) = 0$. This formula is valid only for $f(z) \neq 0$, but then can also be applied on the torus delta functions since the compactification torus is invariant under θ. We find

$$\begin{aligned} G(\lambda) &= \frac{1}{N^2} \sum_{z_i} \sum_{k=0}^{N-1} \frac{1}{|\det^\perp(\mathbb{1} - \theta^k)|} \int d^d x \int d^{D-d} z \, \delta^d(x - x') \times \\ &\quad \times \delta^{D-d}(z - z_i) \, \underline{tr} \left[\Theta^k \, \lambda \, \tilde{\gamma}_D \, e^{(\not{D}/M)^2} \right] , \end{aligned} \tag{3.3.29}$$

where by $\det^\perp$ we denote that the determinant is only to be taken over non–invariant directions (otherwise there is no restriction due to the original torus delta function). We further used $z \to z'$ and see that then the z_i are all solutions to $z = \theta^k z + \Lambda$. But those are known already, since this is nothing but the definition of fixed points/planes. Correspondingly then $|\det^\perp(\mathbb{1} - \theta^k)|$ gives nothing but the number of fixed points. We have converted the anomaly expression into a sum of anomaly expressions evaluated at the branes.

The expression (3.3.29) clearly works very well whenever there are no invariant internal directions in the k–th twisted sector. However this is not always the case. Consider $k = 0$ for definiteness. Then all internal directions are invariant and there is no restriction on z. The integrals run over the complete D dimensional space, therefore signaling a D dimensional anomaly contribution. In every invariant direction we just see a torus and anomalies do not distinguish between a toroidal compactified or non–compact direction. But then we do not have to evaluate further. We just undo all manipulations performed so far to trace the computation back to a torus anomaly. For $k = 0$ this is very easy and we find

$$G(\lambda)^{k=0}(x, z) = \frac{1}{N} \overline{Tr} \left[\lambda \, \tilde{\gamma}_D \, e^{(\not{D}/M)^2} \right] = \frac{1}{N} G^D(\lambda)(x, z), \tag{3.3.30}$$

with $\overline{Tr}$ denoting the trace over ten dimensional spacetime and gauge indices as well as the trace over the torus invariant Hilbert space. We find exactly $1/N$ times the D dimensional anomaly on a torus or equivalently the D dimensional anomaly in D uncompactified dimensions.

We have up to now only considered the case $k = 0$. However because the whole argumentation always works as long as the product structure of two–tori is guaranteed (which is the case throughout this work) we can generalize: Concerning the local anomalies every direction invariant under a given twist can be treated on the same footing as uncompactified ones. But uncompactified and invariant directions together are nothing but the brane directions, that we will denote by $d^{||}$. Therefore the local anomalies are really confined to the branes. This is one of the central results of this work.

The result can be summarized in the more intuitive form

$$\begin{aligned} G(\lambda)(x,z) &= \frac{1}{N}\int \Big\{ I^1_D + \\ &+ \sum_{z_{\rm brane}(k,l)} \frac{d^{d^{||}}z\, J^1_{d^{||}}|_{{\rm brane}(k,l)}}{|\det{}^{\perp}(\mathbb{1}-\theta^k)|}\delta^{d^{||}}(z - z_{\rm brane}(k,l)) \Big\} ,\end{aligned} \tag{3.3.31}$$

with

$$J^1_{d^{||}}|_{{\rm brane}(k,l)} \equiv \overline{tr}[\Theta^k(k,l)\,\lambda\tilde{\gamma}_D] \; . \tag{3.3.32}$$

Note that the trace $\overline{tr}$ now runs over ten dimensional spacetime and gauge indices but over the Hilbert space of $d^{||}$ dimensional states living at the (k,l)–brane and being invariant under Θ^k, i.e. the states of the (k,l)–brane spectrum (like in chapter 2 k denotes the k–th twisted sector and l labels the fixed points/planes contained in it). Since the expression is evaluated at the (k,l)–brane the adjoint representation in which iF is residing is reduced to that of the local unbroken gauge group, analogously the local Lorentz group is broken. The D dimensional chirality operator can be decomposed into a $d^{||}$ and $D - d^{||}$ dimensional one. It should be clear that by summing over the broken indices and taking orbifold phases and internal chirality operator into accout, the $J^1_{d^{||}}$'s will be proportional to the usual $I^1_{d^{||}}$'s obtained by tracing over $d^{||}$ dimensional spacetime and gauge indices. Since the phase factors are model dependent they will be evaluated in the next subsection.

Up to now we have only considerd bulk states but not twisted states. However, from the discussion of twisted states before it is clear that they only contribute to the local anomalies at the corresponding brane. They are already $d^{||}$ dimensional and will therefore be included with help of delta–functions into the anomaly expressions[11].

[11] That the twisted states are included like this is nevertheless an assumption. In a complete string computation the existence of winding modes could lead to delocalization effects for twisted states. In fact tadpole calculations recently performed in [95, 96] seem to hint in this direction.

3.3.3 Local Anomalies

We turn to the evaluation of local anomalies for the different orbifold examples considerd in this thesis.

Since the procedure is always the same we just consider the simple example of $T^4/\mathbb{Z}_2$ for explanation and then just give the general result. The chiral ten dimensional bulk matter that is subject to the orbifold projector in the untwisted sector is those of the free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory. We only consider the gaugino for now.

Using (3.3.31) we see that the untwisted anomaly can be written as

$$G_{\mathbf{Ad}_{[0]}} = \frac{1}{2}\frac{1}{2}\int I^1_{10}|_{\mathbf{Ad}_{[0]}} + \frac{1}{2}\cdot\frac{1}{16}\sum_l J^1_6|_{\mathrm{brane}(l)}\, d^6z\delta^6(z-z_l) \; . \tag{3.3.33}$$

Remember that for $k=1$ there are 16 branes (therefore the multiplicity) labeled by l, all of them six dimensional. $\frac{1}{2}I^1_{10}|_{\mathbf{Ad}_{[0]}}$ is nothing but the ten dimensional gaugino anomaly. One factor 1/2 is due to the Majorana condition again, the other factor of 1/2 reflects that the volume of the orbifold is half of that of the torus (and the anomaly of the torus is that of the free theory as we have seen).

It remains to work out J^1_6 as defined in (3.3.32). Let $\mathbf{Ad}_{[0]}$ denote the adjoint representation of $\mathrm{E}_8 \times \mathrm{E}_8$. As has been shown in subsection 2.1.3 the ten dimensional gaugino transforms as a $\mathbf{8_s}$ under the spacetime little group. In the spin basis we can write it as

$$|\,\mathbf{Ad}_{[0]}\,\rangle\,|\,\tfrac{\alpha_2\alpha_3\alpha_4}{2}, \tfrac{\alpha_2}{2}, \tfrac{\alpha_3}{2}, \tfrac{\alpha_4}{2}\,\rangle \; , \tag{3.3.34}$$

with $\alpha_i = \pm 1$. The product of minus signs is always even as required by ten dimensional left–handedness. The six dimensional chiralities are given by the product $\alpha_1\alpha_2 (= \alpha_3\alpha_4)$. Now we have to remember how $\Theta^k(k,l)$ acts on the states. From subsection 2.9.2 we get that its eigenvalues at the (k,l) brane are given by

$$\exp 2\pi i(q\cdot(kv + m^i(k,l)a_i) + \tilde{q}\phi) \; . \tag{3.3.35}$$

This equation holds in general. Let $V \equiv kv + m^i(k,l)a_i$ for simplicity. Then we know that w.r.t. the unbroken local gauge group at the brane the adjoint of $\mathrm{E}_8 \times \mathrm{E}_8$ decomposes as

$$\mathbf{Ad}_{[0]} \to \mathbf{Ad}_{[\mathbf{V}]} + \mathbf{R}^{(\mathbf{1})}_{[\mathbf{V}]} \tag{3.3.36}$$

according to table (2.3) and the concept of brane spectra. Therefore applied to the gaugino in question we see that the following eigenvalues arise

$$\begin{aligned} \Theta^k(k,l)|\,\mathbf{Ad}_{[\mathbf{V}]}\,\rangle\,|\,\tfrac{\alpha_2\alpha_3\alpha_4}{2}, \tfrac{\alpha_2}{2}, \tfrac{\alpha_3}{2}, \tfrac{\alpha_4}{2}\,\rangle &\quad\to\quad e^{2\pi i[0+\frac{1}{4}(\alpha_3-\alpha_4)]} \\ \Theta^k(k,l)|\,\mathbf{R}^{(\mathbf{1})}_{[\mathbf{V}]}\,\rangle\,|\,\tfrac{\alpha_2\alpha_3\alpha_4}{2}, \tfrac{\alpha_2}{2}, \tfrac{\alpha_3}{2}, \tfrac{\alpha_4}{2}\,\rangle &\quad\to\quad e^{2\pi i[\frac{1}{2}+\frac{1}{4}(\alpha_3-\alpha_4)]} \end{aligned} \tag{3.3.37}$$

The factor $\exp(2\pi i(1/2))$ leads to an opposite sign w.r.t both states. The remaining sum is easily computed (six dimensional chirality included)

$$\sum_{\alpha_3,\alpha_4}(\alpha_3\alpha_4)e^{2\pi i\frac{1}{4}(\alpha_3-\alpha_4)}=4\ . \tag{3.3.38}$$

It is now important to keep track of the remaining d.o.f.. The spacetime ones left over here are only $\alpha_2=\pm1$ and therefore 2. We see that here we are dealing with half–hyper multiplets and not with hyper multiplets for which the anomalies were computed in subsection 3.3.1. Therefore we have to correct by a factor 1/2. However, we have now summed up all internal quantum numbers so that the remaining trace is just that of I_6^1. In summary we see that

$$J_6^1|_{\mathrm{brane}(l)}=4\cdot\frac{1}{2}\left(I_6^1|_{\mathbf{Ad}_{[\mathbf{V(l)}]}}-I_6^1|_{\mathbf{R}^{(\mathbf{1})}_{[\mathbf{V(l)}]}}\right)\ . \tag{3.3.39}$$

Inserting this into (3.3.33) we finally get

$$G_{\mathbf{Ad}_{[\mathbf{0}]}}=\int\left\{\frac{1}{2}\frac{1}{2}I_{10}^1|_{\mathbf{Ad}_{[\mathbf{0}]}}+\frac{1}{16}\sum_l\left(I_6^1|_{\mathbf{Ad}_{[\mathbf{V(l)}]}}-I_6^1|_{\mathbf{R}^{(\mathbf{1})}_{[\mathbf{V(l)}]}}\right)d^6z\delta^6(z-z_l)\right\}\ . \tag{3.3.40}$$

Repeating the same exercise for the ten dimensional gravitino and dilatino one finds for the total untwisted anomaly

$$G_{\mathrm{untw}}=\int\left\{\frac{1}{2}I_{10}^1+\frac{1}{16}\sum_l I_6^1|_{z_l}^{\mathrm{untw}}\,d^6z\delta^6(z-z_l)\right\}\ , \tag{3.3.41}$$

and by inclusion of the six dimensional twisted states

$$G_{\mathrm{tw}}=\int\sum_l I_6^1|_{z_l}^{\mathrm{tw}}\,d^6z\delta^6(z-z_l) \tag{3.3.42}$$

finally

$$G_{\mathrm{total}}=\int\left\{\frac{1}{2}I_{10}^1+\frac{1}{16}\sum_l I_6^1|_{z_l}\,d^6z\delta^6(z-z_l)\right\}\ , \tag{3.3.43}$$

where I_{10}^1 is the anomaly corresponding to the global zero–mode anomaly polynomial of the free heterotic string in ten dimensions (3.3.1) and $I_6^1=I_6^{1\mathrm{untw}}+16I_6^{1\mathrm{tw}}$ the one corresponding to the six dimensional global zero–mode anomaly polynomial (3.3.5). By $|_{z_l}$ we imply that the polynomial has to be evaluated w.r.t. the brane spectrum present at the brane in question.

