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Exercise 3
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Exercises on 'Elementary Particle Physics'

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1. *More transformation properties of SU(N) spinors*

   Define a spinor ψ in the N dimensional representation (the fundamental) of $SU(N)$. It transforms like
   $$\psi \to \psi' = U\psi \ ,$$
   where $U \in \mathrm{SU(N)}$ (i.e. the elements of U are complex!). Consider a different spinor $\tilde{\psi}$ transforming under a representation $\tilde{U}$
   $$\tilde{\psi} \to \tilde{\psi}' = \tilde{U}\tilde{\psi} \ .$$
   We construct the term $\tilde{\psi}^T\psi$ and force it to be invariant under rotations
   $$(\tilde{\psi}')^T\psi' = (\tilde{U}\tilde{\psi})^T U\psi = \tilde{\psi}^T(\tilde{U}^T U)\psi \stackrel{!}{=} \tilde{\psi}^T\psi \ .$$
   With unitarity of U we find
   $$\tilde{U} = U^{-1T} = U^* \ .$$
   Therefore $\tilde{\psi}$ transforms exactly like ψ^* and $\psi^\dagger\psi$ is invariant. We call U^* the conjugated representation of U (Note that there are cases where both representations are equivalent, e.g. in the SU(2) case - cf. Exercise 1, Homework 2a).

   Denote the spinors ψ with upper indices ψ^i and $\tilde{\psi}^T = \psi^\dagger$ with lower indices ψ_i. Then the transformations take the form
   $$\psi'^i = U^i{}_j\psi^j$$
   $$\psi'_i = \psi_j (U^\dagger)^j{}_i$$

   (a) Show that δ^i_j is invariant under these transformations.

   (b) Show that $\epsilon^{i_1 \dots i_N}$ is invariant. Note, that therefore ϵ is a metric for spinors, i.e. we can use it to raise and lower indices!

   Analogous to higher order tensors one now can construct higher order spinors.

   (c) Show that a symmetric (antisymmetric) 2-spinor T^{ij} transforms into a symmetric (antisymmetric) 2-spinor again.

These parts are irreducible, i.e. a given 2-spinor can be decomposed according to $T^{ij} = T^{(ij)} + T^{[ij]}$, where (ij) and $[ij]$ denote symmetrization and antisymmetrization of the indices respectively. For higher order spinors (here corresponding to product representations) this gives us a nice way to extract irreducible representations, according to their transformation properties! We just have to symmetrize and antisymmetrize by contracting with the appropriate 'tensors' ϵ and δ. However, one has to take care about traces (why?).

Example: Consider the flavour SU(3) $q_i = (u, d, s) \sim \mathbf{3}$ and $\bar{q}^i = (\bar{u}, \bar{d}, \bar{s}) \sim \bar{\mathbf{3}}$. Then baryons are combinations $q_i q_j q_k \sim \mathbf{3} \times \mathbf{3} \times \mathbf{3}$ and mesons correspond to $q_i \bar{q}^k \sim \mathbf{3} \times \bar{\mathbf{3}}$.

2. *The SU(3) quark model*

   (a) Use the above method to show that the irreducible parts in $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}}_\mathbf{A} + \mathbf{6}_\mathbf{S}$ (A/S (anti-)symmetrized) are given by

$$\begin{aligned}\bar{\mathbf{3}}_\mathbf{A} &\sim \frac{1}{\sqrt{2}}\epsilon^{ijk} q_j q_k \ , \\ \bar{\mathbf{6}}_\mathbf{S} &\sim \frac{1}{\sqrt{2}}(q_i q_j + q_j q_i)\end{aligned}$$

   and the irreducible parts in $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$ by

$$\begin{aligned}\mathbf{1} &\sim \frac{1}{\sqrt{3}}\bar{q}^k q_k \ , \\ \bar{\mathbf{8}} &\sim \bar{q}^i q_j - \frac{1}{3}\delta^i_j \bar{q}^k q_k \ .\end{aligned}$$

   (b) Use the results of 2a) to express the irreps of

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8_s} + \mathbf{8_a} + \mathbf{10_S}$$

   by the quarks q_i. The lowercase indices s/a are used to distinguish the octets and tell if the states are (anti-) symmetric in the first two quarks.

   (c) It's well kown that the nucleons p, n are in an octet of SU(3). Compute the wavefunctions $p_{s,a}, n_{s,a} \in \mathbf{8}_{s,a}$ which possess the quantum numbers of p, n.

