QFT at finite Temperature

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   - Classical Partition Function
   - The Quantum Mechanical Partition Function
   - High Temperature Limit
Content

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2. Landau-Ginzburg Theory
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Structure

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The Classical Partition Function

The Classical Partition Function is

\[ Z = \sum_i e^{-\beta E_i} = \prod_n \int dp_n dq_n e^{-\beta E(p, q)} \]

Where \( E(p, q) = \sum_n (1/2m)p_n^2 + V(\{q_n\}). \)
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Now in the field theoretical limit:

<table>
<thead>
<tr>
<th>discrete</th>
<th>( n \in \mathbb{Z} )</th>
<th>( \sum_n )</th>
<th>integrals</th>
<th>( \prod_n \int dq_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>continous</td>
<td>( x \in \mathbb{R}^d )</td>
<td>( \int d^d x )</td>
<td>integral</td>
<td>( \int \mathcal{D}\varphi )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>parameter</th>
<th>( n \in \mathbb{Z} )</th>
<th>( x \in \mathbb{R}^d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>particles</td>
<td>( q_n )</td>
<td>field</td>
</tr>
<tr>
<td>sum</td>
<td>( \sum_n )</td>
<td>( \int d^d x )</td>
</tr>
</tbody>
</table>
Path Integral

We get $V(\{q_n\}) \rightarrow \frac{1}{2}(\partial \varphi)^2 + V(\varphi)$.

This leads to the Euclidean path integral in $d$ Dimensions

$$Z = \int \mathcal{D}\varphi \ e^{-\frac{1}{\hbar} \int d^d x \left( \frac{1}{2}(\partial \varphi)^2 + V(\varphi) \right)}$$

where $\beta$ is replaced by $\frac{1}{\hbar}$.
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**Result**

Euclidean QFT in $d$-dimensional spacetime is equivalent to classical statistical mechanics in $d$-dimensional space.
Quantum Statistical Mechanics

The quantum partition function is

\[ Z = \text{tr} \left( e^{-\beta H} \right) = \sum_n \langle n | e^{-\beta H} | n \rangle \]

This looks like

\[ \langle F | e^{-iHt} | I \rangle = \int \mathcal{D}q \ e^{i \int_0^t d\tau L(q)} \quad \text{where} \quad q(0) = I, \ q(t) = F \]

where we have \( \beta \) instead of \( it \) and \( q(0) = q(\beta) \) because of \( I = F \).
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\[ Z = \oint \mathcal{D}\varphi \, e^{-\int_0^\beta d\tau \int d^Dx \mathcal{L}(\varphi)} \]

with \( D \) the number of space dimensions. The \( \oint \) is supposed to indicate that \( \varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta) \).
The $T = 0$ Limit

For $T \to 0$, that is $\beta \to \infty$, we get

$$Z = \int \mathcal{D}\varphi \; e^{-\int d^{D+1}x \mathcal{L}(\varphi)}$$

the normal euclidean path integral in $(D+1)$ dimensions.
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the normal euclidean path integral in $(D+1)$ dimensions.

Result

Euclidean QFT in $(D+1)$-dimensional spacetime is equivalent to quantum statistical mechanics in $D$-dimensional space in the low temperature limit.
Feynman Rules

- Assume Fourier transformation of time $e^{i\omega \tau}$
- $\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta) \Rightarrow \omega_n = \frac{2\pi n}{\beta}$ where $n \in \mathbb{Z}$
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Propagator: $\frac{1}{\omega^2 + k^2} \rightarrow \frac{1}{(2\pi T)^2 n^2 + k^2}$

$T \rightarrow \infty \Rightarrow$ only contribution for $n = 0$
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Result

Euclidean QFT in D-dimensional spacetime is equivalent to high temperature quantum statistical mechanics in D-dimensional space. Thus we get the *classical limit* for high temperatures.
Phase Transitions

Definitions and Question

An *n*-th order phase transition is a thermodynamic state in which an *n*-th derivative of the potential $F$ has a discontinuity while lower order derivatives are continuous.