Similar considerations for the cases $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$ lead to analogous results[12]. We summarize the total local anomaly contribution by

$$G_{\text{total}} = \int \left\{ \frac{1}{N} I_{10}^1 + \sum_k \frac{1}{|\det^{\perp}(\mathbb{1} - \theta^k)|} \sum_l I_{d^{\|}}^1|_{z_{k,l}}\, d^{d^{\|}} z\, \delta^{d^{\|}}(z - z_{k,l}) \right\} , \tag{3.3.44}$$

where $d^{\|}$ is the dimensionality of the (k,l)–brane with l labeling the fixed points/planes in the k–th twisted sector. Note, that $|\det^{\perp}(\mathbb{1} - \theta^k)|$ for the toroidal orbifolds considered here corresponds to the number of fixed points w.r.t. non–invariant directions only. $I_{d^{\|}}^1|_{z_{k,l}}$ is the global zero–mode anomaly evaluated for the (k,l)–brane spectrum.

We now want to discuss some implications of this result:

- First it is interesting to see (3.3.44) in the context of the relation between brane spectra and zero–mode spectra as considered in subsection 2.9.4. By integrating out the internal dimensions we loose all but the d dimensional contributions of (3.3.44). Additionally the projections at all different d dimensional branes get those of the zero–mode one so that the number of identical (zero–mode) spectra exactly offsets the factor one over number of fixed points. We get the expected naive result $I_d^1|_{\text{zero–modes}}$ where the anomaly is evaluated for the zero–modes.

- Now consider an orbifold without Wilson lines. In this case it was often argued in the literature that since all branes are indistinguishable from each other the contributions from the untwisted sector spread homogenously over all fixed points (anomaly inflow). This is reflected in (3.3.44) since then there is no dependence on l anymore. However, the naive argumentation gets lost as soon as Wilson lines are switched on, because they render the fixed points distinguishable from each other. From (3.3.44) we see that the Wilson lines only enter via $|_{z_{k,l}}$ and so in the evaluation of anomalies at the branes, the factor $1/\#(\text{fp})$ is independent of l. This result is far from being obvious and really goes beyond what can be said from the zero–mode spectra.

- It is interesting to note that all local anomalies appearing on orbifolds with Wilson lines are readily determined by the computations of global zero–mode anomalies of orbifolds without Wilson lines (as have been computed in subsection 3.3.1 for the examples considerd in this thesis). Therefore no further work is needed.

- Although we have not explicitly checked (3.3.44) for other cases we strongly believe that it holds for any heterotic orbifold[13].

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[12] For $T^6/\mathbb{Z}_4$ some care is needed due to the appearance of the additional $T^2/\mathbb{Z}_2$ orbifold projector at some branes. For details see [48].

[13] To be precise for any heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ orbifold with gauge shift and shift embedded Wilson

3.4 The Green–Schwarz Mechanism

Although up to now we extensively computed global and local anomalies we have not yet cancelled them. So the interim conclusion would still be that all we are doing is inconsistent because local gauge and/or Lorentz symmetries are spoiled in the low energy effective field theories. Fortunately this is not the final conclusion. As has already been obsorved in [71] for the ten dimensional case there is a correction to the low energy effective action that cancels the occuring anomalies. This goes under the name Green–Schwarz mechanism. Later on it has been shown in [72] and [73] that a remnant of this Green–Schwarz mechanism is still at work in the zero–mode limit of four and six dimensional orbifold compactifications and renders them consistent. We present this mechanism in the following and then introduce a local version of it that is beautifully compatible with the local picture developed in this work.

3.4.1 Free Heterotic $E_8 \times E_8$ String

We start with the original ten dimensional Green–Schwarz mechanism as introduced in [71]. Consider the gauge part of the ten dimensional low energy effective action (3.1.2) again (dilaton background fixed). It is given by

$$S \sim \int d^{10}x\sqrt{-G}\left[-\frac{1}{2}|H_3|^2 - \frac{1}{2}\mathrm{tr}(|F_2|^2)\right] , \qquad (3.4.1)$$

where we absorbed the couplings into the fields for simplicity. Then the term $|H_3|^2$ contains a coupling of the two form tensor B_2 with two gluons as used in the left vertex of figure 3.2. The next observation from (3.1.4) is that $\delta_{\Lambda_0} B_2$ is

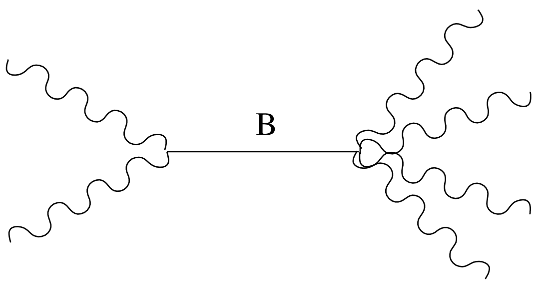


Figure 3.2: Anomalous diagram due to an exchange of the massless two form field B_2.

part of a descent chain (3.2.9), i.e.

$$\delta_{\Lambda_0} B_2 = \omega^1_{YM\,2} \ , \quad d\omega^1_{YM\,2} = \delta_{\Lambda_0}\omega_{YM\,3} \ , \quad d\omega_{YM\,3} = \mathrm{tr}(iF)^2 \ . \qquad (3.4.2)$$

lines but without antisymmetric background tensor.

This led to the postulation of an additional interaction term

$$S_{\text{GS}} = \int d^{10}x\, B_2 X_8 \tag{3.4.3}$$

with X_8 defined in (3.3.3). From the structure of X_8 we see that this interaction term produces vertices like the right one depicted in figure 3.2. Even more, since X_8 is gauge invariant the variation of S' really cancels an anomaly with anomaly polynomial $\text{tr}(iF)^2 X_8 = (1/30)\text{Tr}(iF)^2 X_8$, which is exactly what we were looking for. The so called Green–Schwarz (or Chern–Simons) interaction S_{GS} given in (3.4.3) has not been considered up to now, because it is just not part of the minimal $D = 10, \mathcal{N} = 1$ SUGRA theory. However, we see that anomaly cancellation enforces its inclusion. A further check necessary for the compatibility of S_{GS} with the minimal action considerd so far is that it is invariant under one–form gauge transformations, but this follows trivially by partial integration.

Note that we have not yet cancelled the complete ten dimensional anomaly but only the part $-\text{tr}(iF)^2 X_8$. However, from the knowledge so far it is easy how to cancel the remaining $\text{tr}R^2 X_8$ as well. We simply change the trasformation law (3.1.4) of B_2 under local Lorentz transformations, so that for a combined gauge/local Lorentz transformation

$$\delta B_2 = \omega^1_{YM\,2} - \omega^1_{L\,2} \tag{3.4.4}$$

with $\omega^1_{L\,2}$ the local Lorentz counterpart of $\omega^1_{YM\,2}$. Accordingly the three form field strength H_3 defined in (3.1.3) has to be changed according to

$$H_3 = dB_2 - \omega_{YM\,3} + \omega_{L\,3} \tag{3.4.5}$$

with $\omega_{L\,3}$ the Lorentz Chern–Simons form. Again this change in the field strength is not in contradiction to the low energy effective action without the change. The spin connection appearing in $\omega_{L\,3}$ is proportional to the derivative of the vielbein. Therefore $\omega_{L\,3}$ is higher order in derivatives w.r.t. the Yang–Mills counterpart and that is the reason why it was not present in the lowest order approximation.

With this adaptions finally all ten dimensional anomalies are cancelled and the low energy effective theory is consistent. By virtue of (3.4.4) this so called Green–Schwarz mechanism can cancel anomalies corresponding to an anomaly polynomial of the type $(\text{tr}R^2 - \text{tr}(iF)^2)X_D$ a structure we have always encountered so far. This signals the importance of factorization of anomaly polynomials for anomaly cancellation.

3.4.2 Global Green–Schwarz Mechanism on Orbifolds

Instead of the ten dimensional low energy effective action we now have to consider $d = 6$ and $d = 4$ dimensional ones. We will not discuss them in detail here, but just state that they can be obtained from the ten dimensional one by dimensional reduction and restriction to those fields that survive the orbifold projections (see e.g. [49, 16]). It turns out that then the structure is always like in ten dimensions with B_2 and its fieldstrength H_3 still induced 2– and 3–forms (the indices of the original forms are restricted to d dimensions).

Exactly like in ten dimensions we add a Green–Schwarz action

$$S_{\mathrm{GS}} = \int B_2 X_{d-2} \ , \tag{3.4.6}$$

where X_4 and X_2 are given in (3.3.9) and (3.3.17) respectively. Again its variation cancels the remaining anomalies due to their factorization as long as we change the variation of B_2 according to

$$\delta B_2 = \left(\sum_i c_i \omega^1_{YM\,2\,i} \right) - \omega^1_{L\,2} \ , \tag{3.4.7}$$

where i runs over the gauge group factors G_i and c_i are the constants given in table 3.2. Analogously the Lorentz Chern–Simons now depends on the spin connection in d dimensions. Note that this formula includes the original ten dimensional case (3.4.4).

There is one subtlety here: Since the d dimensional B_2 is a remnant of the ten dimensional one, we are not free to choose its variation in the way we want. It has to be compatible with the ten dimensional variation. It is easiest to discuss this issue by going back the descend equations for (3.4.7). Then compatibility requires that

$$\mathrm{tr}(iF)^2_{\mathrm{E}_8\times\mathrm{E}_8} \to \sum_i c_i \mathrm{tr}(iF)^2_{G_d} \ , \tag{3.4.8}$$

where G_d is the gauge group left unbroken by the orbifolding. The coefficients c_i must coincide with those obtained by anomaly polynomial factorization listed in table (3.2). However, as a consequence of (3.3.11) and the trace decomposition (A.6.30) it follows[14] that

$$c_i = \frac{Q_2(G_i)}{g(G_i)} \ , \tag{3.4.9}$$

which is exactly the same result as (3.3.10) encountered by anomaly polynomial factorization in six dimensions. It can easily be checked that it is valid for all remaining groups and in the four dimensional case as well. Therefore the variations of B_2 are compatible with the originial ten–dimensional one.

Again the final result is that all d dimensional anomalies cancel either directly or due to factorization by a Green–Schwarz mechanism.

[14]It follows directly for semi–simple groups. For a U(1) factor one has to replace $Q_2(\mathrm{U}(1))/g(\mathrm{U}(1))$ by $\frac{1}{k_q}$ with $k_q = 2q^2$ the level and drop the trace in order to fix normalization again.

3.4.3 Local Green–Schwarz Mechanism on Orbifolds

With the results obtained so far it is now finally very easy to write down a local version of the Green–Schwarz mechanism. We have seen that away from the branes the orbifolded theory looks eactly like the free heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string, contributing $(1/N)$ times (with N the order of $\mathbb{Z}_N$) the factorized ten dimensional anomaly. Accordingly in the bulk the original ten dimensional Green–Schwarz mechanism has to be at work. At a given (k,l)–brane of longitudinal dimension $d^{||}$ on the other hand we have a $d^{||}$ dimensional anomly corresponding to that of the brane spectrum, which is exactly $1/\#(\mathrm{fp})$ times the zero–mode anomaly of an orbifold model with gauge shift $kv + m^i(k,l)a_i$ and without Wilson lines. But for the models without Wilson lines we have explicitly seen that the anomaly polynomials factorize so that they can be cancelled by a $d^{||}$ dimensional Green–Schwarz mechanism. Therefore, a Green–Schwarz action that accounts for all local anomalies finally reads

$$S_{\mathrm{GS}}^{\mathrm{local}} = \int B_2|_{G^{\mathrm{local}}} X_8^{\mathrm{local}} \tag{3.4.10}$$

with

$$\begin{aligned} X_8^{\mathrm{local}} &= \frac{1}{N} X_8 + \\ &+ \sum_{k,l:d^{||}=6} \frac{1}{\#(\mathrm{fp})} X_4|_{z_{k,l}}\, \delta^4(z - z_{k,l}) d^4 z \\ &+ \sum_{k,l:d^{||}=4} \frac{1}{\#(\mathrm{fp})} X_2|_{z_{k,l}}\, \delta^6(z - z_{k,l}) d^6 z\ . \end{aligned} \tag{3.4.11}$$

The equation is splitted into bulk contributions (first line) and contributions of six and four dimensional branes (second and third line respectively). Like in (3.3.44) $\#(\mathrm{fp})$ refers to the number of fixed points in the $d^{\perp} = 10 - d^{||}$ directions transversal to the brane. The forms X_8, X_4 and X_2 refer to the equations (3.3.3), (3.3.9) and (3.3.17) respectively. They depend on the brane (bulk) spectrum and have to be evaluated correspondingly. Like before $z_{k,l}$ denotes the (k,l)–brane position.

