QFT at finite Temperature

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Structure

1. Path Integral and Partition Function
   - Classical Partition Function
   - The Quantum Mechanical Partition Function
   - High Temperature Limit
2. Landau-Ginzburg Theory
3. Application to Superconductivity
4. Outlook: Renormalization Group

The Classical Partition Function

The Classical Partition Function is

$$Z = \sum_i e^{-\beta E_i} = \prod_n \int dp_n \, dq_n \; e^{-\beta E(p,q)}$$

Where $E(p,q) = \sum_n (1/2m) p_n^2 + V(\{q_n\})$.

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Now in the field theoretical limit:

discrete		→	continous	
parameter	$n \in \mathbb{Z}$	→	parameter	$x \in \mathbb{R}^d$
particles	q_n	→	field	$\varphi(x)$
sum	$\sum_n$	→	integral	$\int d^d x$
integrals	$\prod_n \int dq_n$	→	path integral	$\int \mathcal{D}\varphi$

Path Integral

We get $V(\{q_n\}) \rightarrow \frac{1}{2}(\partial\varphi)^2 + V(\varphi)$.

This leads to the Euclidean path integral in d Dimensions

$$Z = \int \mathcal{D}\varphi \; e^{-\frac{1}{\hbar}\int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + V(\varphi)\right)}$$

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Result

Euclidean QFT in d-dimensional spacetime is equivalent to classical statistical mechanics in d-dimensional space.

Quantum Statistical Mechanics

The quantum partition function is

$$Z = tr\left(e^{-\beta H}\right) = \sum_n \langle n| e^{-\beta H} |n\rangle$$

This looks like

$$\langle F| e^{-iHt} |I\rangle = \int \mathcal{D}q \; e^{i\int_0^t d\tau L(q)} \quad \text{where} \quad q(0) = I, \; q(t) = F$$

where we have β instead of it and $q(0) = q(\beta)$ because of $I = F$.

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$$Z = \oint \mathcal{D}\varphi \; e^{-\int_0^\beta d\tau \int d^D x \mathcal{L}(\varphi)}$$

with D the number of space dimensions. The $\oint$ is supposed to indicate that $\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta)$.

The $T = 0$ Limit

For $T \to 0$, that is $\beta \to \infty$, we get

$$Z = \int \mathcal{D}\varphi \; e^{-\int d^{D+1}x \mathcal{L}(\varphi)}$$

the normal euclidean path integral in (D+1) dimensions.

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Result

Euclidean QFT in (D+1)-dimensional spacetime is equivalent to quantum statistical mechanics in D-dimensional space in the low temperature limit.

Feynman Rules

- Assume Fourier transformation of time $e^{i\omega\tau}$
- $\varphi(\vec{x},0) = \varphi(\vec{x},\beta) \quad \Rightarrow \quad \omega_n = \frac{2\pi n}{\beta}$ where $n \in \mathbb{Z}$

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Result

Euclidean QFT in D-dimensional spacetime is equivalent to high temperature quantum statistical mechanics in D-dimensional space. Thus we get the classical limit for high temperatures.

Structure

Phase Transitions

Definitions and Question

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Consider a first order phase transition with a discontinuity in $\Psi = \left(\frac{\partial F}{\partial E}\right)_T$ which only occurs below a certain temperature T_c.

- Call Ψ the order parameter.
- Call E the exciter.
- Call the state $(T = T_c,\ E = 0)$ a critical point.

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What is the T-dependence of the order parameter below T_c?

Critical Exponents

$$\tau = \frac{T - T_c}{T_c}$$

Definition

Describe the T-dependence by power laws

- $\Psi(T) = \left(\frac{\partial F}{\partial E}\right)_T \propto |\tau|^{\beta}$
- $\chi(T) = \left(\frac{\partial^2 F}{\partial E^2}\right)_T \propto |\tau|^{-\gamma}$
- $c_E(T) = \left(\frac{\partial^2 F}{\partial T^2}\right)_E \propto |\tau|^{-\alpha}$

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The powers α, β and γ are then called critical exponents. They give us a full characterization of the relevant thermodynamics at the critical point. Now how can we compute them?

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Notation for densities:

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Ansatz

Look at a small region around the critical point. Ask for $\psi \to -\psi$ symmetry. Taylor expansion in ψ:

$$f = f_0 + a|\psi|^2 + b|\psi|^4 + \ldots$$

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$$f = f_0 + a|\psi|^2 + b|\psi|^4 + \ldots$$

$b > 0$ is requested for stability of the system. Comparison to Higgs-Potential:

$$V = \mu^2|\phi|^2 + \lambda|\phi|^4$$

ϕ has two minima for $\mu^2 < 0$ at $\pm\sqrt{\frac{-\mu^2}{2\lambda}}$

The First Critical Exponent

We want one minimum for $T > T_c$ but two minima for $T < T_c$. This can only be achieved by a being T-dependent

$$a = \sum_{n=-\infty}^{\infty} a_n \tau^n$$

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- minimal power n_{min} dominates, therefore it must be odd
- $\exists n < 0 : \; a_n \neq 0 \;\Rightarrow$ first order phase transition, not wanted
- we require $n_{min} \in \mathbb{N}$ odd. For simplicity: $n_{min} = 1$

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Result

$$|\psi| = \sqrt{\frac{-a}{2b}} = \sqrt{\frac{a_1}{2b}} \, |\tau|^{0,5} \quad \Rightarrow \quad \beta = 0,5$$

The Other Critical Exponents

Plugging in the result for $\psi(\tau)$ we get

$$f \propto \tau^2 \quad \Rightarrow \quad c_E(T) \propto \left(\frac{\partial^2 f}{\partial T^2}\right)_E \propto |\tau|^0$$

this means $\alpha = 0$.

