The Optical Theorem and Partial Wave Unitarity

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What is the **Optical Theorem**?

**Motivation:**

The Optical Theorem:

\[
\text{Im } f(\theta = 0) = \frac{|k|}{4\pi} \sigma_{\text{tot}}
\]  

(1)

Is there a more general concept behind this?

⇒ Answer: **Yes**!

**short:** The optical theorem follows directly from the unitarity of the S-matrix.

⇒ How can we define the S-matrix in a physical meaningful way?

Let’s take a look at this...
Definition of one- and many-Particle-States

One-particle-state:

\[ |\phi(t)\rangle = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{\phi(k)}{\sqrt{2E(k)}} |k(t)\rangle \]  

(2)

Multi-particle-state:

\[ |\phi_1\phi_2...\rangle(t) \equiv |\{\phi_f\}\rangle(t) = \prod_f \int \frac{d^3k_f}{(2\pi)^3} \cdot \frac{\phi_f(k_f)}{\sqrt{2E_f}} |\{k_f\}\rangle(t) \]  

(3)

with \[ |\{k_f\}\rangle \equiv |k_1k_2...\rangle \]
Definition of one-and many-Particle-States

One-particle-state:

\[ |\phi\rangle(t) = \int \frac{d^3 k}{(2\pi)^3} \cdot \frac{\phi(k)}{\sqrt{2E(k)}} |k\rangle(t) \]  \hspace{1cm} (2)

Multi-particle-state:

\[ |\phi_1\phi_2...\rangle(t) \equiv |\{\phi_f\}\rangle(t) = \prod_f \int \frac{d^3 k_f}{(2\pi)^3} \cdot \frac{\phi_f(k_f)}{\sqrt{2E_f}} |\{k_f\}\rangle(t) \]  \hspace{1cm} (3)

with \( |\{k_f\}\rangle \equiv |k_1k_2...\rangle\)
The starting point

Question:
What is the transition probability for the scattering of two particles A and B into a many-particle-state $|\{\phi_f(t)\}\rangle$?

transition probability

$$ \Rightarrow \mathcal{P}(t_2, t_1) = \left| \left\langle \{\phi_f\}(t_2) | \phi_A \phi_B(t_1) \right\rangle \right|^2 $$

(4)

$\Rightarrow$ We have to compute $\left\langle \{p_f\}(t_2) | k_A k_B(t_1) \right\rangle$
Definition of the S-matrix

The S-matrix relates the incoming particles (coming from $t_1 \to -\infty$) and the outgoing particles (going to $t_2 \to +\infty$).

**Definition: The S-matrix:**

$$\langle \{p_f\} | S | k_A k_B \rangle \equiv \lim_{t \to +\infty} \langle \{p_f\}(t) | k_A k_B(-t) \rangle$$

**Some properties of the S-matrix:**

- $S$ have to be unitary ($S^\dagger S = 1$)
- $S$ is the identity if the particles do not interact between each other
Definition of the T-matrix

Definition: The T-matrix:

\[ S \equiv 1 + iT \] (6)

So, T is the interesting part of the interaction process (⇒ shows if something interacts).

Some properties of the T-matrix:

- S unitary ⇔ \( S^\dagger S = 1 \) ⇒ \( T^\dagger T = -i(T - T^\dagger) \)
- \( T = 0 \) if the particles do not interact between each other

Define the invariant matrix-element \( M(k_A k_B \rightarrow \{p_f\}) \) by

\[ \langle \{p_f\}|iT|k_A k_B \rangle = iM(k_A k_B \rightarrow \{p_f\}) (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f) \] (7)

\( M \) is proportional to the scattering amplitude \( f \).
How does the Cross-Section $\sigma$ depends on $\mathcal{M}$?

Transition probability

At first we decide the probability that the initial state $|\phi_A\phi_B\rangle$ becomes scattered into the final momentum-state $|\{p_f\}\rangle$ (that means in a small region $\prod_f d^3 p_f$).