The variation of B_2 under a combined gauge/local Lorentz transformation is given by

$$\delta B_2|_{G^{\mathrm{local}}} = \left[\sum_i c_i \omega^1_{YM\,2\,i}\right]_{G^{\mathrm{local}}} - \omega^1_{L\,2}\ , \tag{3.4.12}$$

and depends via the coefficients c_i's on the local unbroken gauge group G^{local} (including $\mathrm{E}_8 \times \mathrm{E}_8$ in the bulk). For all different possible group factors they are given in table (3.2).

We close with a discussion of the local Green–Schwarz mechanism: In contrast to the global versions discussed previously in the local version one and the same B_2 field has to cancel anomalies in different dimensions simultaneously, B_2 itself the bulk anomalies, and its different projections the brane anomalies. That this is possible is a consequence of the fact that the c_i do not possess any brane dimension dependence (which would in principle have been allowed in the global versions). Only therefore and because of factorization of the local anomaly polynomials a local implementation of the Green–Schwarz mechanism was possible.

We have checked the local mechanism for $T^4/\mathbb{Z}_2$, $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$ explicitly, but strongly believe that it works for other examples as well. In especially the non–prime $T^6/\mathbb{Z}_4$ case was a very strong crosscheck because it contains four as well as six dimensional branes. There are no branes of different dimensionality within the whole plethora of consistent $T^{4/6}/\mathbb{Z}_N$ orbifold compactifications leading to $\mathcal{N} = 1$ SUSY in six or four dimensions. On the other hand (3.4.10) is a hybrid equation, in the sense that for the B_2 part we just have to restrict to the local gauge group, i.e. the brane model, whereas for the brane contributions of X_8^{local} we have to give the brane dimensional anomalies explicit (by inclusion via delta functions). Why do we not get the lower dimensional anomalies explicitly from boundary terms via integration? In fact for $T^4/\mathbb{Z}_N$ work in this direction has been done (see e.g. [97, 98]). However, for more complicated orbifolds like $T^6/\mathbb{Z}_N$ the performance of these integrations gets notoriously difficult. Therefore we used the (local) spectra for backwards engeneering and so obtained our result. It would be very interesting to link both approaches since in fact they should coincide.

Chapter 4

Conclusions and Outlook

We close with a summary of the most important new results contained in this thesis and finally give an outlook on possible future research.

We developed a local perspective on heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ orbifold compactifications with Wilson line backgrounds in section 2.9.2. A brane world like picture was found. It was shown that the local massless spectrum at a brane is completely determined by the zero–mode spectrum of a (different) orbifold without Wilson lines. Each brane spectrum was shown to be consistent due to a level matching condition corresponding to the brane. In this consistency check we found that the heterotic level matching condition as used in the literature has to be supplemented by a stronger version for subsectors of non–prime orbifolds. As an important application of the local picture we commented on the use of (discrete) Wilson lines for the construction of phenomenologically interesting models: We argued that once the compactification radii are fixed at a non–vanishing value our universe has to be situated at a 3–brane in internal space. But at such a brane only gauge groups of orbifold models without Wilson lines are observable. Therefore, the zero–mode gauge group of models with non–vanishing Wilson lines loses its traditional meaning for phenomenology.

We then computed local anomalies for the specific examples $T^4/\mathbb{Z}_2$, $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$ in sections 3.3.2 and 3.3.3. The result of these computations (3.3.44) turned out to be in complete accordance with the local picture. Furthermore we showed in section 3.4.3 that all remaining anomalies can be cancelled by a new local version of the well–known Green–Schwarz mechanism (3.4.10). Since from the local perspective anomalies of different dimensionalities had to be cancelled at the same time, this cancellation was non–trivial and served as an important crosscheck for the picture. The two–form fields that appeared in the local Green–Schwarz actions were shown to be the local projections of the original ten dimensional two–form field of the free heterotic theory.

Putting these results back into the context of the introduction, we see that also under inclusion of Wilson lines all local anomalies are cancelled. This is due to the very high amount of symmetry, especially modular invariance, present in string theories. Therefore the use of local anomaly cancellation for the (re)introduction of anomalous "twisted states" in field theory orbifolds[1] is in agreement with heterotic string theory. Furthermore, heterotic orbifold compactifications on four dimensional orbifolds could be used as a tool to produce anomaly free six dimensional low energy models that can again serve as starting points for field theory orbifold GUTs. Unfortunately it is not possible to draw conclusions for five dimensional field theory orbifold GUTs directly. This is due to the fact that the orbifolds studied in this work are always built from a product of two–tori and that the examined anomalies only occur in even dimensions.

There are several possibilities for future research. An example for a more abstract direction would be to reconsider blow–ups of heterotic orbifold singularities in the presence of Wilson lines. It would be very interesting to examine whether the blow–up of a singularity with Wilson line can always be traced back to a blow–up of a singularity without Wilson line, as suggested by the brane spectra. Such an examination could be done along the lines of [45]. Many phenomenologically interesting research areas are opened up by possible reconsiderations of zero–mode results from a local perspective. One example of such an area is e.g. the discussion of Fayet-Iliopoulos tadpoles that in four dimensions are automatically generated by an anomalous U(1) [85, 99, 100, 89]. In fact research in this direction has already been done for field theory orbifolds [101, 102, 103, 34, 35] as well as heterotic orbifolds with Wilson lines [47]. In fact there are very interesting results from string computations [95, 96] indicating new brane physics. Another possibility is to reexamine threshold corrections from a local perspective (for a zero–mode discussion see [104]).

[1]Remember that anomaly considerations alone are never sufficient to determine twisted states uniquely.

Appendix A

Excerpts from Group Theory

Section A.1 gives the basic definitions of Lie algebras. Section A.2 is devoted to spinor representations and properties of certain Clifford algebras. In the following section the Cartan–Weyl basis, roots, weights and Cartan diagrams are introduced. They are used heavily in the main text. Up to this point we adopt the conventions of [16]. Then we list some properties of simply–laced groups. This list is essential in the sense that we fix the ordering of roots within a Dynkin diagram and therfore what we take as representation and what as conjugated representation. These conventions are used in the main text. Section A.4 defines what we understand as Cartan symmetry breaking and exploits some simple properties. There are some explanations about the computation of U(1) charges in this context, since we are not aware of a nice account somewhere else in the literature. Some examples of Cartan symmetry breaking are given. They are used in the main text. Section A.4.4 is a very special list of branching rules needed for the relation between six and four dimensional brane spectra within one and the same $T^6/\mathbb{Z}_4$ model. It is included for completeness. Finally section A.6 deals with some definitions and general results needed for anomaly polynomials. All trace identities used in the main text are listed here. The results and trace identities are taken from [43, 84].

A lot of results of this appendix are given without derivations. For introductury reading concerning group theory we refer the reader to [105, 106], for deeper information about topological invariants to [107, 81] and references therein.

A.1 Basic definitions

Consider a Lie algebra g of a Lie group G generated by a set of generators T^A with $A = 1, \ldots, \dim(\mathsf{g})$. Then we take the mapping from the algebra to the group to be $\exp(i\omega_A T^A)$, with ω_A real parameters. The generators fulfill

$$[T^A, T^B] = i f^{AB}{}_C T^C \ , \tag{A.1.1}$$

where $f^{AB}{}_C$ are the structure constants of g.

All Lie groups we encounter in this work will be products of compact simply–laced simple Lie groups times U(1)'s. Therefore we will from now on take G to be compact, simply–laced and simple.

A matrix representation r of g is given by a set of matrices $t^A_{r,ij}$ satisfying (A.1.1) with the same structure constants as those of g. (For the adjoint representation $t^A = T^A$.) The Killing metric η^{AB}, which can be used to raise and lower indices, is defined by

$$\eta^{AB} = \frac{1}{2g}\mathrm{tr}(T^A T^B) \ , \tag{A.1.2}$$

where g is called dual Coxeter number, which will be defined later.

A.2 Clifford Algebras and Spinors

Let Γ^μ be a set of (Dirac) matrices. Then

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \tag{A.2.1}$$

is called a Clifford algebra. There are only two cases relevant for this work. Either g is the d–dimensional Minkowski metric η or just the Euclidean metric δ. We will treat the Minkowskian case first and comment on the Euclidean case later.

Let $d = 2k + 2$ be even and define

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1) \ , \tag{A.2.2}$$

$$\Gamma^{I\pm} = \frac{1}{2}(\Gamma^{2I} \pm i\Gamma^{2I+1}) \ , \tag{A.2.3}$$

where I runs from 1 to k. Then, the only nonvanishing anticommutator of these matrices is given by

$$\{\Gamma^{I+}, \Gamma^{J-}\} = \delta^{IJ} \ , \tag{A.2.4}$$

allowing an interpretation of the $\Gamma^{I\pm}$ as creation and annihilation operators, respectively. The $\Gamma^{I\pm}$ are nilpotent of order 2. Let $\chi_{\mathbf{0}}$ be a lowest weight state, i.e. a state annihilated by all annihilation operators. A general state $\chi_{\mathbf{s}}$ can then be obtained by either acting with a creation operator once (due to the nilpotency) in a certain dimension, or not. Let

$$\chi_{\mathbf{s}} \equiv \left(\prod_{I=0}^{k}(\Gamma^{I+})^{s_I+1/2}\right)\chi_{\mathbf{0}} \tag{A.2.5}$$

be labeled by $\mathbf{s} = (s_0, s_1, \dots, s_k)$ with $s_I = \pm 1/2$. This is called spin basis. Clearly $\chi_{\mathbf{0}}$ corresponds to the $\chi_{\mathbf{s}}$ with all $s_I = -1/2$. In total there are 2^{k+1}

possibilities of choosing signs in $\mathbf{s}$ and therefore states. Consider $d = 2$. In the basis (A.2.5) the Dirac matrices read

$$\Gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad \Gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{A.2.6}$$

The Dirac matrices in $d = 2k + 2$ can then be stated recursively via

$$\Gamma^{\hat{\mu}} = \gamma^{\hat{\mu}} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} , \; \Gamma^{d-2} = \mathbb{1}_{2^k} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \; \Gamma^{d-1} = \mathbb{1}_{2^k} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \tag{A.2.7}$$

with $\hat{\mu}$ running from 0 to $d-3$, $\gamma^{\hat{\mu}}$ the Dirac matrices in $d-2$ dimensions and $\mathbb{1}_{2^k}$ the identity matrix in 2^k dimensions. The 2×2 blocks in (A.2.7) are understood to act on the 'new' spin entry s_k.

Let

$$\Sigma^{\mu\nu} \equiv -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] . \tag{A.2.8}$$

The $\Sigma^{\mu\nu}$ generate and satisfy the Lorentz algebra of $\mathrm{SO(d-1,1)}$. They represent the T^A's of A.1. The generators of the subset $\Sigma^{2I,2I+1}$ pairwise commute with each other and can therefore be diagonalized simultaneously. By defining the spin operators

$$S_I \equiv i^{\delta_{I,0}} \Sigma^{2I,2I+1} = \Gamma^{I+}\Gamma^{I-} - 1/2 \tag{A.2.9}$$

the states (A.2.5) are eigenvectors of the spin operators (A.2.9) with eigenvalues s_I. Since the eigenvalues are half–integer valued, the states (A.2.5) form a spinor representation, the Dirac representation.

Let Γ denote the chirality operator

$$\Gamma \equiv i^{-k}\Gamma^0\Gamma^1 \dots \Gamma^{d-1} = 2^{k+1} S_0 S_1 \dots S_k , \tag{A.2.10}$$

with the last equality valid in the spin basis. Then

$$(\Gamma)^2 = \mathbb{1} , \quad \{\Gamma, \Gamma^\mu\} = 0 , \quad [\Gamma, \Sigma^{\mu\nu}] = 0 . \tag{A.2.11}$$

The eigenvalues of the chirality operator are clearly given by ± 1, which is nothing but the product of signs of spins in the spin basis. Because the chirality operator commutes with the Lorentz generators, states with opposite chiralities will never mix. Therefore, the 2^{k+1} dimensional Dirac representation, as a reducible representation of the Lorentz algebra, reduces to two independent 2^k dimensional Weyl representations, one with chirality $+1$, and one with -1.

