Large $N$

Markus Gross

Universität Bonn

Seminar Relativistic Quantum Field Theory
05/11/2006
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Summary
Motivation

Looking for an approximation scheme for QCD …

- coupling constant $g$ not good expansion parameter in low energy regime $\mu$

Suggestion by ’t Hooft:

- generalize $SU(3)$ with 3 colors to $SU(N)$ with $N$ colors
- hope that theory simplifies for large $N$
- obtain new expansion parameter: $1/N$
Consider QCD Lagrangian with SU($N$) gauge group:

$$\mathcal{L} = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{f=1}^{N_F} (\bar{q}_i)_f (i\partial - m_f) (q^i)_f$$

- $D_\mu = \partial_\mu + ig A_\mu$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$
- $N_F$ flavor (anti-)quark fields $q^i$ ($\bar{q}_i$) in fundamental representation ($i = 1, \ldots, N$)
- gluon field $(A_\mu)_j^i = A_\mu^a (T^a)_j^i$: hermitian traceless $N \times N$ matrix ($a = 1, \ldots, N^2 - 1$)

But so far no explicit $N$ dependence . . .
Large N QCD

Hint: consider renormalization group flow of QCD:

$$\mu \frac{dg}{d\mu} = \left( -\frac{11}{3}N + \frac{2}{3}N_F \right) \frac{g^3}{16\pi^2} + O(g^5)$$

⇒ does not have a sensible large $N$ limit

Solution

▶ replace:

$$g \longrightarrow \frac{g}{\sqrt{N}}$$

Obtain:

$$\mu \frac{dg}{d\mu} = \left( -\frac{11}{3N} + \frac{2}{3N}N_F \right) \frac{g^3}{16\pi^2} + O(g^5)$$
Large N QCD

Replace $g \rightarrow g/\sqrt{N}$ in $\mathcal{L}$, and for convenience, rescale the fields:

- $A_\mu \rightarrow \frac{\sqrt{N}}{g} A_\mu$
- $q \rightarrow \sqrt{N}q$

**SU(N) Lagrangian:**

$$
\mathcal{L} = N \left[ -\frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \bar{q}_i (i\sigma^\mu - m)^j q^j \right]
$$

Note: $g$ does not occur in $D_\mu$ and $F_{\mu\nu}$ anymore

**Counting rules:**

Read off $N$-dependence of vertices and propagators:

- all *vertices* $\propto N$
- all *propagators* $\propto \frac{1}{N}$
Double-Line Notation

Reorganize Feynman diagrams to visualize color flow

Propagators:

- **quark**: \( \langle T q^i(x) \bar{q}_j(y) \rangle = \delta^i_j S_F(x - y) \)

  \[ i \quad q \quad j \quad i \quad \bar{q} \quad j \]

- **gluon**: \( \langle T A_{\mu j}^i(x) A_{\nu k}^k(y) \rangle = \langle T A_{\mu}^a(x) A_{\nu}^b(y) \rangle (T^a)^i_j (T^b)^k_l \)
  
  \[ \langle T A_{\mu}^a(x) A_{\nu}^b(y) \rangle \delta^{ab} (T^a)^i_j (T^b)^k_l = (\delta^i_j \delta^k_l - \frac{1}{N} \delta^i_j \delta^k_l) D_{\mu \nu}(x - y) \]

  \((\ast)\) drops out for \( U(N) \)

  \[ A_{\mu}^i \quad i \quad l \]

  \[ j \quad k \quad j \]

  Group theoretically: \( A_{\mu j}^i \) transforms as \( q^i \bar{q}_j \)

  For simplicity: from now on **consider** \( U(N) \) instead of \( SU(N) \)!
Double-Line Notation

Vertices

- **quark-gluon:** \( \bar{q}_i \gamma^\mu q^j A_{\mu j} \)

- **3-gluons:** \( A_{\mu j} A_{\nu k} \partial_{\mu} A_{\nu i} \)

- **4-gluons:** \( A_{\mu j} A_{\nu k} A_{\mu l} A_{\nu i} \)
Double-Line Notation - Examples

Can now determine $N$-dependence of an arbitrary Feynman diagram:

$\propto \frac{1}{N^3}$

→ Basic reason: $N$ times more intermediate gluon states than quark states to sum over
Diagram Rules

How does this nontrivial $N$ dependence help simplifying QCD analysis?