Therefore:

$$P(AB \rightarrow \{p_f\}, b) = \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} |\langle \{p_f\}|\phi_A\phi_B\rangle(b)|^2$$

(8)

with $b$ as impact-parameter. Definition of the cross-section

$$\sigma = \frac{N_{sc}}{n_B N_A} = \int d^2 b P(b)$$

(9)

($N_{sc} \triangleq \#$ scattered particles, $n_B \triangleq$ number density, $N_A \triangleq \#$ incoming particles)
How does the Cross-Section $\sigma$ depends on $\mathcal{M}$?

\[
\sigma_{\text{tot}} = \left( \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^4(P - \sum_f p_f) \right) \times |\mathcal{M}(p_A p_B \rightarrow \{p_f\})|^2 \frac{2E_A 2E_B |v_A - v_B|}{2E_A 2E_B |v_A - v_B|}
\]

with $P = p_A + p_B$ and $v_i = k_i^2 / E_i$, $i = A, B$. Now, we can follow the optical theorem quite easy...
The Optical Theorem:

We know $S^\dagger S = 1 \Rightarrow T^\dagger T = -i(T - T^\dagger)$ an so we can calculate the scattering amplitude for the process $k_1 k_2 \rightarrow p_1 p_2$.

$$
\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \left( \prod_{f=1}^{n} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \langle p_1 p_2 | T^\dagger | \{q_f\} \rangle \langle \{q_f\} | T | k_1 k_2 \rangle
$$

(10)

This gives us

$$
-i(M(k_1 k_2 \rightarrow p_1 p_2) - M^*(p_1 p_2 \rightarrow k_1 k_2))
= \sum_n \left( \prod_{f=1}^{n} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) M(p_1 p_2 \rightarrow \{q_f\}) M^*(k_1 k_2 \rightarrow \{q_f\})
\cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f)
$$

(11)
The optical theorem relates the forward scattering amplitude to the cross-section.

\[ 2 \text{Im} \mathcal{M}(k_1 k_2 \rightarrow k_1 k_2) = \sum_n \left( \prod_{f=1}^{n} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^4(k_1 + k_2 - \sum_f q_f) \right) \]

\[ \equiv \int d\Pi_n \]

\[ \times |\mathcal{M}(k_1 k_2 \rightarrow \{q_f\})|^2 \]
Put in the relation for the total cross-section

\[
2 \text{Im } M(k_1k_2 \rightarrow k_1k_2) = 2E_A 2E_B |v_A - v_B| \sigma_{\text{tot}}
\]  

(12)

go into the CM-system

\[
(p_A + p_B = 0 \Rightarrow E_{\text{CM}} = E_A + E_B, \quad P_{\text{CM}} = p_A = -p_B)
\]

Optical Theorem (Standardform)

\[
\text{Im } M(k_1k_2 \rightarrow k_1k_2) = 2E_{\text{CM}} P_{\text{CM}} \sigma_{\text{tot}}
\]  

(13)
The Optical Theorem for Feynman Diagrams

We can derive $\mathcal{M}$ by the Feynman rules.

⇒ the virtual particles of the propagator yields an imaginary part $i\varepsilon$ if they go on-shell.

Let’s check

$$-i(\mathcal{M}(k_1 k_2 \rightarrow p_1 p_2) - \mathcal{M}^*(k_1 k_2 \rightarrow p_1 p_2))$$

$$= \sum_f \int d\Pi_f \mathcal{M}(p_1 p_2 \rightarrow \{q_f\}) \mathcal{M}^*(k_1 k_2 \rightarrow \{q_f\})$$

$$\cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f)$$

for $\phi^4$-theory diagrams

threshold energy $s_0$ is needed for production of the lightest multi-particle state ($s = E_{CM}^2$, Mandelstam-variable).
The Optical Theorem for Feynman Diagrams

Properties of $\mathcal{M}$

\[
\mathcal{M}(s) = [\mathcal{M}(s^*)]^* \quad s < s_0 \quad (\mathcal{M}(s) \text{ analytic!})
\]

\[\Rightarrow \quad \text{Re } \mathcal{M}(s + i\epsilon) = \text{Re } \mathcal{M}(s - i\epsilon), \quad s > s_0\]

\[\text{Im } \mathcal{M}(s + i\epsilon) = -\text{Im } \mathcal{M}(s - i\epsilon), \quad s > s_0 \Rightarrow \text{discontinuity}\]

Attention!