For odd $d = 2k + 3$, one can just take $\Gamma^\mu = \Gamma^{\hat{\mu}}$ ($\hat{\mu}$ running from 0 to $2k + 2$) as before, and add $\Gamma^d = \Gamma$ to the set of Dirac matrices. Since then $\Sigma^{\mu d}$ does not commute with Γ^d anymore, the (still 2^{k+1} dimensional) Dirac representation is already irreducible.

A.3 The Cartan–Weyl Basis

In the last section we worked out some representations of $\mathrm{SO(d-1,1)}$. This procedure can be generalized and leads to the Cartan–Weyl basis of generators. Split the generators T^A in a maximal set of commuting generators H^I with $I = 1, \ldots, \mathrm{rank}(g)$ and other generators E^α. The H^I span the so–called Cartan subalgebra of g. The E^α can be chosen such that

$$\begin{array}{rcll} [H^I, H^J] & = & 0 & \\ [H^I, E^\alpha] & = & \alpha^I E^\alpha & \\ [E^\alpha, E^\beta] & = & \pm E^{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ & = & \frac{2}{|\alpha|^2}\,\alpha\cdot H & \text{if } \alpha = -\beta \\ & = & 0 & \text{otherwise} \end{array} \tag{A.3.1}$$

where the α^I's are called roots (there is a one–to–one correspondence between a root α and E^α). The contractions within $\alpha \cdot H$ and $|\alpha|^2 = \alpha \cdot \alpha$ are take to be with respect to the euclidean metric. In fact we will normalize the generators in such a way, that $|\alpha|^2 = 2$, which is always possible for the simply–laced groups we are dealing with. In addition, for this kind of algebras it can be shown that the dual Coxeter number g introduced earlier is given by

$$g = \frac{\dim \mathsf{g}}{\mathrm{rank}\mathsf{g}} - 1 \ . \tag{A.3.2}$$

By definition the matrices t_r^I representing the H^I can be diagonalized simultaneously. Their eigenvalues ω^I can be combined into rank$g$ dimensional vectors again. They are called weights. Clearly, the roots are the weights of the adjoint representation. If we compare this with the last section, we see, that the S_I correspond to this t^I's, that k was the rank, and that the vectors $\mathbf{s}$ are nothing but the weights of certain representations, Dirac (all signs allowed), positive chirality spinor (product of signs positive) and negative chirality spinor (product of signs negative) representations of the algebra of $\mathrm{SO(d-1,1)}$.

For the weights (including roots) a half–ordering can be introduced by defining that a weight is positive, if and only if its first non–vanishing entry is positive. Then a weight λ is bigger as a weight ω, $\lambda > \omega$ exactly if $\lambda - \omega$ is positive.

The roots can be expanded in a basis of so–called simple roots α_i, with $i = 1, \ldots, \mathrm{rank}\mathsf{g}$. For the algebras considered in this work the simple roots can be and are defined in the following way: Take the smallest positive root to the set of simple roots and go through the roots left over in ascending order. Add every root to the set of simple roots, which is linearly independent to those already contained in the set of simple roots. From the set of simple roots the so–called Cartan matrix is defined via

$$C_{ij} \equiv \alpha_i^I \alpha_j^I \ . \tag{A.3.3}$$

The diagonal entries are clearly 2. For the class of algebras considered here, the off–diagonal entries are either 0 (roots are said to be not connected) or -1 (roots are said to be connected). The Cartan matrix can be visualized by so–called Dynkin diagrams: Represent every root by a filled dot and connect two dots by a line, if and only if the corresponding roots are connected (scalar product is -1). The correspondence between Cartan matrix (or equivalently the Dynkin diagram) and a Lie algebra is unique.

The weights can be expanded in a basis of so–called fundamental weights λ^i. The coefficients in this expansion are called Dynkin–labels. The fundamental weights can be computed from the inverse Cartan matrix via

$$\lambda^i \equiv (C^{-1})^{ij}\alpha_j \ . \tag{A.3.4}$$

They therefore span a dual basis to the simple roots ($\lambda^i\alpha_j = \delta^i_j$). Using this, the Dynkin–labels d_i of a given weight ω are given by $d_i = \omega \cdot \alpha_i$. It can be shown, that they are always integers.

The set of weights of an irreducible representation contain exactly one weight which is bigger than all the others. It is called highest weight. Then all weights can be obtained from the highest weight by subtracting roots in an appropriate, well–defined way, leading to weight–diagrams. Therefore an irreducible representation is in one–to–one correspondence to a highest weight.

The root lattice $\Gamma_{\mathbf{g}}$ of a Lie algebra is defined to be the integer span of all roots (or equivalently all simple roots). Let ω be a weight of some representation (it can but must not be the highest weight). Then the ω-weight lattice is given by $\omega + \Gamma_{\mathbf{g}}$, so just the root lattice offset by one weight. Then the weight lattice is defined to be the union of all ω-weight lattices. Clearly the adjoint-weight lattice is nothing but the root lattice.

A.3.1 Properties of Simply–laced Groups

Let r denote the rank of the algebra. In addition to the $\mathrm{E_r}$ groups the only other simply–laced Lie groups are $\mathrm{SU(r+1)}$ and $\mathrm{SO(2r)}$, which are sometimes called $\mathrm{A_r}$ and $\mathrm{D_r}$ respectively. Therefore the simply–laced groups are denoted as ADE groups as well. Concerning the notation in the following: The dimension and dual Coxeter numbers are given in terms of the rank of the algebra in question. The dimension of the algebra is always the dimension of its adjoint representation. The numbers at the dots of the Dynkin diagrams label the simple roots 1 for α_1, etc.. We use these labels as presented here to define the standard ordering of simple roots used throughout this work. There are ambiguities in the ordering whenever a Dynkin diagram has an outer symmetry, e.g. for $\mathrm{A_3}$ there are two roots which are only connected to one other. We fix these ambiguities by supporting sufficient simple root labels by explicit ordering prescriptions, e.g. for $\mathrm{A_3}$ again $1(<3)$, i.e. $\alpha_1 < \alpha_3$. Concerning the roots underlined entries are understood to be permuted. The selected representations are given in terms of the Dynkin labels of its highest weight. They are as usual written by their dimension in boldface. We only list those representations appearing in this work.

- $\mathrm{A}_{\mathrm{r}\geq 2} \cong \mathrm{SU(r+1)}$

Dimension: $\dim \mathsf{g} = r^2 + 2r$
Dual Coxeter: $g = r + 1$

Dynkin diagram:

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

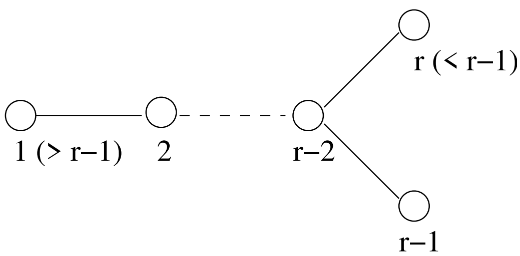
Roots: $(\underline{+1, -1, 0^{r-1}})$

Important reps: $[0^{i-1}, 1, 0^{r-i}] \cong \mathbf{R}$ or $\bar{\mathbf{R}}$ with $R = \binom{r+1}{i}$
(We take the reps with $i \leq (r+1)/2$ as $\mathbf{R}$)

- $\mathrm{D}_{\mathrm{r}\geq 4} \cong \mathrm{SO(2r)}$

Dimension: $\dim \mathsf{g} = 2r^2 - r$
Dual Coxeter: $g = 2(r-1)$

Dynkin diagram:



Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}$$

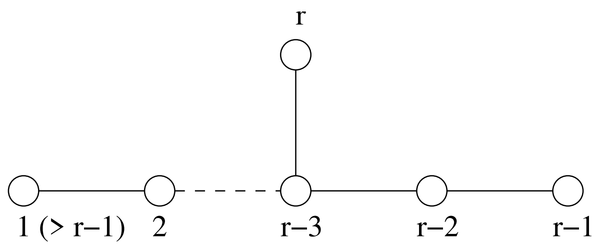
Roots: $(\underline{\pm 1, \pm 1, 0^{r-2}})$

Important reps: $[1, 0^{r-1}] \cong \mathbf{R_v}, R = 2r$ (fundamental/vector)
$[0^{r-2}, 1, 0] \cong \mathbf{R_s}, R = 2^{r-1}$ (pos. chirality spinor)
$[0^{r-2}, 0, 1] \cong \mathbf{R_c}, R = 2^{r-1}$ (neg. chirality spinor)

- $\mathrm{E}_{\mathrm{r}=8/7/6}$

Dimension: $\dim \mathfrak{g} = 248/133/78$
Dual Coxeter: $g = 30/18/12$

Dynkin diagram:



Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 0 & 2 \end{pmatrix}$$

Roots for E_8: $(\underline{\pm 1, \pm 1, 0^6})$, $\quad (\underbrace{\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}}_{\#(-)\ \text{even}})$

Some reps for E_8: $[1, 0^7] \cong \mathbf{248}$ (fundamental and adjoint)
Some reps for E_7: $[1, 0^6] \cong \mathbf{56}$ (fundamental)
$[0^5, 1, 0] \cong \mathbf{133}$ (adjoint)
Some reps for E_6: $[1, 0^5] \cong \mathbf{27}$ (fundamental)
$[0^4, 1, 0] \cong \overline{\mathbf{27}}$ (antifundamental)
$[0^4, 0, 1] \cong \mathbf{78}$ (adjoint)

A.3.2 Classification of Shift Vectors

We have seen in subsection 2.4.2 that two shift embedded gauge transformations V and V' are equivalent if either their generalized[1] shift vectors are related via

[1] V might contain Wilson line as well as gauge shift contributions.

an E_8 root lattice shift (2.4.11) or via a (chain of) Weyl reflections (2.4.12). This will be the basis for our classification.

The E_8 roots are given as roots and weights of the positive chiral spinor representation of SO(16). Since a gauge shift V of a $\mathbb{Z}_N$ orbifold has to fulfill $NV \cdot \alpha_i = 0 \mod 1$ for all roots α_i, it follows that NV is an element of the E_8 root lattice, as this lattice is self–dual. Since all E_8 roots have length–squared 2, it follows that for a Weyl reflection $V'^2 = V^2$.

A useful application are the Weyl reflections at the $SO(16)$ roots

$$(V_1,\ V_2,\ V_3, \ldots) \cong r_{(\ 1, \pm 1,\ 0^6)}(V_1,\ V_2,\ V_3, \ldots) = (\mp V_2, \mp V_1,\ V_3, \ldots). \quad \text{(A.3.5)}$$

Hence we see that by interchanging two shift elements, or replacing two shift elements by minus those elements, equivalent shifts are obtained. In particular, if a shift has at least one zero, the sign of all other entries is irrelevant.

A.3.3 $\mathbb{Z}_2$ Gauge Shifts

First of all, if V is an element of the E_8 root lattice, the gauge group will not be broken, and hence be equal to E_8. Since the E_8 lattice is even, such a lattice vector fulfills $2V^2 = 0 \mod 4$. Assume that V is not an E_8 root lattice vector. To classify these $\mathbb{Z}_2$ gauge shifts we note that for all E_8 roots w we have $(V + w)^2 = V^2 \mod 1$, since $(w, V) = 0, 1/2 \mod 1$ and $w^2 = 2$. From this it follows that for equivalent $\mathbb{Z}_2$ gauge shifts $V^2 \mod 1$ is equal, since the sum of squared entries is always invariant under Weyl reflections. This completes the description of the $\mathbb{Z}_2$ gauge shift classification, the three possibilities are given in table A.1. A standard shift can be defined by a shift with maximal number of zeros and all entries positive, which is the form we used in the table.

Number	$2V^2$	Shift $2V$
0	0	(00000000)
1	1, 3	(11000000)
2	2	(20000000)

Table A.1: Shift classification for $\mathbb{Z}_2$: general shifts are completely determined by the number $2V^2 \mod 4$.