Given an arbitrary diagram, one can see...

1. additional internal gluon lines don’t change $N$ dependence

$$
\frac{1}{N} \quad \Rightarrow \quad \frac{1}{N} \quad \propto \quad \frac{1}{N}
$$

2. internal quark loops are suppressed by $\frac{N_F}{N}$

$$
\frac{1}{N} \quad \Rightarrow \quad \frac{1}{N} \quad \propto \quad \frac{N_F}{N^2}
$$
Diagram Rules

3. **non-planar** diagrams are suppressed by $\frac{1}{N^2}$

→ fewer index loops compared to corresponding planar diagram!
Graph Topology

Consider first only vacuum-to-vacuum graphs. Denote:

\[ L \] no. of index loops
\[ P \] no. of quark and gluon propagators
\[ V \] no. of vertices

Then

\[ \mathcal{O}(\text{Graph}) \sim N^{L-P+V} = N^\chi \]

Construct 2d orientable surface from a double-line graph:

1. loops → faces, propagators → edges, vertices → vertices
2. identify edges when on the same double-line propagator
3. give orientation according to arrows on perimeter

Thus \( \chi \) is the Euler characteristic.
Every 2d orientable surface is topologically equivalent to a 2-sphere with holes and handles:
Graph Topology

Therefore: \( \chi = 2 - 2H - B \)

with
- \( H \) no. of handles stuck onto the sphere
- \( B \) no. of boundaries (holes) in the sphere

But also: \( B = \) no. of quark loops

Conclusion:

\[
\mathcal{O}(\text{Graph}) \sim N^{2 - 2H - B}
\]

- **l.o. graphs**: \( H = 0 \Leftrightarrow \text{planar}, \ B = 0 \Leftrightarrow \text{no quark loops} 
- **l.o. graphs with quark dependence**: \( H = 0 \Leftrightarrow \text{planar}, \ B = 1 \Leftrightarrow \text{one single quark loop on the outer edge} 

Why only on the outer edge?

Because: 

\[ \overset{\equiv}{\text{would be "non-planar" too}} \]
Mesons

To create a meson: apply to the vacuum a quark bilinear $B$

$$B \in \{ q\bar{q},\ \gamma^{\mu}\bar{q},\ F_{\mu\nu}\bar{q},\ldots \}$$

Interactions of $n$ mesons $\rightarrow$ conn. Greens function $\langle B_1 \ldots B_n \rangle$

To use our previous counting rules...

- replace action: $S \rightarrow S + N \sum_i b_i B_i$
  - then $\langle B_1 \ldots B_n \rangle = \frac{1}{(iN)^n} \left. \frac{\partial^n W}{\partial b_1 \ldots \partial b_n} \right|_{b_i=0}$

  with $W = \sum$(connected vacuum-to-vacuum graphs)

Example: $\langle B_1 B_2 B_3 \rangle \sim \hat{=} = \frac{1}{(iN)^n} \left. \frac{\partial^n W}{\partial b_1 \ldots \partial b_n} \right|_{b_i=0}$

Example:

