Physikalisches Institut
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Theoretische Physik

Exercise 1
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WS 01/02

Exercises on 'Elementary Particle Physics'

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This exercise is devoted to repeat some basic stuff about group theory.
Here and for all future we sum over indices appearing twice. Further we set $\hbar = c = 1$.

1. *Elementary definitions and properties of groups and algebras*

   (a) Clarify briefly the terms group, Lie group and Lie algebra. When is a Lie group called simple/semisimple? How do Lie groups and Lie algebras correspond to each other?

   (b) What is a group homomorphism, what a group isomorphism? When is a Lie group called compact?

   (c) What are the definitions of the groups $O(n)$, $U(n)$, $SO(n)$ und $SU(n)$? Where do these groups appear in physics?

2. *Representations of groups and algebras*

   (a) Define the term representation of a group. What are equivalent, reducible, completely reducible and irreducible representations and what is their meaning? How are these terms defined for Lie algebras? What are product representations? What is *Schur*s lemma?

   (b) Comment briefly: unitary representations are completely reducible (or irreducible respectively).

   (c) Explain the term adjoint representation.

3. $SU(2)$ *und* $SO(3)$

   (a) Give a convenient basis for the Lie algebra of $SU(2)$ and of $SO(3)$. Show, by constructing the algebras, that (the groups) $SU(2)$ and $SO(3)$ are locally isomorphic.

   (b) Examine the global correspondence of these groups, i.e. give an explicit construction of the homomorphism.

Homework

1. *Groups, representations and algebras*

   (a) Show that the *Jacobi* identity

   $$[X_i,[X_j,X_k]]+[X_j,[X_k,X_i]]+[X_k,[X_i,X_j]]=0 \tag{1}$$

   is fullfilled for arbitrary X_i ($[A,B]:=AB-BA$ is the commutator).

   (b) Show for the algebra

   $$[T_i,T_j]=if^{ijk}T_k, \tag{2}$$

   that the matrices $(\tilde{T^i})_{kj}=if^{ijk}$ generate a representation (the adjoint).
   Hint: Use the *Jacobi* identity (1).

   (c) Show: For D a representation of the group G (with elements g)

   i. the complex conjugated representation $D^*:=(D)^*$ is a representation as well.

   ii. $\tilde{D}(g)\equiv((D(g))^T)^{-1}$ is a representation. If v_i transforms under D und w^i under $\tilde{D}$ then v_iw^i is a group invariant.

   (d) For semisimple groups the generators of the corresponding algebra (2) can always be normalized as $tr(T_iT_j)=-\lambda\delta_{ij}$, $\lambda\in\mathbb{R}$. Show: The structure constants are totally antisymmetric in this basis.

2. *$SU(2)$ and $SO(3)$ once again*

   (a) Show that the fundamental representation **2** (that means two dimensional) of $SU(2)$ and its complex conjugated representation are equivalent.

   (b) Consider the product representation $\mathbf{3}\times\mathbf{3}$ of $SO(3)$. Compute its irreducible representations using

   i. methods of quantum mechanics, i.e. creation and annihilation operators, *Clebsch-Gordon* coefficients, etc.

   ii. the tensormethod.

Updated information about when and where the exercises take place can be found at the URL 'http://www.th.physik.uni-bonn.de/nilles/' (take the 'Exercises' link at the navigation bar).

Facing problems or questions please feel free to visit us: Holger (AVZ 110), Martin (AVZ 125).