A.3.4 $\mathbb{Z}_3$ Gauge Shifts

First, we bring the entries of $3V$ to a standard form: If $3V$ has a half-integer entry, all its entries are half integer, therefore by adding any spinorial root to V all entries of $3V$ become integer. Since $(2, 0^7)$ and its permutations are the sums

Number	$(1/2)(3V)^2$	Shift $3V$
0	0	(00000000)
1	1_2	(11000000)
2	2	(20000000)
3	3	(21100000)
4	1_0	(21111000)

Table A.2: Shift classification for $\mathbb{Z}_3$: general shifts are nearly uniquely determined by the number $(1/2)(3V)^2 \mod 3$. For a value of 1 additionally the sum of the entries of $3V \mod 6$ is needed. It appears as a subscript in the table.

of two roots of SO(16), we infer that the integer valued entries of $3V$ can be restricted to $3V^I = -2, \ldots, 3$. In fact, we may even assume that no entry $3V^I$ is equal 3: If there are two or more entries equal to 3, then by adding the SO(16) root with -1 at two of these entries, they become zero. If there is just one entry equal to 3, there is at least one other entry of $3V^I$ equal to ± 1, otherwise the sum of entries of $3V$ is not even, i.e. $3V$ is not in the E_8 root lattice. Again, by adding an appropriate SO(16) root we can make the 3 entry 0, and turn the ± 1 entry into ± 2. (We have assumed that this procedure has been applied throughout the paper to set all entries $3V^I \in \{-2, -1, 0, 1, 2\}$.)

Once in this standard form, we can consider the sum of the square of the entries of the shift vector. This classifies the shifts almost uniquely, expect when $(3V)^2/2 \mod 3 = 1$. In this case there are two possibilities, but the sum of the entries of $3V \mod 6$ can be used to distinguish between both possibilities. The results are summarized in table A.2.

A.3.5 $\mathbb{Z}_4$ Gauge Shifts

For the classification of $\mathbb{Z}_4$ gauge shifts, we first need to bring them to a standard form: The entries of $4V$ can either all be half-integer, or all be integer. However, if they are half-integer, by addition of a spinorial E_8 root one can always reach integer values. Therefore, we restrict on integer values only. Since $(2, 0^7)$ and its permutations are the sums of two roots of SO(16), we infer that the integer valued entries of $4V$ can be restricted to $4V^I = -3, \ldots, 4$. In fact, we may even assume that either $4V = (4, 0^7)$ or that no entry $4V^I$ is equal to 4: If there are two or more entries equal to 4, then by adding the SO(16) root with -1 at two of these entries, they become zero. If there is just one entry equal to 4, then either $4V = (4, 0^7)$ or there is another entry of $4V^I = -3, \ldots, 3$, for which, by adding a SO(16) root again, we can make the 4 entry 0, and map $4V^I$ back into $4V^I = -3, \ldots, 3$. (We have assumed, that if the shift contains more than one non–vanishing entries, this procedure has been applied throughout the paper to

Number	$8V^2$	Shift $4V$
0	0	(00000000)
1	1	(11000000)
2	2	(20000000)
3	3	(21100000)
4	4_0	(22000000)
5	5	(31000000)
6	6	(22200000)
7	7	(31111100)
8	8	(40000000)
9	4_1	(1111111-1)

Table A.3: Shift classification for $\mathbb{Z}_4$: general shifts are nearly uniquely determined by the number $8V^2 \mod 4$. For a value of 4 additionally the sum of the entries of $2V \mod 2$ is needed. It appears as a subscript in the table.

set all entries $4V^I \in \{-3,-2,-1,0,1,2,3\}$.)

From now on, we would like to exploit the following conditions: 1) Since $4V$ is in the E_8 root lattice, the number of odd entries of $4V$ is always even. 2) If there is at least one entry $4V^J = 0$ it follows from (A.3.5), that signs do not matter anymore; we take them positive. 3) A pair of entries ± 3 can always be mapped to a pair ± 1 by addition of an SO(16) root, so that we can restrict to at most one entry of 3. 4) If there is a entry of 2 in a gauge shift $4V$ with at least one zero, by adding a SO(16) root with -1 at this entry, the 2 gets mapped to -2 which is with the help of 2) equivalent to 2 again, so that all other non–vanishing entries of $4V$ are equivalent modulo 4. The standard form of a gauge shift is defined to be that form with the least possible numbers of 3's and 2's in $4V$. And we require that all entries are positive. This is always possible except when all entries of $4V$ are ± 1. If the number of minus signs is even, they can be made all positive, while in the odd case, one has to keep one -1. The number of 3's and 2's can often be reduced by subtracting a spinorial root, for example we have $(3,1^7)/4 \cong (1^7,-1)/4$.

Once in this standard form, we can again consider the sum of the square of the entries of the shift vector. This classifies the shifts almost uniquely, expect when $(4V)^2 = 8$. In this case there are two possibilities, but the sum of the entries of $2V \mod 2$ can be used to distinguish between both possibilities. The results are summarized in table A.3.

A.4 Cartan Symmetry Breaking

In this section we work out some branching rules for important examples appearing in the main text. We use the method of Cartan Symmetry Breaking, which is naturally at work in orbifold models, i.e. the breaking is governed by some generalized shift vector.

A.4.1 Generalities

Consider the breaking of a group G to some other group $H \subset G$. by one[2] generalized shift vector V. The breaking is realized by a generalized gauge transformation

$$(V) = \exp(2\pi i V^I H_I) \ , \tag{A.4.1}$$

where H_I denotes the Cartan generators of G, I runs from 1 to the rank of G. Since therefore everything works in complete analogy to the discussion around (2.4.7), we will not rediscuss everything. However, from (2.4.9) we clearly see, that Cartan symmetry breaking never reduces the rank and therefore rank$H =$ rank$G$.

To work out the unbroken gauge group we start with the adjoint and therefore with the roots q of G, that are given in subsection A.3.1. Exactly like in the orbifold case the action on the q's is given by $\mathcal{P} \equiv \exp(2\pi i q \cdot V)$. Then the states split with respect to their value under $\mathcal{P}$. Those with $\mathcal{P} = 1$, which is equivalent to

$$q \cdot V \in \mathbb{Z} \ . \tag{A.4.2}$$

give rise to the adjoint representation of the unbroken subgroup H and therefore determine H completely. Within the other groups of states representations w.r.t. H can be worked out in the usual way.

There is a very general relation between representations stemming from an value $\mathcal{P}$ and those from $\mathcal{P}^{-1}$, i.e. they are always conjugate to each other. Assume the eigenvalue of $\mathcal{P}$ is given by $\exp(2\pi i c)$ for some c. This corresponds to the projection

$$q \cdot V = c \mod 1 \tag{A.4.3}$$

analogous to (A.4.2). Then the states from $\mathcal{P}^{-1}$ are subject to the projection

$$q \cdot V = -c \mod 1 \ . \tag{A.4.4}$$

But from this it is clear, that if a set of weights λ fullfill the first projection, exactly the set of weights $-\lambda$ fullfill the second. Now, if λ is the highest weight

[2] There is the possibility of more than one such vectors. This happens for example when Wilson lines are turned on. However, then the breaking can be realized by succesively breaking the group with one shift vector, further and further. Therefore we will just describe one of these steps.

of some representation $\mathbf{R}$, the highest weight λ^* of its conjugate representation $\bar{\mathbf{R}}$ is given by

$$\lambda^* = -(r_{(0)}\lambda) \;, \tag{A.4.5}$$

where $r_{(0)}$ is the longest element of the Weyl group, which is generated by the Weyl reflections at simple roots defined in (2.4.12). But because we learned that we can essentially neglect Weyl reflections, only the minus sign survives, which finishes the proof.

In the twisted sectors of heterotic orbifolds there sometimes occur highest weights of the form $|\, q - V \,\rangle$ with $q \in \Gamma$ and Γ even (implying integrality) and self–dual. The Dynkin labels are computed from products of $|\, q - V \,\rangle$ with simple roots α_i of the unbroken group, which before the breaking were and still are elements of Γ. Then from integrality of Γ the product $q \cdot \alpha_i$ can be neglected. With the same reasoning as above the representation corresponding to the highest weight $|\, q - V \,\rangle$ is always complex conjugate to the representation corresponding to the highest weight $|\, \hat{q} + V \,\rangle$. For the twisted sectors of orbifolds that implies, that the $(N-k)$–th twisted sector is always complex conjugate to the k–th twisted sector.

A.4.2 Abelian Groupfactors

We have seen that the unbroken subgroup is determined by finding a set of simple roots from the roots that survive projection (A.4.2). Let s denote the number of them. If r denotes the rank of G, then sometimes one finds $s < r$, equivalently the rank of the Cartan matrix of the unbroken subgroup is smaller, than that before the breaking. However, in the beginning of subsection A.4.1 we have seen, that Cartan symmetry breaking never reduces the rank. The reason for this mismatch is the presence of Abelian groupfactors, i.e. the missing part of the unbroken group is just not semi–simple.

Assume that $G \to H \times \mathrm{U}(1)^t$, where H is semi–simple with rank s and a set of simple roots α_i with $i = 1, \ldots, s$. Further $t = r - s$. From the structure of the breaking we see, that the U(1) generators must be neutral under the semi–simple part of the unbroken group H, because otherwise H would have been enhanced to some bigger semi–simple group containing H and so eating the U(1). For the same reason, all t U(1) generators have to be orthoganal to each other. Therefore, finding a set of U(1) generators corresponds to finding an orthogonal basis for the eigenspace with eigenvalue 0 of the unbroken Cartan matrix. Note that for $t > 1$ this is not unique, even ignoring scalings.

The next question is how U(1) charges are determined. Let $q_n^I H^I$ be the n–th U(1) generator, $n = 1, \ldots, t$. The key ingredient for finding the charges is the fact, that weights (including the roots themselves again) are additive. Let ω be the weight, whose U(1) charges we would like to determine, let λ^i denote the fundamental weights (w.r.t. to the unbroken simple roots after the breaking) and $d_i = \omega\alpha_i$ be the Dynkin labels of ω. Since the unbroken Cartan matrix has a non–trivial 0 eigenspace, it is clearly not invertible and therefore in general $\omega \neq d_i\lambda^i$. Using the additivity of the weights then the U(1) charges Q_n are implicitly determined, i.e.

$$\omega = d_i\lambda^i + \sum_n Q_n q_n \ . \tag{A.4.6}$$

Using $\lambda^i = (C^{-1})^{ij}\alpha_j$ we see, that the first term of the RHS is clearly orthogonal to all q_n by construction of the U(1) generators. On the other hand all q_n are mutual orthogonal as well. Therefore the m–th U(1) charge Q_m is simple given by

$$Q_m = \frac{\omega \cdot q_m}{q_m \cdot q_m} \ , \tag{A.4.7}$$

where the dots refer to contraction over Cartan indices $I, J, \ldots$ and no sum over m is taken.

Even if the directions of the U(1) generators have been fixed within the 0 eigenspace of the unbroken Cartan matrix, the normalizations have been not. Clearly, (A.4.7) is compatible with the simultaneous rescaling

$$q_m \to c\, q_m \ , \quad Q_m \to (1/c)\, Q_m \tag{A.4.8}$$

for some real constant c. Such a combined rescaling is always possible for U(1)'s. Therefore one has to either state the generators for given breakings explicitly, or write U(1) dependecies in such a way, that the rescalings drops out. We will do both in the main text.

We close this subsection with some comments on a special choice of U(1) generators. Assume there is a one step breaking, i.e. there is just one generalized shift vector V. Then there is only one constraint on the roots and therefore maximally one Abelian subgroup can occur, if at all. Assume this is the case and denote the U(1) generator by q_1. One might be tempted to choose $q_1 = V$. From (A.4.2) we see, that $V \cdot q = 0 \mod 1$. On the other hand the U(1) generator requires $q_1 \cdot q = 0$ and is not defined modulo one. Therefore, $q_1 = V$ might be a valid choice (and in fact for most cases it is). However, sometimes it is not, and therefore this identification should be treated with care.

A.4.3 Breaking of SO(8) for $\phi = (1/2, -1/2)$

To work out the unbroken group[3], we take the roots of SO(8), that are given in subsection A.3.1. The generalized shift vector is now given by $V = \phi$. Then the following roots survive the projection (A.4.2)

$$(\pm 1, \pm 1, 0, 0) \ , \quad (0, 0, \pm 1, \pm 1) \ . \tag{A.4.9}$$

[3]That is, the group unbroken after the breaking.