- $\phi^4$-theory
- \Rightarrow the simplest diagram in our case is a one loop diagram
  (order $\propto \lambda^2$, $s_0 = 2m$)
- the generalization of the result for multi-loop diagrams has been proven by Cutkosky
- \Rightarrow Cutting Rules

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The Optical Theorem for Feynman Diagrams

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Attention!

- $\phi^4$-theory
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- the generalization of the result for multi-loop diagrams has been proven by Cutkosky
- $\Rightarrow$ Cutting Rules
Consider the one-loop diagram

\[ \frac{k}{2} - q \rightarrow \frac{k}{2} + q \quad k = k_1 + k_2 \]
The Optical Theorem for Feynman Diagrams

Loop-correction

\[ i\delta M = \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(k/2 - q)^2 - m^2 + i\epsilon} \cdot \frac{1}{(k/2 + q)^2 - m^2 + i\epsilon} \]  

(14)

Some properties of \( \delta M \):

- For \( k_0 < 2m \) the integral can be calculated and then we increasing \( k_0 \) by analytical continuation
- !! We want to verify, that the integral has a discontinuity across the real axis for \( k_0 > 2m \) !!

\[ \Rightarrow \text{go into the CM-system } k = (k_0, 0) \]
⇒ we obtain four poles \( (E_q^2 = |q|^2 + m^2) \)

\[
q^0 = \frac{1}{2} k^0 \pm (E_q - i\epsilon), \quad q^0 = -\frac{1}{2} k^0 \pm (E_q - i\epsilon)
\]

⇒ only the pole at \( q_0 = -\frac{1}{2} k^0 + E_q - i\epsilon \) will contribute to the discontinuity (close the integration contour in the lower half plane!)

⇒ replace: \[
\frac{1}{(k/2+q)^2-m^2+i\epsilon} \rightarrow -2\pi i\delta((k/2 + q)^2 - m^2)
\]
under the \(dq_0\)-integral
The Optical Theorem for Feynman Diagrams

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\[ q^0 = \frac{1}{2} k^0 \pm (E_q - i\epsilon), \quad q^0 = -\frac{1}{2} k^0 \pm (E_q - i\epsilon) \]

\[-\frac{1}{2} k^0 - E_q \quad \frac{1}{2} k^0 - E_q \quad \frac{1}{2} k^0 + E_q \quad -\frac{1}{2} k^0 + E_q\]

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⇒ replace:

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\frac{1}{(k/2+q)^2-m^2+i\epsilon} \rightarrow -2\pi i \delta((k/2+q)^2 - m^2)
\]

under the \(dq_0\)-integral.
The Optical Theorem for Feynman Diagrams

\[ i\delta M = -2\pi i \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{(k^0 - E_q)^2 - E_q^2} \]

\[ = -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_m^\infty \text{d}E_q E_q |q| \frac{1}{2E_q} \frac{1}{k^0(k^0 - 2E_q)} \]

**Properties**

- \( E_q = k^0/2 \) is a pole of the integrand
- if \( k^0 < 2m \) the pole doesn’t lie in the integration contour
  \( \Rightarrow \mathcal{M} \) is real
- if \( k^0 > 2m \) the pole does lie in the integration contour
  \( \Rightarrow k^0 \) has a small positive or negative imaginary part
The Optical Theorem for Feynman Diagrams

\[ i \delta \mathcal{M} = -2\pi i \lambda^2 \frac{1}{2} \int \frac{d^3 q}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{(k^0 - E_q)^2 - E_q^2} \quad (15) \]

\[ = -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_{m}^{\infty} dE_q E_q \frac{1}{|q|} \frac{1}{2E_q} \frac{1}{k^0(k^0 - 2E_q)} \quad (16) \]

**properties**

- \( E_q = k^0/2 \) is a pole of the integrand
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The Optical Theorem for Feynman Diagrams

Integration contour:

\[ \int \frac{1}{k^0 - 2E_q \pm i\epsilon} = \mathcal{P} \frac{1}{k^0 - 2E_q} \mp i\pi\delta(k^0 - 2E_q) \quad (17) \]

⇒ Thus, the integral has a discontinuity between \( k^2 + i\epsilon \) and \( k^2 - i\epsilon \)!
Thus, the integral has a discontinuity between $k^2 + i\epsilon$ and $k^2 - i\epsilon$!

