Magnetic Monopoles

Roman Schmitz

Seminar on Theoretical Particle Physics
University of Bonn

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Introduction

Dirac Monopoles
  Maxwell’s Equations and Duality
  The magnetic monopole field
  Dirac’s Quantization of Magnetic Charge
  Summary

’t Hooft-Polyakov Monopoles
  What are Solitons ?
  Solitons in the SO(3) model
  ’t Hooft Polyakov Soliton
  The ’t Hooft-Polyakov-Monopole
Motivation: Why magnetic monopoles?

- First idea from Dirac in 1931 (symmetric form of Maxwell-Equations)
- Appear in non-abelian gauge theories with symmetry breakdown
- Possibly particles not yet observed, no experimental evidence up to now!
The Maxwell Equations in terms of the Dual Tensor

Take Maxwell Equations:

$$\partial_{\nu} F^{\mu\nu} = -j^{\mu} \quad dF = 0$$

In terms of the **Dual Tensor** defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

with components:

$$\tilde{F}^{0i} = \frac{1}{2} \epsilon^{0ijk} F_{jk} = -\frac{1}{2} \epsilon^{0ijk} \epsilon_{jkl} B^l = B^i$$

$$\tilde{F}^{ik} = \frac{1}{2} \epsilon^{ijk\nu} F_{\mu\nu} = \frac{1}{2} (\epsilon^{ijk0} F_{k0} + \epsilon^{ij0l} F_{0l} = \epsilon^{ijk} E^k)$$

Maxwell’s equations read:

$$\partial_{\nu} F^{\mu\nu} = -j^{\mu} \quad \partial_{\nu} \tilde{F}^{\mu\nu} = 0$$
Extension of Maxwell’s Equations

To describe Monopoles construct a magnetic 4-current in analogy to the electrical:

\[ k^\mu = (\sigma, \vec{k}) \]

Now Maxwell’s Equations read

\[ \partial_\nu F^{\mu\nu} = -j^\mu \quad \partial_\nu \tilde{F}^{\mu\nu} = -k^\mu \]

in a nice symmetric form and are invariant under the so-called Duality Transformation:

\[ F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} \quad \tilde{F}^{\mu\nu} \mapsto -F^{\mu\nu} \quad j^\mu \mapsto k^\mu \quad k^\mu \mapsto -j^\mu \]
Magnetic Monopoles

Dirac Monopoles

The magnetic monopole field

Magnetic Monopole Field

Magnetic field for a point-source with magnetic charge $g$:

$$\vec{B}(\vec{r}, t) = \frac{g}{4\pi r^2} \cdot \frac{\vec{r}}{r}$$

Problem:

$$\text{div} \vec{B} = \nabla \cdot \frac{g}{4\pi} \cdot \frac{\vec{r}}{r^3} = -\frac{g}{4\pi} \Delta \frac{1}{r} = -\frac{g}{4\pi} \delta(r) \neq 0$$

$$= \nabla (-\frac{1}{r})$$

$$\Rightarrow \nexists \vec{A} \text{ s.t. } \vec{B} = \text{rot} \vec{A}$$

Is this the end of magnetic monopoles?
Solution: The Dirac string

Add an infinitely small, infinitely extended solenoid field (e.g. along the negative z-axis):

\[ \vec{B}_{sol} = \frac{g}{4\pi r^2} \hat{r} + g \cdot \Theta(-z) \delta(x) \delta(y) \hat{z} \]

Verify that the flux is zero by integrating and using Gauss’ theorem. Now:

\[ \vec{B}_{Monopole} = \text{rot} \vec{A}_{sol} - g \cdot \Theta(-z) \delta(x) \delta(y) \hat{z} \]
Dirac’s Quantization of Magnetic Charge

Motion of charged particle in Monopole field:

\[
\frac{d}{dt} L = m [\vec{r} \times \dot{\vec{r}}] = \frac{gq}{4\pi r^3} [\vec{r} \times [\vec{r} \times \vec{r}]] = \frac{gq}{4\pi} \left( \frac{\dot{\vec{r}}}{r} - \frac{\vec{r} (\dot{\vec{r}} \cdot \vec{r})}{r^3} \right) = \frac{d}{dt} \frac{gq\vec{r}}{4\pi r}
\]

Angular momentum of Electromagnetic field:

\[
L_{EM} = \int d^3x [\vec{r} \times [\vec{E} \times \vec{B}]] = \int d^3x \frac{\vec{E}}{r} - \frac{\vec{r}}{r^3} (\vec{r} \cdot \vec{E}) = \int d^3x E^i (\nabla^i \hat{r}) = -\frac{gq\vec{r}}{4\pi r}
\]