Clearly both groups of roots give rise to identical unbroken groups. To find a set of simple roots we proceed like explained in the paragraph before equation (A.3.3), giving (labels chosen for later convenience)

$$\alpha_1 = (1,1,0,0)\ , \quad \alpha_2 = (1,-1,0,0)\ , \quad \alpha_R = (0,0,1,1)\ , \quad \alpha_H = (0,0,1,-1)\ . \tag{A.4.10}$$

The mutual product of all simple roots is zero, so that the Cartan matrix is diagonal and therefore a prouct of rank 1 algebras. These algebras are SU(()2), which can be read off from subsection A.3.1 again. Therefore we face the following breaking:

$$\mathrm{SO}(8) \to \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_H\ . \tag{A.4.11}$$

Next, we identify some representations: Consider a highest weight $(1,0,0,0)$. By multiplying with all simple roots we find the Dynkin labels $[1],[1],[0],[0]$ w.r.t. the unbroken group (A.4.11), which with help of subsection A.3.1 can be identified as a representation $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$. In complete analogy the following correspondeces are found:

$$\begin{aligned}
(1,0,0,0) &\cong (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})\ ,\\
(0,0,1,0) &\cong (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\ ,\\
(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) &\cong (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})\ ,\\
(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}) &\cong (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\ .\\
(0,0,\tfrac{1}{2},\tfrac{1}{2}) &\cong (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})\ ,\\
(\tfrac{1}{2},-\tfrac{1}{2},0,0) &\cong (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})\ .
\end{aligned}$$

A.4.4 Breaking of SO(8) for $\phi = (1/3, 1/3, -2/3)$

For this case the following roots survive

$$(\pm 1,0,0,0)\ , \quad (0,\underline{1,-1},0) \quad (0,\pm(\underline{1,0},-1))\ . \tag{A.4.12}$$

Again we find the simple roots as explained in the paragraph before equation (A.3.3), giving

$$\alpha_1 = (1,0,0,0)\ , \quad \alpha_1' = (0,0,1,-1)\ , \quad \alpha_2' = (0,1,-1,0)\ , \tag{A.4.13}$$

where we first ordered the roots in such a way, that the descending Cartan matrix is block diagonal and then according to the ordering descriptions given in A.3.1. The semi–simple part of the resulting unbroken group is given by $\mathrm{SU}(2) \times \mathrm{SU}(3)_H$, where the SU(3) is the "would–be" holonomy group again. However, the important thing here is that the rank after the breaking is not equal to that of SO(8), namely 4. As shown in subsection A.4.2 this indicates an unbroken U(1). This U(1) will host the R–symmetry. Up to the pure existence we will ignore this $\mathrm{U}(1)_R$.

Summarizing, we face the following breaking:

$$\mathrm{SO}(8) \to \mathrm{SU}(2) \times \mathrm{SU}(3)_H \times \mathrm{U}(1)_R \, . \tag{A.4.14}$$

A.4.5 Breaking of E_8 for $v = (1/3, 1/3, 0^6)$

The E_8 roots are given in A.3.1. The following roots survive the projection

$$(\pm(1,-1), 0^6) \, , \quad (0, 0, \underline{\pm 1, \pm 1, 0^4}) \, , \quad \underbrace{(\pm(\tfrac{1}{2}, -\tfrac{1}{2}), (\pm\tfrac{1}{2})^6)}_{\#(-) \text{ even}} \, . \tag{A.4.15}$$

Again we search for simple roots as explained in the paragraph before equation (A.3.3), giving

$$\begin{aligned}
\alpha_1 &= (0,0,1,-1,0,0,0,0) \, , && \text{(A.4.16)}\\
\alpha_2 &= (0,0,0,1,-1,0,0,0) \, , && \text{(A.4.17)}\\
\alpha_3 &= (0,0,0,0,1,-1,0,0) \, , && \text{(A.4.18)}\\
\alpha_4 &= (0,0,0,0,0,1,-1,0) \, , && \text{(A.4.19)}\\
\alpha_5 &= (0,0,0,0,0,0,1,-1) \, , && \text{(A.4.20)}\\
\alpha_6 &= (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}) \, , && \text{(A.4.21)}\\
\alpha_7 &= (0,0,0,0,0,0,1,1) \, , && \text{(A.4.22)}\\
& && \text{(A.4.23)}
\end{aligned}$$

and therefore correspond to E_7. We already ordered the roots according to the descriptions given in A.3.1. Again, there is a mismatch of one in the ranks before and after the breaking, so the full branching is given by

$$\mathrm{E}_8 \to \mathrm{E}_7 \times \mathrm{U}(1) \, . \tag{A.4.24}$$

This time we work out some U(1) charges in detail. The generator q_1 of the U(1) must be orthogonal to the simple roots. Here, the gauge shift v gives us a correct choice, i.e. $q_1 = c\,(1,1,0,0,0,0,0,0)$ for some real constant c. Clearly by construction the adjoint of E_7 is neutral under the U(1) (the weights are the unbroken roots). E.g. for the projection $p \cdot v = 2/3 \mod 1$ one finds two highest weights. Firstly $(1,1,0^6)$, which is a singlet under E_7 since it is collinear with q_1. Applying (A.4.7) the U(1) charge turns out to be $(2c)/(2c^2) = 1/c$. Secondly one finds a **56** under E_7 for the highest weight $(0,-1,1,0^5)$. Its U(1) charge turns out to be $(-c)/(2c^2) = -1/(2c)$. Since the projection $p \cdot v = 1/3 \mod 1$ gives rise to complex conjugated representations, the corresponding U(1) charges are just minus those already computed. We now choose c in such a way, that the smallest charge in the untwisted sector is 1, fixing $c = -1/2$. Summarizing we find the following branching for the untwisted states

$$\mathrm{E}_8 \to \mathrm{E}_7 \times \mathrm{U}(1) \tag{A.4.25}$$

$$\mathbf{248} \to (\mathbf{133})_0 + (\mathbf{1})_{-2} + (\mathbf{56})_1 + (\mathbf{1})_2 + (\mathbf{56})_{-1} \,. \tag{A.4.26}$$

Note that summing up the dimensions on the RHS gives rise to 247. The missing d.o.f. corresponds to the U(1) generator.

A.5 Branching Rules for $T^6/\mathbb{Z}_4$ Orbifolds without Wilson lines

Let V be a generalized gauge shift associated to a $\mathbb{Z}_4$ breaking of $\mathrm{E}_8 \times \mathrm{E}_8$. Then the generalized gauge shift $2V$ is associated to a $\mathbb{Z}_2$ breaking of $\mathrm{E}_8 \times \mathrm{E}_8$. All branchings are listed and the notation corresponds to the representations (2.8.2) as they appear within the tables 2.5 and 2.10.

Model–No. $G_{[2V]} \to G_{[V]}$				
Untwisted sector:				
$\mathbf{Ad}_{[2V]}$	$\to$	$\mathbf{Ad}_{[V]}$	$+$	$\frac{1}{2}\mathbf{R}^{(2)}_{[V]}\ (+\frac{1}{2}\overline{\mathbf{R}}^{(2)}_{[V]})$
$\mathbf{R}^{(1)}_{[2V]}$	$\to$	$\mathbf{R}^{(3)}_{[V]}$	$+$	$(\overline{\mathbf{R}}^{(3)}_{[V]})$
Twisted sector:				
$\mathbf{S}_{[2V]} + 4\mathbf{D}_{[2V]}$	$\to$	$\mathbf{T}_{[2V]}$	$+$	$(\overline{\mathbf{T}}_{[2V]})$
1. $\mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8{}' \to (\mathrm{E}_6 \times \mathrm{U}(1)) \times \mathrm{SU}(2) \times \mathrm{E}_8{}'$				
$(\mathbf{133},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{78},\mathbf{1};\mathbf{1})_0 + (\mathbf{1},\mathbf{3};\mathbf{1})_0$ $+(\mathbf{1},\mathbf{1};\mathbf{248})_0 + (\mathbf{1},\mathbf{1};\mathbf{1})_0$	$+$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_{-2}$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{27}},\mathbf{2};\mathbf{1})_1 + (\mathbf{1},\mathbf{2};\mathbf{1})_{-3}$	$+$	$(^-)$
$(\mathbf{56},\mathbf{1};\mathbf{1}) + 4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_1 + (\mathbf{1},\mathbf{1};\mathbf{1})_{-3}$ $+2(\mathbf{1},\mathbf{2};\mathbf{1})_0$	$+$	$(^-)$
2. $\mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8{}' \to (\mathrm{E}_6 \times \mathrm{U}(1)) \times \mathrm{SU}(2) \times \mathrm{E}_7{}' \times \mathrm{SU}(2)'$				
$(\mathbf{133},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{78},\mathbf{1};\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})_0$ $+ (\mathbf{1},\mathbf{1};\mathbf{133},\mathbf{1})_0 + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})_0$ $+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_0$	$+$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1},\mathbf{1})_{-2}$ $+\frac{1}{2}(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})_0$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{27}},\mathbf{2};\mathbf{1},\mathbf{1})_1 + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{-3}$	$+$	$(^-)$
$(\mathbf{56},\mathbf{1};\mathbf{1}) + 4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\mathbf{27},\mathbf{1};\mathbf{1},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_3$ $+ 2(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_0$	$+$	$(^-)$
3. $\mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8{}' \to (\mathrm{E}_6 \times \mathrm{U}(1)) \times \mathrm{SU}(2) \times \mathrm{SO}(16)'$				
$(\mathbf{133},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{78},\mathbf{1};\mathbf{1})_0 + (\mathbf{1},\mathbf{3};\mathbf{1})_0$ $+(\mathbf{1},\mathbf{1};\mathbf{120})_0 + (\mathbf{1},\mathbf{1};\mathbf{1})_0$	$+$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_{-2}$ $+ \frac{1}{2}(\mathbf{1},\mathbf{1};\mathbf{128}_c)_0$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{27}},\mathbf{2};\mathbf{1})_1 + (\mathbf{1},\mathbf{2};\mathbf{1})_{-3}$	$+$	$(^-)$
$(\mathbf{56},\mathbf{1};\mathbf{1}) + 4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{27}},\mathbf{1};\mathbf{1})_1 + (\mathbf{1},\mathbf{1};\mathbf{1})_{-3}$ $+ 2(\mathbf{1},\mathbf{2};\mathbf{1})_0$	$+$	$(^-)$

4.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SO}(14)\times \mathrm{U}(1))\times \mathrm{E}_7{}'\times \mathrm{U}(1)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{91};\mathbf{1})_{0,0}+(\mathbf{1};\mathbf{133})_{0,0}$ $+\ 2(\mathbf{1};\mathbf{1})_{0,0}$	$+$	$(\mathbf{14}_v;\mathbf{1})_{2,0}$ $+\ (\mathbf{1};\mathbf{1})_{0,2}$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\mathbf{64}_s;\mathbf{1})_{1,0}+(\mathbf{1};\mathbf{56})_{0,1}$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\mathbf{14}_v;\mathbf{1})_{0,1}+(\mathbf{1};\mathbf{1})_{2,-1}$ $+\ (\mathbf{1};\mathbf{1})_{-2,-1}$	$+$	$(\overline{\ })$

5.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SO}(14)\times \mathrm{U}(1))\times (\mathrm{SO}(12)'\times \mathrm{SU}(2)')\times \mathrm{U}(1)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{91};\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1};\mathbf{66},\mathbf{1})_{0,0}$ $+\ (\mathbf{1};\mathbf{1},\mathbf{3})_{0,0}+2(\mathbf{1};\mathbf{1})_{0,0}$	$+$	$(\mathbf{14}_v;\mathbf{1})_{2,0}+(\mathbf{1};\mathbf{1},\mathbf{1})_{0,2}$ $+\ \frac{1}{2}(\mathbf{1};\mathbf{32}_c,\mathbf{2})_{0,0}$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\mathbf{64}_s;\mathbf{1},\mathbf{1})_{1,0}+(\mathbf{1};\mathbf{32}_s,\mathbf{1})_{0,-1}$ $+(\mathbf{1};\mathbf{12}_v,\mathbf{2})_{0,1}$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\mathbf{14}_v;\mathbf{1},\mathbf{1})_{0,1}+(\mathbf{1};\mathbf{1},\mathbf{1})_{2,-1}$ $+\ (\mathbf{1};\mathbf{1},\mathbf{1})_{-2,-1}$	$+$	$(\overline{\ })$