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Example:  \langle B_1 B_2 B_3 \rangle \sim \hat{=} = \frac{1}{(iN)^n} \left. \frac{\partial^n W}{\partial b_1 \ldots \partial b_n} \right|_{b_i=0}
```

Example:

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Example:
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Example:

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Example:
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Example:

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Example:
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Mesons - Diagram Rules

Conclude:

\[
\text{l.o. interaction graphs} = \text{l.o. vacuum graphs with bilinears inserted into quark loop}
\]

⇒ order of a graph now: \( \langle B_1 \ldots B_n \rangle \propto N^{(1-n)} \)

Assumption:
QCD shows confinement for arbitrary large \( N \)
- all states made by the \( B_i \)'s are \( \text{SU}(N) \) singlets

Transition amplitude \( \langle BB \rangle \) should be \( \sim O(1) \) for arbitrary \( N \)
⇒ use properly normalized operators \( B_i' = N^{\frac{1}{2}} B_i \)

Finally
\[
\langle B_1' \ldots B_n' \rangle \propto N^{1-\frac{n}{2}}
\]
Mesons - Diagram Rules

Claim:
To leading order in $1/N$,

$$\langle B'_1 \ldots B'_n \rangle = \sum \text{(meson tree diagrams)}$$

$\Rightarrow$ a $B'_i$ creates only a single particle

Heuristical understanding:
Look at intermediate states in a planar diagram:

$$\sim \bar{q}_l A^l_k A^k_j A^j_i q^i$$

cannot be broken up to color singlets
Mesons - Diagram Rules

Proof (by contradiction):

We know: "a $B'_i$ creates only a single particle"
⇔ "the only singularities of $\langle B'_1 \ldots B'_n \rangle$ are simple poles"

Consider 2-point function $\langle B'_i B'_j \rangle$:

1. assume $B'_1$ creates two color-singlet particles (a,b), amplitude $\sim \mathcal{O}(1)$

2. particles reflected ($B'_2$, $B'_1$) and finally absorbed ($B'_4$) 
   all amplitudes also $\sim \mathcal{O}(1)$ by crossing symmetry

3. thus get singularity of $\sim \mathcal{O}(1)$ in 4-point function

4. but $\langle B'_1 \ldots B'_4 \rangle \sim N^{1-\frac{4}{2}} = \frac{1}{N}$
Mesons - Interactions

By reduction formula: \( S_{\{n \text{ particles}\}} \propto \langle B'_1 \ldots B'_n \rangle \propto N^{1-\frac{n}{2}} \)

- **Leading order 2-point function:** \( \sim O(1) \)

- **Leading order 3-point function:** \( \sim O(1/\sqrt{N}) \)

- **Leading order 4-point function:** \( \sim O(1/N) \)
Mesons - Phenomenology

It seems: we have rewritten QCD as an effective theory of weakly interacting hadrons...

- effective coupling constant $\sim \frac{1}{\sqrt{N}}$
- l.o. in $1/N$ is tree approximation to this theory

Behavior for large $N$

- Mesons stable and noninteracting for $N \rightarrow \infty$
- infinite number of mesons

Why? → Can expand 2-point function as sum over 1-particle resonances:

$$\int d^x e^{iqx} \langle B'_1(x)B'_2(0) \rangle = \sum_i \frac{Z_i}{q^2 - m_i^2}$$

Now: l.h.s. is known $\sim \log(q^2) \Rightarrow$ r.h.s. must be infinite sum
Predictions for reality \((N=3)\)

- leading order scattering amplitudes \(= \sum\) tree diagrams with physical hadrons exchanged
  \(\rightarrow\) similarity to successful "Regge phenomenology"
- multiparticle decays of unstable mesons preferably through two body states
- suppression of the \(q\bar{q}\) sea in mesons
- suppression of \(q\bar{q}q\bar{q}\) exotics
Mesons - Phenomenology

Justification for OZI (Zweig) rule: 
"flavor disconnected processes are suppressed"

suppressed

allowed

Looks in double line notation:

Count color loops ⇒ expect branching ratio: \( \frac{\Gamma_{OZI \text{ suppressed}}}{\Gamma_{OZI \text{ allowed}}} \propto \frac{1}{N^2} \)