In QM quantized angular momenta in units of \(\frac{n}{2}\):

\[
\frac{eg}{4\pi} = \frac{n}{2}
\]
Remark: Dirac string unobservable

Aharonov-Bohm-Effect changes phase factors of wavefunctions if $A$ given, but $B = 0$. Consider 2 paths around Dirac string. Condition to observe no Dirac string:

$$|\psi_1 + \psi_2|^2 = \left| \exp \left( iq \int_1 \vec{A} \cdot d\vec{l} \right) \cdot \psi_1 + \exp \left( iq \int_2 \vec{A} \cdot d\vec{l} \right) \cdot \psi_2 \right|^2$$

interference terms with exponents can differ by $n 2\pi$:

$$2\pi n = \left( iq \int_1 \vec{A} \cdot d\vec{l} \right) - \left( iq \int_2 \vec{A} \cdot d\vec{l} \right) = q \oint \vec{A} \cdot d\vec{l} = qg$$

That means the Dirac string is unobservable because of the quantization condition.
Magnetic coupling strength

From the quantization condition

\[ \frac{qg}{4\pi} = \frac{n}{2} \]

one can estimate the magnetic coupling constant. Coupling of 2 monopoles will be \( \sim g^2 \), so:

\[ \sim g^2 \sim \frac{n^2}{4} \cdot \left(\frac{4\pi}{q^2}\right)^2 \sim \frac{n^2}{4} q^2 \left(\frac{4\pi}{q^2}\right)^2 \underbrace{1/\alpha^2}_{\text{1/\alpha^2}} \]

The means the magnetic coupling between two monopoles is about 10^4 times stronger than the electrical coupling.
Electromagnetic Duality

In vacuum ($j^\mu = 0$) the "old" Maxwell-Equations are symmetric under the Duality Transformation

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} \quad \text{and} \quad \tilde{F}^{\mu\nu} \mapsto -F^{\mu\nu}$$

which is equivalent to

$$E \mapsto B \quad \text{and} \quad B \mapsto -E$$

With magnetic charges we have a symmetric form that is invariant under the Duality Transformation if

$$j^\mu \mapsto k^\mu \quad \text{and} \quad k^\mu \mapsto -j^\mu$$
Summary

- symmetric form of Maxwell Equations, duality transformation
- describes quantization of electric/magnetic charges by QM
- still have to deal with point-sources and singularities
- Dirac string unobservable
- no mass predictions
What are Solitons?

Solitary waves or so-called Solitons are static finite-energy solutions to the equations of motion that appear in most field theories.

Example: 1+1-dimensional scalar fields with potential

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \quad \text{with} \quad V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2 \]

\[ \cdot \text{from } \mathcal{L}: \text{equivalent to motion of particle in Potential } -V(\phi) \]

\[ \cdot E < \infty \Rightarrow \phi \rightarrow \pm a, \quad T \rightarrow 0 \text{ for } x \rightarrow \infty \]
Solitons in 1+1d

Energy conservation: \( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 = V(\phi) \) leads to solutions

\[ \phi_{\pm}(x) = \pm a \cdot \tanh(\mu x) \]

(kink, anti-kink) with mass \( \mu = \sqrt{\lambda} a \) (symmetry breaking).
Stability and topological conservation law

Soliton mass scale $\sim$ symmetry breaking scale $\rightarrow$ heavy, unstable? 

$$\phi(\infty) - \phi(-\infty) = n \cdot 2a \quad \text{with} \quad n = 0, \pm 1$$

define current by $j_\mu(x) = \epsilon_{\mu\nu} \partial^\nu \phi$

$$\rightarrow Q = \int_{-\infty}^{\infty} j_0(x)dx = \int_{-\infty}^{\infty} (\partial_x \phi(x))dx = n(2a)$$

is the topologically conserved charge. Hence, $n$ is conserved and there should be no transitions between the states and no decay of the soliton to the vacuum.
Generalization to 3+1 dimensions

▶ "sphere" of minima: \( \mathcal{M}_0 = \{ \phi_i = \eta_i | V(\eta_i) = 0 \} \)

▶ finite energy: \( \phi_i^\infty = \lim_{R \to \infty} \phi_i(R\hat{r}) \in \mathcal{M}_0 \)

▶ \( H = \int d^3x \left[ \frac{1}{2} (\partial_0 \phi_i)^2 + \frac{1}{2} (\nabla \phi_i)^2 + V(\phi_i) \right] \) should converge!