Consider a first order phase transition with a discontinuity in $\Psi = \left(\frac{\partial F}{\partial E}\right)_T$ which only occurs below a certain temperature $T_c$. Call $\Psi$ the order parameter. Call $E$ the exciter. Call the state $(T = T_c, E = 0)$ a critical point.

What is the $T$-dependence of the order parameter below $T_c$?
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What is the $T$-dependence of the order parameter below $T_c$?
Critical Exponents

\[ \tau = \frac{T - T_c}{T_c} \]

**Definition**

Describe the \( T \)-dependence by power laws

- \( \Psi(T) = \left( \frac{\partial F}{\partial E} \right)_T \propto |\tau|^\beta \)
- \( \chi(T) = \left( \frac{\partial^2 F}{\partial E^2} \right)_T \propto |\tau|^{-\gamma} \)
- \( c_E(T) = \left( \frac{\partial^2 F}{\partial T^2} \right)_E \propto |\tau|^{-\alpha} \)

The powers \( \alpha, \beta \) and \( \gamma \) are then called **critical exponents**. They give us a full characterization of the relevant thermodynamics at the critical point. Now how can we compute them?
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Notation for densities:

\[ \Psi = \int d^3 x \, \psi \quad F = \int d^3 x \, f \]
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Ansatz

Look at a small region around the critical point. Ask for \( \psi \to -\psi \) symmetry. Taylor expansion in \( \psi \):

\[ f = f_0 + a|\psi|^2 + b|\psi|^4 + ... \]

\( b > 0 \) is requested for stability of the system.
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\( b > 0 \) is requested for stability of the system. Comparison to Higgs-Potential:

\[ V = \mu^2|\phi|^2 + \lambda|\phi|^4 \]

\( \phi \) has two minima for \( \mu^2 < 0 \) at \( \pm \sqrt{-\frac{\mu^2}{2\lambda}} \)
The First Critical Exponent

We want one minimum for $T > T_c$ but two minima for $T < T_c$. This can only be achieved by $a$ being $T$-dependent

$$a = \sum_{n=-\infty}^{\infty} a_n \tau^n$$
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- minimal power $n_{min}$ dominates, therefore it must be odd
- $\exists n < 0 : a_n \neq 0 \Rightarrow$ first order phase transition, not wanted
- we require $n_{min} \in \mathbb{N}$ odd. For simplicity: $n_{min} = 1$
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Result

$$|\psi| = \sqrt{\frac{-a}{2b}} = \sqrt{\frac{a_1}{2b}} |\tau|^{0.5} \Rightarrow \beta = 0.5$$
The Other Critical Exponents

Plugging in the result for $\psi(\tau)$ we get

$$f \propto \tau^2 \Rightarrow c_E(T) \propto \left( \frac{\partial^2 f}{\partial T^2} \right)_E \propto |\tau|^0$$

this means $\alpha = 0$. 
The Other Critical Exponents

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this means $\alpha = 0$. To calculate $\gamma$ we must do a Legendre transformation to fix $E$ and not $\psi$ externally.

$$g = a|\psi|^2 + b|\psi|^4 - \psi E$$

Then we differentiate w.r.t $E$ on both sides and use $\psi = \frac{\partial g}{\partial E}$ and $\chi = \frac{\partial \psi}{\partial E}$. At last setting $E = 0$ we get

$$\chi = \frac{1}{a + 2b|\psi|^2} \propto |\tau|^{-1} \Rightarrow \gamma = 1$$
Second Order Phase Transition

We have at last calculated the critical exponents

- $\alpha = 0$
- $\beta = 0, 5$
- $\gamma = 1$

The last critical exponent leads to a discontinuity of $\chi(T) = \left( \frac{\partial^2 g}{\partial E^2} \right)_T$ so we have by definition a phase transition of second order in the critical point.
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By considering a space dependent order parameter we can compute a correlation function $\langle \psi(x)\psi(0) \rangle$ which goes like $e^{-x/\xi}$ where $\xi = |\tau|^{-\nu}$ is the correlation length with critical exponent $\nu = 0, 5$. 
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Introduction

For the superconductor
- The order parameter is $\psi$.
- The external magnetic field is $H$.
- The conjugate of $H$ is $B$, the magnetic field inside the superconductor.
- The order parameter $\psi(x)$ can vary in space.