Glue states

Same analysis can be applied to glueballs:
Let
\[ G_i \in \{ \text{tr} \ F_{\mu\nu} F^{\mu\nu}, \text{tr} \ F_{\mu\nu} (\ast F)^{\mu\nu} \} \]

Facts:

- **pure glue states**: l.o. graphs = planar, no quark loops
  \[ \Rightarrow \langle G_1 \ldots G_n \rangle \sim N^{2-n} \text{ (already properly normalized)} \]
- **mixed glueball-meson states**: l.o. graphs = planar, one quark loop at boundary
  \[ \Rightarrow \langle B'_1 \ldots B'_m \ G_1 \ldots G_n \rangle \sim N^{1-m/2-n} \]

Conclusions:

- **glueball interaction constant** \( \sim \frac{1}{N} \Rightarrow \text{weakly interacting} \)
- **meson-glueball coupling** \( \langle GB' \rangle \sim \frac{1}{\sqrt{N}} \Rightarrow \text{mixing suppressed} \)
Baryons - Counting Rules

Baryon = $N$ quark color singlet state $\sim \epsilon_{i_1 \cdots i_N} q^{i_1} \cdots q^{i_N}$

Have well-defined large $N$ limit although no. of quarks diverges!

Propagator:

Any connected $k$-body interaction subgraph
$\hat{=} O(N)$ meson graph:

- cut $k$ fermion lines
- but you loose $k$ color index sums

Therefore: $k$-particle interactions $\sim N^{1-k}$
Baryon Masses

$\mathcal{O}(N^k)$ ways of choosing $k$ quarks from an $N$ baryon
$\Rightarrow$ net effect $k$ particle interaction $\sim \mathcal{O}(N)$

In general: diagram with $l$ disconnected pieces is $\sim \mathcal{O}(N^l)$

Baryon mass:

$$M_B = Nm_q + NT_q + \frac{1}{2} N^2 \left( \frac{1}{N} V_{qq} \right)$$

$\sim \mathcal{O}(N)$

Now baryon propagator

$$e^{-iM_B t} = 1 - iM_B t - \frac{1}{2} M_B^2 t^2 + \ldots$$

$l$th term represents $l$ disconnected subdiagrams
$\Rightarrow$ disconnected graph $\sim \mathcal{O}(M_B^l)$
Baryons as Solitons

We have seen:

- QCD coupling constant: $g_s \sim \frac{1}{\sqrt{N}}$
- Baryon mass: $M_B \sim \frac{1}{g_s^2} \to \infty$ for $g_s \to 0$

Compare with 't Hooft-Polyakov monopole: $M_{\text{monopole}} \sim \mathcal{O}(1/\alpha)$

More analogs:

for large $N$ respectively small $\alpha$ . . .

- baryon size and shape independent of $N$
  $\Leftrightarrow$ size and shape of the monopole independent of $\alpha$
- mesons become non-interacting, baryons still interact
  $\Leftrightarrow$ $e^+$, $e^-$ non-interacting, but m.-m. and m.-e$^\pm$ interaction still possible

Baryons $\sim$ solitons in weakly coupled theory of strong interactions
Justification for $1/N$ expansion

Why consider at all large $N$?

- understand **tree approximation** ($\approx$ large $N$) of QCD first before study **loop corrections** ($\approx$ finite $N$)

How good is $1/N$ expansion for $N = 3$?

- depends on coefficients of the **suppressed terms**:
  - quark loops $\mathcal{O}(1/N)$ often unimportant in phenomenology
    \[ \Rightarrow \text{expansion really in terms of } 1/N^2 = 1/9 (!) \]
- explains a lot of **observations**

Compare to QED:

- electric charge actually is $e = \sqrt{4\pi\alpha} \approx 0.3$
- correct expansion parameter **found** to be $\frac{e^2}{4\pi}$

Lastly: $1/N$ is the only known expansion parameter of QCD in low energy regime
Master Field

As far, only studied overall $N$ dependence of the theory

- want to calculate at least leading term in $1/N$
- sum all planar diagrams? → hopeless

Hint:
consider large $N$ behavior of

$$G = \text{gauge invariant operator made up of gauge fields}$$

- remember: $\langle G_1 \ldots G_n \rangle_C \propto N^{2-n}$
- $G' \equiv G/N$ has well defined v.e.v. for $N \to \infty$: $\langle G' \rangle_C \propto 1$

Compute variance of $G'$:

$$\langle (G' - \langle G' \rangle)^2 \rangle = \langle G' G' \rangle - \langle G' \rangle \langle G' \rangle$$

$$= \langle G' G' \rangle_C$$

$$= \mathcal{O}(1/N^2) \xrightarrow{N \to \infty} 0$$
Master Field