This is equivalent to replacing the original propagator by a delta-distribution:

$$
\frac{1}{(k/2 - q)^2 - m^2 + i\epsilon} \rightarrow -2\pi i\delta((k/2 - q)^2 - m^2)
$$
The Optical Theorem for Feynman Diagrams - Cutting Rules

**Cutting Rules:**

Look again at

\[ i\delta M = \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(k/2 - q)^2 - m^2 + i\epsilon} \frac{1}{(k/2 + q)^2 - m^2 + i\epsilon} \]

and relabel the momenta at the two propagators with \( p_1 \) and \( p_2 \).

\( \Rightarrow p_1 = k/2 - q, \ p_2 = k/2 + q. \)

1. Replace:

\[ \int \frac{d^4q}{(2\pi)^4} = \int \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} (2\pi)^4 \delta(4)(p_1 + p_2 - k) \]

\[ \Rightarrow i\delta M = \frac{\lambda^2}{2} \int \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} (2\pi)^4 \delta(4)(p_1 + p_2 - k) \]

\[ \times \frac{1}{p_1^2 - m^2 + i\epsilon} \frac{1}{p_2^2 - m^2 + i\epsilon} \]
The Optical Theorem for Feynman Diagrams

The Optical Theorem for Feynman Diagrams - Cutting Rules

2. Replace:

\[
\frac{1}{p_i^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta(p_i^2 - m^2)
\]

This gives us in order \( \lambda^2 = |M(k)|^2 \)

\[
2i \text{Im} \delta M(k) = \frac{i}{2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} |M(k)|^2 (2\pi)^4 \delta(p_1 + p_2 - k) \quad (18)
\]

\[
2i \text{Im}
\]

\[
= \frac{i}{2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} \left| \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} \right|^2 \delta^{(4)}(p_1 + p_2 - k)
\]

this verifies the optical theorem to order \( \lambda^2 \) in \( \phi^4 \)-theory
The Optical Theorem for Feynman Diagrams - Cutting Rules

Cutting Rules

1. Cut through the diagram in \textit{all possible} ways such that the cut propagator can simultaneously be put \textit{on shell}.

2. For each cut, replace \( \frac{1}{(p^2 - m^2 + i\epsilon)} \rightarrow -2\pi i\delta(p^2 - m^2) \) in each cut propagator, then perform the loop integrals.

3. Sum the contributions of all possible cuts.

\textit{Cutkosky} proved this method in general.

Using these cutting rules, it is possible to check the optical theorem for all orders in perturbation theory.
Cutting Rules

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Cutkosky proved this method in general. Using these cutting rules, it is possible to check the optical theorem for all orders in perturbation theory.
The Optical Theorem for Feynman Diagrams - Example

Bhabha-scattering:

\[(a) \quad 2 \text{Im} \left( \begin{array}{c}
\end{array} \right) = \int d\Pi \left| \begin{array}{c}
\end{array} \right|^2\]

\[(b) \quad 2 \text{Im} \left( \begin{array}{c}
\end{array} \right) = \int d\Pi \left| \begin{array}{c}
\end{array} \right|^2\]

Two contributions to the optical theorem for Bhabha-scattering.
Partial Wave Unitarity

For the $M$-Matrix we have found

$$-i(M(k_1k_2 \rightarrow p_1p_2) - M^*(p_1p_2 \rightarrow k_1k_2))$$

$$= \sum_n \left( \prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) M(p_1p_2 \rightarrow \{q_f\})M^*(k_1k_2 \rightarrow \{q_f\})$$

$$\cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f) \quad (19)$$

Choose particle $k_2$ at rest
Consider spinless particles $\Rightarrow \phi$-independence
$\Rightarrow$ scattering angle $\theta \Rightarrow M(k_1k_2 \rightarrow p_1p_2) = M_{ij}(s, \theta)$ where
$i = k_1 + k_2$ and $j = p_1 + p_2$ denotes the initial- and final-state.
Partial Wave Unitarity

\[
\begin{align*}
\Rightarrow M_{ij}(s, \theta) & \equiv 8\pi s^{1/2} f_{ij}(s, \theta) \\
& = 8\pi s^{1/2} \sum_{l=0}^{\infty} (2l + 1) M_{ij,l}(s) P_l(\cos \theta)
\end{align*}
\]  

(20)

(21)

with \( f_{ij}(s, \theta) \) as the scattering amplitude.