6.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SO}(10)\times \mathrm{SU}(4))\times \mathrm{E}_7{}'\times \mathrm{U}(1)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{45},\mathbf{1};\mathbf{1})_0+(\mathbf{1},\mathbf{15},\mathbf{1})_0$ $+\ (\mathbf{1},\mathbf{1};\mathbf{133})_0+(\mathbf{1},\mathbf{1};\mathbf{1})_0$	$+$	$\frac{1}{2}(\mathbf{10}_v,\mathbf{6};\mathbf{1})_0$ $+\ (\mathbf{1},\mathbf{1};\mathbf{1})_2$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\mathbf{16}_c,\overline{\mathbf{4}};\mathbf{1})_0+(\mathbf{1},\mathbf{1};\mathbf{56})_1$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\mathbf{10}_v,\mathbf{1};\mathbf{1})_{-1}+(\mathbf{1},\mathbf{6};\mathbf{1})_1$	$+$	$(\overline{\ })$

7.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SO}(10)\times \mathrm{SU}(4))\times (\mathrm{SO}(12)'\times \mathrm{SU}(2)')\times \mathrm{U}(1)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{45},\mathbf{1};\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{15},\mathbf{1},\mathbf{1})_0$ $+\ (\mathbf{1},\mathbf{1};\mathbf{66},\mathbf{1})_0+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})_0$ $+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_0$	$+$	$\frac{1}{2}(\mathbf{10}_v,\mathbf{6};\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_2$ $+\frac{1}{2}(\mathbf{1},\mathbf{1};\mathbf{32}_c,\mathbf{2})_0$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\mathbf{16}_c,\overline{\mathbf{4}};\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1};\mathbf{32}_s,\mathbf{1})_{-1}$ $+(\mathbf{1},\mathbf{1};\mathbf{12}_v,\mathbf{2})_1$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\mathbf{10}_v,\mathbf{1};\mathbf{1},\mathbf{1})_{-1}+(\mathbf{1},\mathbf{6};\mathbf{1},\mathbf{1})_1$	$+$	$(\overline{\ })$

8.	$\mathrm{E}_7\times \mathrm{SU}(2)\times \mathrm{E}_8{}' \to \mathrm{SU}(8)\times \mathrm{SU}(2)\times \mathrm{E}_8{}'$			
$(\mathbf{133},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{63},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})$ $+\ (\mathbf{1},\mathbf{1};\mathbf{248})$	$+$	$\frac{1}{2}(\mathbf{70},\mathbf{1};\mathbf{1})$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{28}},\mathbf{2};\mathbf{1})$	$+$	$(\overline{\ })$
$(\mathbf{56},\mathbf{1};\mathbf{1})+4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\mathbf{28},\mathbf{1};\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1})$	$+$	$(\overline{\ })$

9.	$\mathrm{E}_7\times \mathrm{SU}(2)\times \mathrm{E}_8{}' \to \mathrm{SU}(8)\times \mathrm{SU}(2)\times (\mathrm{E}_7{}'\times \mathrm{SU}(2)')$			
$(\mathbf{133},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{63},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{133},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	$+$	$\frac{1}{2}(\mathbf{70},\mathbf{1};\mathbf{1},\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{28}},\mathbf{2};\mathbf{1},\mathbf{1})$	$+$	$(\overline{\ })$
$(\mathbf{56},\mathbf{1};\mathbf{1})+4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{28}},\mathbf{1};\mathbf{1},\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$	$+$	$(\overline{\ })$

10.	$\mathrm{E}_7\times \mathrm{SU}(2)\times \mathrm{E}_8{}' \to \mathrm{SU}(8)\times \mathrm{SU}(2)\times \mathrm{SO}(16)'$			
$(\mathbf{133},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{248})$	$\to$	$(\mathbf{63},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})$ $+(\mathbf{1},\mathbf{1};\mathbf{120})$	$+$	$\frac{1}{2}(\mathbf{70},\mathbf{1};\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{1};\mathbf{128}_c)$
$(\mathbf{56},\mathbf{2};\mathbf{1})$	$\to$	$(\overline{\mathbf{28}},\mathbf{2};\mathbf{1})$	$+$	$(\overline{\ })$
$(\mathbf{56},\mathbf{1};\mathbf{1})+4(\mathbf{1},\mathbf{2};\mathbf{1})$	$\to$	$(\mathbf{28},\mathbf{1};\mathbf{1})+2(\mathbf{1},\mathbf{2};\mathbf{1})$	$+$	$(\overline{\ })$

11.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SU}(8)\times \mathrm{U}(1))\times (\mathrm{E}_6{}'\times \mathrm{U}(1)')\times \mathrm{SU}(2)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{63};\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1};\mathbf{78},\mathbf{1})_{0,0}$ $+\ (\mathbf{1};\mathbf{1},\mathbf{3})_{0,0}+2(\mathbf{1};\mathbf{1},\mathbf{1})_{0,0}$	$+$	$(\mathbf{28};\mathbf{1},\mathbf{1})_{2,0}$ $+\ (\mathbf{1};\overline{\mathbf{27}},\mathbf{1})_{0,-2}$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\overline{\mathbf{56}};\mathbf{1},\mathbf{1})_{1,0}+(\mathbf{8};\mathbf{1},\mathbf{1})_{-3,0}$ $+(\mathbf{1};\overline{\mathbf{27}},\mathbf{2})_{0,1}+(\mathbf{1};\mathbf{1},\mathbf{2})_{0,-3}$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\overline{\mathbf{8}};\mathbf{1},\mathbf{2})_{-1,0}$	$+$	$(\overline{\ })$

12.	$\mathrm{SO}(16)\times \mathrm{E}_7{}'\times \mathrm{SU}(2)' \to (\mathrm{SU}(8)\times \mathrm{U}(1))\times \mathrm{SU}(8)'\times \mathrm{SU}(2)'$			
$(\mathbf{120};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{133},\mathbf{1})$ $+(\mathbf{1};\mathbf{1},\mathbf{3})$	$\to$	$(\mathbf{63};\mathbf{1},\mathbf{1})_0+(\mathbf{1};\mathbf{63},\mathbf{1})_0$ $+\ (\mathbf{1};\mathbf{1},\mathbf{3})_0+(\mathbf{1};\mathbf{1},\mathbf{1})_0$	$+$	$(\mathbf{28};\mathbf{1},\mathbf{1})_2$ $+\frac{1}{2}(\mathbf{1};\mathbf{70},\mathbf{1})_0$
$(\mathbf{128}_c;\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{56},\mathbf{2})$	$\to$	$(\overline{\mathbf{56}};\mathbf{1},\mathbf{1})_1+(\mathbf{8};\mathbf{1},\mathbf{1})_{-3}$ $+(\mathbf{1};\overline{\mathbf{28}},\mathbf{2})_0$	$+$	$(\overline{\ })$
$(\mathbf{16}_v;\mathbf{1},\mathbf{2})$	$\to$	$(\mathbf{8};\mathbf{1},\mathbf{2})_{1,0}$	$+$	$(\overline{\ })$

A.6 Anomaly Polynomials in Various Dimensions

The anomaly polynomials I_{D+2} of different fields relevant for anomalies in D dimensions are determined in terms of topological invariants. In this appendix these invariants are defined and the anomaly polynomials for different particles in different dimensions are listed. We give no derivations and/or explanations at this stage but refer the reader to the literature [80, 108, 82, 83] instead (a textbook introduction can be found e.g. in [81]).

A.6.1 Basic Definitions and General Results

It turns out to be convenient to define

$$I_{D+2} \equiv -i(2\pi)^{-D/2}\hat{I}_{D+2} \ . \tag{A.6.1}$$

The total Chern character is defined by

$$\mathrm{ch}_{\mathbf{r}}(iF) \equiv \mathrm{tr}_{\mathbf{r}} \exp\left(\frac{iF}{2\pi}\right) \ , \tag{A.6.2}$$

where the trace has to be taken in representation $\mathbf{r}$. We will sometimes use the renormalized $\hat{\mathrm{ch}}_{\mathbf{r}}(iF) \equiv \mathrm{tr}_{\mathbf{r}} \exp(iF)$ instead. As the Chern character is a sum of p forms with different p's it has to be taken formally again. The jth Chern character is defined as the jth term in the formal expansion of the exponential, i.e.

$$\mathrm{ch}_j(iF) \equiv \frac{1}{j!}\mathrm{tr}\left(\frac{iF}{2\pi}\right)^j \ . \tag{A.6.3}$$

Consider $\mathrm{tr}(R^{2m})$. Then R is represented by an antisymmetric $D \times D$ matrix and can always be brought to the form

$$R = \begin{pmatrix} 0 & x_1 & & & & & 0 & \\ -x_1 & 0 & & & & & & \\ & & 0 & x_2 & & & & \\ & & -x_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & x_{D/2} \\ & 0 & & & & & -x_{D/2} & 0 \end{pmatrix} \tag{A.6.4}$$

by an orthogonal transformation. With this it is clear that

$$\mathrm{tr}(R^{2m}) = 2(-1)^m \sum_1^{D/2} x_i^{2m} \ . \tag{A.6.5}$$

We use the inverse of this relation to assign

$$\sum_i x_i^{2m} \to \frac{1}{2}(-1)^m \mathrm{tr}(R^{2m}) \ . \tag{A.6.6}$$

For later convenience we define

$$Y_{2m} = \sum_{i=1}^{D/2} (\tfrac{1}{2}x_i)^{2m} \to \frac{1}{2}(-\frac{1}{4})^m \mathrm{tr}(R^{2m}) \ . \tag{A.6.7}$$

Then the topological invariants are easiest defined as polynomials in those Y_{2m}.

The roof genus (or Dirac genus) $\hat{A}(R)$ is defined by

$$\begin{aligned}
\hat{A}(R) &= \prod_{i=1}^{D/2} \left(\frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i} \right) && \text{(A.6.8)}\\
&= \prod_i (1 + \frac{(\frac{1}{2}x_i)^2}{3!} + \frac{(\frac{1}{2}x_i)^4}{5!} + \dots)^{-1} && \text{(A.6.9)}\\
&= 1 - \frac{1}{6}Y_2 + \frac{1}{180}Y_4 + \frac{1}{72}Y_2^2 && \text{(A.6.10)}\\
&\quad - \frac{1}{2835}Y_6 - \frac{1}{1080}Y_2Y_4 - \frac{1}{1296}Y_2^3 + \dots \ , && \text{(A.6.11)}
\end{aligned}$$

where the expression has been expanded up to 12–forms.

The Hirzebruch L–polynomial $L(x)$ is defined by

$$\begin{aligned}
L(x) &= -\frac{1}{8} \prod_{i=1}^{D/2} \left(\frac{x_i}{\tanh x_i} \right) && \text{(A.6.12)}\\
&= -\frac{1}{8} - \frac{1}{6}Y_2 + (\frac{7}{45}Y_4 - \frac{1}{9}Y_2^2) && \text{(A.6.13)}\\
&\quad + \frac{1}{2835}(-496Y_6 + 588Y_2Y_4 - 140Y_2^3) + \dots \ , && \text{(A.6.14)}
\end{aligned}$$

where the expression has been expanded up to 12–forms again.

Then, the anomaly polynomials for chiral (left–handed) spin 1/2 fields in representation $\mathbf{r}$, chiral (left–handed) spin 3/2 fields and self–dual rank $D/2$ tensors are given by

$$\begin{aligned}
\hat{I}_{1/2,\mathbf{r}} &= \hat{\mathrm{ch}}_{\mathbf{r}}(iF)\hat{A}(R) \ , && \text{(A.6.15)}\\
\hat{I}_{3/2} &= \hat{A}(R)(-1 + 2\sum_{i=1}^{D/2} \cosh x_i) \ , && \\
&= \hat{A}(R)(D - 1 + 4Y_2 + \frac{4}{3}Y_4 + \frac{8}{45}Y_6 + \dots) \ , && \text{(A.6.16)}\\
\hat{I}_A &= L(x) \ , && \text{(A.6.17)}
\end{aligned}$$

respectively. Switching the chirality results in a minus sign. In all expressions the $D+2$ form part has to be extracted. For neutral spin 1/2 fields we denote the anomaly polynomial by $\hat{I}_{1/2} = \hat{A}(R)$. It corresponds to the definition (A.6.15) with $\mathrm{ch}(iF) = 1$ and so is consistent with it.