▶ \( (\nabla \phi)^2 = \left( \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \varphi} \hat{\varphi} + \frac{1}{r \sin \varphi} \frac{\partial \phi}{\partial \theta} \hat{\theta} \right)^2 \) transverse \( \sim r^{-2} \)

▶ add gauge fields s.t. \( D_i \phi \sim r^{-2} \) and \( A_i^a \sim \phi_i \sim r^{-1} \) makes integral convergent (Derrick 1964)
The Georgi-Glashow-SO(3) model

Consider SO(3)-model with Higgs-Triplet in adj. representation:

\[ \mathcal{L} = \frac{1}{2} (D^\mu \phi)^a (D_\mu \phi)^a - \frac{1}{4} F^a_{\mu\nu} F^{a}_{\mu\nu} - V(\phi) \]

with the potential, fields and cov. derivatives given by:

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e \epsilon^{abc} A^b_\mu A^c_\nu \]

\[ (D_\mu \phi)^a = \partial_\mu \phi^a - e \epsilon^{abc} A^b_\mu \phi^c \]

\[ V(\phi) = \frac{\lambda}{4} (\phi^2 - a^2)^2 \]
Breakdown $SO(3) \sim SU(2) \rightarrow SO(2) \sim U(1)$ via ground state:

$$\phi = (0, 0, a)$$

gives 2 massive gauge bosons and a massless (photon). Now identify:

$$F^0_i = E^i, \quad F^{ij}_3 = -\epsilon^{ijk} B^k$$

Minima of potential form a sphere:

$$\mathcal{M}_0 = \{ \phi = \eta | \eta^2 = a^2 \}$$
t’Hooft-Polyakov Ansatz

We need $D_i \phi \sim r^{-2}$ and $A^a_i \sim \phi_a \sim r^{-1}$ for $H$ to converge. In addition we want $\phi^\infty_i = \lim_{R \to \infty} \phi_i = \eta_i = a \cdot \hat{r}$.

Ansatz by ’t Hooft and Polyakov (1974):

$$\phi_b = \frac{r^b}{er^2} H(aer) \quad A^i_b = -\epsilon_{bij} \frac{r^j}{er^2} (1 - K(aer)) \quad A^0_b = 0$$

Energy of system given by Hamiltonian:

$$E = \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2} \left[ \xi^2 \left( \frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left( \xi \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2} (K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \right]$$

with $\xi = aer$. 
Magnetic Monopoles

't Hooft-Polyakov Monopoles

't Hooft Polyakov Soliton

\[ E = \frac{4\pi a}{e} \int_{0}^{\infty} \frac{d\xi}{\xi^2} \left[ \xi^2 \left( \frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left( \xi \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2} (K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \right] \]

determine EOM for H,K by variation of E w.r.t H and K:

\[ \xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2 - 1) \]
\[ \xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \]

Our asymptotic condition \( \phi_i^\infty = \eta_i = a \cdot \hat{r} \) implies:

\[ H \sim \xi \text{ for } \xi \to \infty \]

The terms \( (K^2 H^2) \) and \( \frac{1}{\xi^2} (K^2 - 1)^2 \) imply:

\[ K \to 0 \text{ for } \xi \to \infty \text{ and } H \leq O(\xi) \quad K^2 - 1 \leq O(\xi) \]
The mass of this (static) solution is given by its Energy, the integral can be solved numerically and is \(\approx 1\), so we have:

\[
M \approx \frac{4\pi a}{e}
\]

so the mass is set by vev of the scalar field which is also a scale for symmetry breaking \(SO(3) \rightarrow SO(2)\).
The magnetic field

Plugging the Ansatz into $F_{a}^{ij}$ we get after several $\epsilon^{ijk}$-Terms cancel out:

$$F_{a}^{ij} = \epsilon^{ijk} \frac{r^{k} r^{a}}{er^{4}} = \frac{1}{aer^{3}} \epsilon^{ijk} r^{k} \phi_{a} \quad \text{with} \quad \phi_{a} = \frac{ar^{a}}{r}$$

so at large distances we get:

$$\vec{B} = \frac{g}{4\pi} \frac{\vec{r}}{r^{3}} \quad \text{with} \quad g = -\frac{4\pi}{e}$$
Size of the monopole

The monopole has finite size as can be seen below. For large $\xi$ we have $H \to \xi$ and $K \to 0$:

$$\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2 - 1) \Rightarrow \frac{d^2 K}{d\xi^2} = K$$

$$\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \Rightarrow \frac{d^2 h}{d\xi^2} = \frac{2\lambda}{e^2} h \quad h = H - \xi$$

with solutions:

$$K \sim e^{-\xi} = e^{-(ea)r} \quad H - \xi \sim e^{-(2\lambda)^{\frac{1}{2}}ar}$$

The prefactors are the masses $\mu = (2\lambda)^{\frac{1}{2}}a$ of the scalar and $M = ea$ of the gauge bosons after breaking the symmetry.
Summary

- At large distances behaves like Dirac monopole
- finite size and smooth structure, no point charge
- classical mass of order of symmetry breaking
- no singularities like Dirac string