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this means $\alpha = 0$. To calculate γ we must do a Legendre transformation to fix E and not ψ externally.

$$g = a|\psi|^2 + b|\psi|^4 - \psi E$$

Then we differentiate w.r.t E on both sides and use $\psi = \frac{\partial g}{\partial E}$ and $\chi = \frac{\partial \psi}{\partial E}$. At last setting $E = 0$ we get

$$\chi = \frac{1}{a + 2b|\psi|^2} \propto |\tau|^{-1} \quad \Rightarrow \quad \gamma = 1$$

Second Order Phase Transition

We have at last calculated the critical exponents

- $\alpha = 0$
- $\beta = 0,5$
- $\gamma = 1$

The last critical exponent leads to a discontinuity of $\chi(T) = \left(\frac{\partial^2 g}{\partial E^2}\right)_T$ so we have by definition a phase transition of second order in the critical point.

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By considering a space dependent order parameter we can compute a correlation function $\langle \psi(x)\psi(0) \rangle$ which goes like $e^{-x/\xi}$ where $\xi = |\tau|^{-\nu}$ is the correlation length with critical exponent $\nu = 0,5$.

Structure

Introduction

For the superconductor

- The order parameter is ψ.
- The external magnetic field is H.
- The conjugate of H is B, the magnetic field inside the superconductor.
- The order parameter $\psi(x)$ can vary in space.

We get an energy term for B which is $(F_{ij})^2$. The space dependence of ψ gives a term $|\vec{\nabla}\psi|^2$. Introducing a gauge field for a charged ψ we get $\left|(\vec{p} - e^*\vec{A})\psi\right|^2$.

Potential for the Superconductor

The total energy is

$$f - f_0 = ((\partial_j - ie^* A_j)\psi)^+((\partial_j - ie^* A_j)\psi) + a|\psi|^2 + b|\psi|^4 + \frac{1}{4}F_{ij}F_{ij} + \ldots$$

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In comparison, the Higgs-Lagrangian is

$$\mathcal{L}_{Higgs} = ((\partial_\mu - ieA_\mu)\phi)^+((\partial^\mu - ieA^\mu)\phi) - \mu^2|\phi|^2 - \lambda|\phi|^4 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

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Thus we get as energy in the superconducting case

$$f - f_0 = \left(e^* \sqrt{\frac{-a}{2b}}\right)^2 \vec{A}^2 + \frac{1}{4}(F_{ij})^2 + \ldots$$

Meissner Effect

- For $\vec{B}$ constant we have $\vec{A}^2(\vec{x}) = \frac{\vec{B}^2 \vec{x}^2 \sin^2\theta}{4}$
- Thus the energy density rises quadratically with the distance.
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- The $\vec{B}$-field is expelled. This is called Meissner Effect.

Result

By spontaneous symmetry breaking we get a term $\propto \vec{A}^2$ in the Lagrangian, which resembles very much the gauge boson mass terms we know from the Higgs mechanism in particle physics. In fact Landau-Ginzburg theory was developed long before the Higgs mechanism. It can be translated to particle physics due to the equivalence between statistical mechanics and QFT which we saw before.

Structure

Critical Point and ϕ^4-Theory

Consider the euclidean Lagrangian for a ϕ^4-Theory in d dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\rho_m M^2 \phi^2 + \frac{1}{4}\lambda M^{d-4}\phi^4$$

where M is the renormalization scale. ρ_m and λ are then dimensionless.

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- Similar to Landau-Ginzburg energy for a ferromagnet
- $\Rightarrow$ we can look at the renormalization group for a ϕ^4-Theory and use the results to describe our critical point.
- For $m = 0$ we have a fixed point of the renormalization group:

$$\lambda = \begin{cases} 0 & \text{for } d \geq 4 \\ \lambda_* = \frac{16\pi^2}{3}(4-d) & \text{for } d < 4 \end{cases}$$

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$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + \beta_m \frac{\partial}{\partial \rho_m} + n\gamma \right] G^{(n)} = 0$$

- Solution $\overline{\rho}_m(p) = \rho_m \left(\frac{M}{p} \right)^{\frac{1}{\nu}}$ where $\frac{1}{\nu} = 2 - \frac{4-d}{3}$

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- This gives $\xi \sim \rho_m^{-\nu} \sim |\tau|^{-\nu}$ with $\nu = 0,6$ for $d = 3$
- For comparison, the measured value is $\nu \approx 0,64$ so we have got a more realistic critical exponent here than in Landau-Ginzburg theory ($\nu = 0,5$).

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- For comparison, the measured value is $\nu \approx 0,64$ so we have got a more realistic critical exponent here than in Landau-Ginzburg theory ($\nu = 0,5$).
- Up to now we considered only first order in $(d - 4)$. Much more realistic results are achieved in higher orders.

Summary

- Central Message: QFT is equivalent to statistical mechanics.
- Landau-Ginzburg theory describes second order phase transitions by T-dependent symmetry breakdown. It was adopted in the Higgs effect.
- In superconductivity we can use Landau-Ginzburg to explain the Meissner effect.
- The renormalization group can be used in statistical mechanics to calculate critical exponents that describe a second order phase transition.

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