If we put \( f_{ii}(s, 0) \equiv f(0) \)

Prove: Optical Theorem

\[
\text{Im } f(0) = \left| \frac{P_{\text{CM}}}{4\pi} \right| \sigma_{\text{tot}}
\]

(22)
Two-Particle Partial Wave Unitarity:
If we consider only elastic scattering \((i=j)\)

\[
\text{Im } \mathcal{M}_1 = \sum_k p_k |\mathcal{M}_{k,1}|^2 \tag{23}
\]

and that all \(k\)-channels are closed at low energies \((p_k = p)\)

\[
\text{Im } \mathcal{M}_1 = p |\mathcal{M}_1|^2 \tag{24}
\]

\[
\Rightarrow \mathcal{M}_l = \frac{1}{p} e^{i\delta_l} \sin \delta_l
\]

where \(\delta_l\) denotes the scattering-phase for the \(l\)-th partial wave
Partial Wave Unitarity

The differential cross-section is in general given by

$$\frac{d\sigma_{ij}}{d\Omega} = \frac{1}{16\pi^2} \frac{p'}{p} \frac{1}{4s} |M_{ij}(s, \theta)|^2$$  \hspace{1cm} (25)

Using

$$M_{ij}(s, \theta) = 8\pi s^{1/2} \sum_{l=0}^{\infty} (2l + 1)M_{ij,l}(s)P_l(\cos \theta)$$
we get

\[
\sigma_{ij} = 4\pi \frac{p'}{p} \sum_l (2l + 1)|M_{ij,l}|^2 \equiv \sum_l \sigma_{ij,l}
\]  

For pure elastic scattering at low energies

Partial total cross-section

\[
\sigma_l = \frac{4\pi}{p^2} (2l + 1) \sin^2 \delta_l
\]
Two-Particle Partial Wave Unitarity

Partial Wave Unitarity

And for the case: A and B carry spin

Partial total cross-section for spin 1/2 particles

\[
\sigma_j = 4\pi \frac{2j + 1}{(2s_1 + 1)(2s_2 + 1)} \sum_{\lambda_1' \lambda_2' \lambda_1 \lambda_2} |M_j(\lambda_1' \lambda_2'; \lambda_1 \lambda_2; s)|^2 \quad (28)
\]

we \( \lambda_i \) are the initial and \( \lambda'_i \) the final helicities
short outlook: $W^+ W^+$-scattering $\Rightarrow$ need higgs-boson!
It is possible to show

\[-i\mathcal{M}_{\gamma Z^0+\pi^0}(s) = -ig^2 \left( \frac{s}{M_W^2} \right) + 2 \right] \propto s \quad (29)\]

but from the optical theorem follows for an large \(s\)

\[|\mathcal{M}(s)| < 16\pi \frac{q^2}{t_0} (\ln s)^2 \text{ (result by [ItZu])} \quad (30)\]

\[\Rightarrow \text{there must be a counter-term in eq. (29) to cancel the } s\]
put in the Higgs-Boson $h_0$

$$-i\mathcal{M}(s) = -i(\mathcal{M}_{\gamma+Z^0+\times}(s) + \mathcal{M}_{h_0}(s)) = -ig^2 \left[ 4 + \frac{1}{2} \left( \frac{M_{h_0}}{M_W} \right)^2 \right]$$

⇒ OK!
1. We have proved, that the optical theorem follows directly from the unitarity of the S-matrix.

2. Proving the optical theorem for Feynman diagrams in $\phi^4$-theory we had found the cutting rules.

3. We have derived an equation for the partial total cross-section for bosonic and fermionic particles from the principles of the optical theorem.
1. We have proved, that the optical theorem follows directly from the unitarity of the S-matrix.

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Summary

1. We have proved, that the optical theorem follows directly from the unitarity of the S-matrix.
2. Proving the optical theorem for Feynman diagrams in $\phi^4$-theory we had found the cutting rules.
3. We have derived an equation for the partial total cross-section for bosonic and fermionic particles from the principles of the optical theorem.
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