What does this imply?

Path integral for pure U(N) gauge theory:

\[ \langle G'_1 \ldots G'_n \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu e^{-N \int d^4x \left[ \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} \right]} G'_1 \ldots G'_n \]

Compare:

If \( f(x) \) has minimum at \( a \):

\[ f(x) = f(a) + \frac{1}{2} f''(a)(x - a)^2 + \mathcal{O} ((x - a)^3) \]

Then for large \( N \):

\[ \int dx \ e^{-N f(x)} = e^{-N f(a)} \left( \frac{2\pi}{N f''(a)} \right)^{1/2} e^{-\mathcal{O}(1/\sqrt{N})} \]

Master field \( \vec{A}_\mu \):

For \( N \to \infty \): path integral determined by extremal field configuration \( \vec{A}_\mu \in \{ UA_\mu U^{-1} - iU \partial_\mu U^{-1} | U \in U(N) \} \)

\[ \Rightarrow \langle G'_1(A_\mu(x_1)) \ldots G'_n(A_\mu(x_n)) \rangle = G'_1(\vec{A}_\mu(x_1)) \ldots G'_n(\vec{A}_\mu(x_n)) \]
Master Field

Properties of $\bar{A}_\mu$:

- Four hermitian $'\infty \times \infty'$ matrices
- Expected to be spacetime independent!
  
  Reason: action and measure are translationally invariant.

\[ \bar{A}_\mu(x) = e^{iP \cdot x} \bar{A}_\mu(0) e^{-iP \cdot x} \]

Perform gauge transformation $U = e^{iP \cdot x}$:

\[ \Rightarrow \bar{A}_\mu(x) = \bar{A}_\mu(0) + P_\mu \]

Also: $\bar{F}_{\mu\nu} = [\bar{A}_\mu, \bar{A}_\nu]$

Solution of large N QCD by finding four '$\infty \times \infty'$ matrices!
Matrix Model

How to deal with ’$\infty \times \infty$’ matrices?

Solvable model:

QCD in $0 + 0$ dimensional spacetime

→ Evaluate in large N limit:

$$\langle \text{tr } g(A) \rangle = \frac{1}{Z} \int dA e^{-N \text{tr} V(A) \text{ tr } g(A)}$$

where

- $A$: hermitian $N \times N$ matrix
- $g(A)$ gauge invariant function of $A$
- $dA = \prod dA_{ab}, \quad a, b = 1 \ldots N$
- $dA_{ab}dA_{ba} = d(\text{Re}A_{ab})d(\text{Im}A_{ba})$
- $V(A)$ gauge inv. function of $A$ (e.g. $V(A) = \frac{1}{2} M^2 A^2 + A^4$)
Matrix Model

Procedure:
1. write $A = U^\dagger \Lambda U$, with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$
2. use $dA = dU(\prod_k d\lambda_k) \prod_{i \neq j} (\lambda_i - \lambda_j)$

Arrive at:

$$\langle \text{tr} \, g(A) \rangle = \frac{1}{Z} \int (\prod_i d\lambda_i)(\sum_j g(\lambda_j)) e^{-N \sum_k V(\lambda_k) + \sum_{m \neq n} \log |\lambda_m - \lambda_n|}$$

Remark (Dyson): $Z$ = partition function of classical 1-dimensional gas with particle positions $\lambda_i$

Consider further:

$$S_{\text{eff}} = N \sum_k V(\lambda_k) - \sum_{m \neq n} \log |\lambda_m - \lambda_n|$$
Matrix Model

Density of eigenvalues: $\rho(\lambda) \equiv \frac{1}{N} \sum_i \delta(\lambda - \lambda_i), \quad \int d\lambda \rho(\lambda) = 1$

Rewrite $S_{\text{eff}}$:

$$S_{\text{eff}} = N^2 \left[ \int d\lambda \rho(\lambda) V(\lambda) - \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'| \right]$$

For $N \to \infty$: $\langle \text{tr} g(A) \rangle$ is dominated by the minimal $\rho$:

$$V'(\lambda) = 2 P \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}$$

- solve for $\rho$ to get d.o.e. for the master matrix
- to first order then $\langle \text{tr} g(A) \rangle = N \int d\lambda \rho(\lambda) g(\lambda) \bar{A}$

E.g. for $V(A) = \frac{1}{2} m^2 A^2 \to$ Wigner Semi-Circle distribution:

$$\rho(\lambda) = \frac{2}{\pi(4/m^2)^2} \sqrt{(4/m)^2 - \lambda^2}$$
Summary

- Right QCD expansion parameter is $1/N$
- Large $N$ QCD gets simple (tree graphs)
- Mesons appear as particles, Baryons as solitons
- Explains many of strong interaction phenomena (often the only known general explanation)
- For $N = 3$ it might be not such a bad approximation
- Summation of planar diagrams seems not feasible
- Master field is hopeful solution candidate
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