A.6.2 Trace Decompositions

In this subsection we consider relations between different traces of the form

$$\mathrm{tr}_{\mathbf{r}}(iF)^m_{\mathbf{Ad}} \approx \mathrm{tr}_{\mathbf{r}}(iF)^m_G\ , \tag{A.6.18}$$

where $\mathbf{Ad}$ denotes the algebra in which the field stregth iF is living and $\mathbf{r}$ the representation of this algebra over which the trace is taken. Instead of $\mathbf{Ad}$ we sometimes just write the corresponding Lie group G.

Since we have to deal with a product of groups from Cartan symmetry breaking very often, we start with considering $G = G_1 \times G_2$ and $r = (r_1, r_2)$. It is then convenient to reexpress the traces w.r.t. both representations and group factors seperately. Consider first the easy case that $\mathbf{r_1} = \mathbf{1}$ is a singlet. Then clearly it does not contribute at all and so

$$\mathrm{tr}_{(\mathbf{1},\mathbf{r_2})}(iF)^m_{G_1\times G_2} = \mathrm{tr}_{\mathbf{r_2}}(iF)^m_{G_2}\ . \tag{A.6.19}$$

An analogous expression holds for $\mathbf{r_2} = \mathbf{1}$ of course. However, if $(\mathbf{r_1}, \mathbf{r_2})$ are both non–trivial one has to include combinatorial factors. First it is clear that the structure of the traces has to be given by

$$\mathrm{tr}_{(\mathbf{r_1},\mathbf{r_2})}(iF)^m_{G_1\times G_2} = c(m,k)\mathrm{tr}_{\mathbf{r_1}}(iF)^k_{G_1}\mathrm{tr}_{\mathbf{r_2}}(iF)^{m-k}_{G_2} \tag{A.6.20}$$

for some constant $c(m,k)$, that are determined by cyclicity of the traces on both sides of the equation. The trace on the LHS has $m!$ permutations, wether those on the RHS have $k!(m-k)!$. It follows that $c(m,k) = \frac{m!}{k!(m-k)!} = \binom{m}{k}$ gives the right counting. Note that the simple examples with the singlets are included in this formula.

By recursion the procedure can be extended to $G = G_1 \times G_2 \times \cdots \times G_n$ with $\mathbf{r} = (\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n})$ yielding

$$\mathrm{tr}_{(\mathbf{r_1},\mathbf{r_2},\ldots,\mathbf{r_n})}(iF)^m_{G_1\times G_2\times\cdots\times G_n} = \frac{m!}{k_1!k_2!\ldots k_n!}\mathrm{tr}_{\mathbf{r_1}}(iF)^{k_1}_{G_1}\mathrm{tr}_{\mathbf{r_2}}(iF)^{k_2}_{G_2}\ldots\mathrm{tr}_{\mathbf{r_n}}(iF)^{k_n}_{G_n} \tag{A.6.21}$$

with $k_1 + k_2 + \cdots + k_n = m$. Using this result it is from now on enough to consider G to contain exactly one group.

The next task is to compare traces over different representations of one and the same algebra. We are not going to derive all identities here but give simple examples instead. We consider the adjoint representation $\mathbf{Ad}$ of SU(N). It is very convenient to exploit the following relations valid for the Chern characters (A.6.2):

$$\mathrm{ch}_{\mathbf{r_1}\times\mathbf{r_2}}(iF) = \mathrm{ch}_{\mathbf{r_1}}(iF)\cdot \mathrm{ch}_{\mathbf{r_2}}(iF)\ , \quad \mathrm{ch}_{\mathbf{r_1}+\mathbf{r_2}}(iF) = \mathrm{ch}_{\mathbf{r_1}}(iF) + \mathrm{ch}_{\mathbf{r_2}}(iF)\ , \tag{A.6.22}$$

In terms of fundamental and antifundamental representations $\mathbf{Ad} = \mathbf{N}\times\overline{\mathbf{N}} - \mathbf{1}$. Pluggin this into (A.6.22) and comparing the equal order forms of both sides yields

$$\begin{aligned}
\mathrm{Tr}(iF)_{\mathrm{SU(N)}} &= 0\ , &\text{(A.6.23)}\\
\mathrm{Tr}(iF)^2_{\mathrm{SU(N)}} &= 2N\,\mathrm{tr}(iF)^2_{\mathrm{SU(N)}}\ , &\text{(A.6.24)}\\
\mathrm{Tr}(iF)^3_{\mathrm{SU(N)}} &= 0\ , &\text{(A.6.25)}\\
\mathrm{Tr}(iF)^4_{\mathrm{SU(N)}} &= 2N\,\mathrm{tr}(iF)^4_{\mathrm{SU(N)}} + 6\,[\mathrm{tr}(iF)^2_{\mathrm{SU(N)}}]^2\ , &\text{(A.6.26)}
\end{aligned}$$

with Tr and tr denoting trace over adjoint and fundamental representation respectively. Here we used that $\mathrm{tr}_{\mathbf{N}}(iF)^k = (-1)^k\mathrm{tr}_{\overline{\mathbf{N}}}(iF)^k$ and $\mathrm{tr}_r(1) = \dim(r)$. Similarly all such identities can be derived.

In the following all identities needed for the different purpuses are listed. The ones relevant for anomaly cancellation in ten dimensions are

$$\begin{aligned}
\mathrm{Tr}(iF)^6_{\mathrm{E_8}} &= \tfrac{1}{7200}\,[\mathrm{Tr}(iF)^2_{\mathrm{E_8}}]^3\ ,\\
\mathrm{Tr}(iF)^4_{\mathrm{E_8}} &= \tfrac{1}{100}\,[\mathrm{Tr}(iF)^2_{\mathrm{E_8}}]^2\ ,
\end{aligned}\tag{A.6.27}$$

The identities needed in six dimensions are given by

$$\begin{aligned}
\mathrm{Tr}(iF)^2_{\mathrm{E_7}} &= 3\,\mathrm{tr}(iF)^2_{\mathrm{E_7}}\\
\mathrm{Tr}(iF)^2_{\mathrm{SU(N)}} &= 2N\,\mathrm{tr}(iF)^2_{\mathrm{SU(N)}}\\
\mathrm{Tr}(iF)^2_{\mathrm{E_8}} &= 30\,\mathrm{tr}(iF)^2_{\mathrm{E_8}}\\
\mathrm{Tr}(iF)^2_{\mathrm{SO(N)}} &= (N-2)\,\mathrm{tr}(iF)^2_{\mathrm{SO(N)}}\\[1ex]
\mathrm{Tr}(iF)^4_{\mathrm{E_7}} &= \tfrac{1}{6}\,[\mathrm{tr}(iF)^2_{\mathrm{E_7}}]^2\\
\mathrm{Tr}(iF)^4_{\mathrm{SU(N)}} &= 2N\,\mathrm{tr}(iF)^4_{\mathrm{SU(N)}} + 6\,[\mathrm{tr}(iF)^2_{\mathrm{SU(N)}}]^2\\
\mathrm{Tr}(iF)^4_{\mathrm{E_8}} &= \frac{1}{100}\,[\mathrm{tr}(iF)^2_{\mathrm{E_8}}]^2\\
\mathrm{Tr}(iF)^4_{\mathrm{SO(N)}} &= (N-8)\,\mathrm{tr}(iF)^4_{\mathrm{SO(N)}} + 3\,[\mathrm{tr}(iF)^2_{\mathrm{SO(N)}}]^2\\[1ex]
\mathrm{tr}_{\mathbf{2^{r-1}}}(iF)^2_{\mathrm{SO(2r)}} &= 2^{r-4}\,\mathrm{tr}(iF)^2_{\mathrm{SO(2r)}}\\
\mathrm{tr}_{\mathbf{2^{r-1}}}(iF)^4_{\mathrm{SO(2r)}} &= -2^{r-5}\,\mathrm{tr}(iF)^4_{\mathrm{SO(2r)}} + 3\;2^{r-7}\,[\mathrm{tr}(iF)^2_{\mathrm{SO(2r)}}]^2
\end{aligned}\tag{A.6.28}$$

Relevant identities for anomaly cancellation in four dimensions.

$$\begin{aligned}
\mathrm{tr}_{\binom{\mathbf{N}}{\mathbf{2}}}(iF)^3_{\mathrm{SU(N)}} &= (N-4)\,\mathrm{tr}(iF)^3_{\mathrm{SU(N)}}\\
\mathrm{tr}_{\binom{\mathbf{N}}{\mathbf{3}}}(iF)^3_{\mathrm{SU(N)}} &= \tfrac{1}{2}(N^2-9N+18)\,\mathrm{tr}(iF)^3_{\mathrm{SU(N)}}
\end{aligned}\tag{A.6.29}$$

Now consider the branching of some simple group G into the semi–simple group $G \to H_1 \times H_2 \times \cdots \times H_n$. Then the identity

$$\frac{1}{g(G)}\mathrm{Tr}(iF)^2_G \to \sum_{i=1}^{n} \frac{1}{g(H_i)}\mathrm{Tr}(iF)^2_{H_i} \tag{A.6.30}$$

holds, where the traces are taken over the adjoint representations and weighted by the dual Coxeter numbers.

Appendix B

Some Supersymmetric Multiplets

This appendix gives a list of some supersymmetric (SUSY) multiplets used in this work. They are taken from [109]. Very nice general introductions to SUSY and SUGRA can be found in [110, 111, 112].

B.1 $D = 10$, $\mathcal{N} = 1$

There are 16 supersymmetries, corresponding to a single Majorana–Weyl fermion, which we take to have positive chirality. The massless spacetime little group in ten dimensions is SO(8). There are only two massless supersymmetric multiplets:

Multiplet	SO(8) rep	Content
SUGRA	$\mathbf{8_v} \times \mathbf{8_v}$ $+ \mathbf{8_v} \times \mathbf{8_c}$	graviton, dilaton, antisymmetric tensor, spartners
Vector	$\mathbf{8_v}$ $+ \mathbf{8_c}$	gauge bosons (in adjoint of gauge group), gauginos (in adjoint of gauge group)

B.2 $D = 6$, $\mathcal{N} = 1$

There are 8 supersymmetries. The multiplets are classified w.r.t. the massless spacetime little group in six dimensions $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \cong \mathrm{SO}(4) \subset \mathrm{SO}(1,5)$ and the R–parity group $\mathrm{SU}(2)_R$. There are the following massless supersymmetric multiplets[1]: More information about six dimensional super multipletes can be found e.g. in [113, 114, 91]. One result used in the main text is, that half–hyper multiplets are only allowed if they transform in pseudo–real representations under the gauge group.

[1] sd/asd stands for (anti–)selfdual.

Multiplet	$\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ rep	Content
SUGRA	$(\mathbf{3}, \mathbf{3}; \mathbf{1}) + (\mathbf{3}, \mathbf{1}; \mathbf{1})$ $+ (\mathbf{3}, \mathbf{2}; \mathbf{2})$	graviton, antisym. tensor (asd), spartners
Tensor	$(\mathbf{1}, \mathbf{3}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{1})$ $+ (\mathbf{1}, \mathbf{2}; \mathbf{2})$	antisym. tensor (sd), dilaton, spartners
Vector	$(\mathbf{2}, \mathbf{2}; \mathbf{1})$ $+ (\mathbf{2}, \mathbf{1}; \mathbf{2})$	gauge bosons, gauginos
Half–hyper	$(\mathbf{1}, \mathbf{1}; \mathbf{2})$ $+ (\mathbf{1}, \mathbf{2}; \mathbf{1})$	half–hyper, spartners
Hyper	$2\ (\mathbf{1}, \mathbf{1}; \mathbf{2})$ $+ 2\ (\mathbf{1}, \mathbf{2}; \mathbf{1})$	hyper, spartners

B.3 $D = 4$, $\mathcal{N} = 1$

There are 4 supersymmetries. The multiplets are classified w.r.t. the massless spacetime little group in four dimensions $\mathrm{SO}(2) \subset \mathrm{SO}(1,3)$. $\mathrm{U}(1)_R$ charges are omitted. There are the following massless supersymmetric multiplets

Multiplet	SO(2) rep	Content
SUGRA	(-2 + -3/2) + (3/2 + 2)	graviton, gravitino
Vector	(-1 + -1/2) + (1/2 + 1)	gauge bosons, gauginos
Chiral	(-1/2 + 0) + (0 + 1/2)	chiral matter, spartners or dilaton/axion, spartners

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