We get an energy term for $B$ which is $(F_{ij})^2$. The space dependence of $\psi$ gives a term $|\vec{\nabla}\psi|^2$. Introducing a gauge field for a charged $\psi$ we get $\left| (\vec{p} - e^* \vec{A})\psi \right|^2$. 
Potential for the Superconductor

The total energy is

\[ f - f_0 = ((\partial_j - ie^* A_j)\psi)^+((\partial_j - ie^* A_j)\psi) + a|\psi|^2 + b|\psi|^4 + \frac{1}{4} F_{ij} F_{ij} + ... \]
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In comparison, the Higgs-Lagrangian is

\[ \mathcal{L}_{\text{Higgs}} = ((\partial_\mu - ieA_\mu)\phi)^+((\partial^\mu - ieA^\mu)\phi) - \mu^2|\phi|^2 - \lambda|\phi|^4 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

Only difference: Euclidean ↔ Minkowski Space
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Thus we get as energy in the superconducting case

\[ f - f_0 = \left( e^* \sqrt{-\frac{a}{2b}} \right)^2 \vec{A}^2 + \frac{1}{4}(F_{ij})^2 + \ldots \]
Meissner Effect

- For $\vec{B}$ constant we have $\vec{A}^2(\vec{x}) = \frac{\vec{B}^2 \vec{x}^2 \sin^2 \theta}{4}$
- Thus the energy density rises quadratically with the distance.
- The $\vec{B}$-field is expelled. This is called Meissner Effect.
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Result

By spontaneous symmetry breaking we get a term $\propto A^2$ in the Lagrangian, which resembles very much the gauge boson mass terms we know from the Higgs mechanism in particle physics. In fact Landau-Ginzburg theory was developed long before the Higgs mechanism. It can be translated to particle physics due to the equivalence between statistical mechanics and QFT which we saw before.
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Critical Point and $\phi^4$-Theory

Consider the euclidean Lagrangian for a $\phi^4$-Theory in $d$ dimensions:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \rho_m M^2 \phi^2 + \frac{1}{4} \lambda M^{d-4} \phi^4$$

where $M$ is the renormalization scale. $\rho_m$ and $\lambda$ are then dimensionless.
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where $M$ is the renormalization scale. $\rho_m$ and $\lambda$ are then dimensionless.

- Similar to Landau-Ginzburg energy for a ferromagnet
- $\Rightarrow$ we can look at the renormalization group for a $\phi^4$-Theory and use the results to describe our critical point.
- For $m = 0$ we have a fixed point of the renormalization group:

$$\lambda = \begin{cases} 
0 & \text{for } d \geq 4 \\
\lambda_* = \frac{16\pi^2}{3} (4 - d) & \text{for } d < 4
\end{cases}$$
For $d < 4$: fixed point = critical point!
Consider only $m \approx 0$, that means $T \approx T_c$ because the mass evolves away from the fixed point
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Solve the Callan-Symanzik equation in $d < 4$

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + \beta m \frac{\partial}{\partial \rho_m} + n \gamma\right] G^{(n)} = 0$$

Solution $\bar{\rho}_m(p) = \rho_m \left(\frac{M}{p}\right)^{\frac{1}{\nu}}$ where $\frac{1}{\nu} = 2 - \frac{4-d}{3}$
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Correlation length $\xi \sim p_0^{-1}$ where $\bar{\rho}_m(p_0) = 1$

This gives $\xi \sim \rho_m^{-\nu} \sim |\tau|^{-\nu}$ with $\nu = 0, 6$ for $d = 3$

For comparison, the measured value is $\nu \approx 0, 64$ so we have got a more realistic critical exponent here than in Landau-Ginzburg theory ($\nu = 0, 5$).
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Up to now we considered only first order in $(d - 4)$. Much more realistic results are achieved in higher orders.
Central Message: QFT is equivalent to statistical mechanics.

Landau-Ginzburg theory describes second order phase transitions by \( T \)-dependent symmetry breakdown. It was adopted in the Higgs effect.

In superconductivity we can use Landau-Ginzburg to explain the Meissner effect.

The renormalization group can be used in statistical mechanics to calculate critical exponents that describe a second order phase transition.
Literature

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