   (d) The spins of the quarks couple via

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{2_s} + \mathbf{2_a} + \mathbf{4_S} \ .$$

   Compute the $S_3 = 1/2$ states in $\mathbf{2_s}$ and $\mathbf{2_a}$.

   (e) Discuss the allowed $SU(3) \times SU(2)_{\text{spin}}$ baryon spectrum, if we demand complete symmetry for wavefunctions (without color) under permutations of the quark fields. For the nucleons $|p\rangle, |n\rangle$ the only possibilities are combinations of $(\mathbf{8_s}, \mathbf{2_s})$ and $(\mathbf{8_a}, \mathbf{2_a})$. Why? Show that only the combination $(\mathbf{8_s}, \mathbf{2_s}) + (\mathbf{8_a}, \mathbf{2_a})$ is completely symmetric. Construct the $SU(3) \times SU(2)_{\text{spin}}$ wavefunctions of the nucleons.

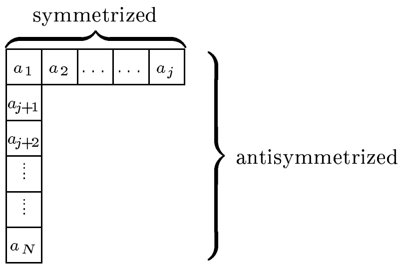
(f) In the constituent model the magnetic moment of the nucleons $N = p, n$ is given by

$$\mu(N) = \langle N \uparrow | \hat{\mu}\hat{\sigma}_3 | N \uparrow \rangle \ ,$$

where $\hat{\mu}|u\rangle = \mu_u|u\rangle$, $\hat{\mu}|d\rangle = \mu_d|d\rangle$ and $\hat{\sigma}_3|\uparrow\rangle = |\uparrow\rangle, \hat{\sigma}_3|\downarrow\rangle = -|\downarrow\rangle$. The quarks possess a magnetic moment $\mu_i = Q_i \frac{e}{2m_i}$. Assume that $m_u \approx m_d$ and use the wavefunctions constructed above to compute the ratio $\mu(p)/\mu(n)$.

3. *Young tableaux or product representations at Legoland*

For extracting the irreducible parts of product representations there is a very nice graphical method using *Young* tableauxs. This technique is mathematically well founded, but we will only give a 'physics for pharmacist' kind of introduction here[1] Consider an arbitrary tensor (or higher order spinor) $T_{a_1 \ldots a_N}$. We sort the indices according to if they are symmetric or antisymmetric. For every index we write a box $\square$ and we concatenate the boxes horizontally if the corresponding indices are symmetric and vertically if they are antisymmetric:



Further examples: $T_{[ij]}$: $\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$, $T_{(ij)}$: $\begin{array}{|c|c|}\hline i & j \\ \hline \end{array}$, $T_{(i[j)k]}$: $\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.

We only allow for so called standard *Young* tableauxs. That means that the tableaux must be left aligned and the number of boxes in the i th row must be greater or equal to the number of boxes in the $(i + 1)$ th row.

Let us apply this method for an **N** of SU(N), i.e. consider a **3** of SU(3) (a quark). It's given by q_i, one index, so it's *Young* tableaux is given by $\square$. Whats with an element of $\bar{3}$? We learned that we have to use ϵ to raise and lower indices, therefore an element $\bar{q}^i$ of $\bar{3}$ is represented by $\bar{q}^i \sim \epsilon^{ijk} q_j q_k$, i.e. we are left with two antisymmetric indices. These are of course written like $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$. This generalizes to the general **N** case.

Product representations: Consider the $\mathbf{3} \times \mathbf{3}$. Then we just combine the boxes in all possible ways but with the constraint that they must form standard *Young* tableauxs again, i.e.

$$\square \times \square \quad = \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} .$$

The resulting *Young* tableauxs correspond to irreducible representations and one can read of their dimensions. To do this we need two different ways of labeling.

[1]The interested student (who is not?) can find details e.g. in 'Greiner/Mueller Theoretische Physik Vol. 5' or in 'Fulton/Harris Representation theory - A first course' or in 'Cheng/Li Gauge theory of EPP'

i) Write N in the upper left corner. Fill up the first row from the left to the right with increasing numbers ($N+1$, $N+2$, etc.). Fill in the first column from top to bottom with decreasing numbers ($N-1$, $N-2$, etc.). Fill in the other boxes using the same scheme, e.g. if the box left to the box you want to label is labeled by K then you have to label with $K+1$...

If you labeled all boxes, multiply all numbers and denote the product by, say, s.

Example $N = 3$ again:

$$\begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \Rightarrow s = 3 \cdot 4 \cdot 2 = 24\ .$$

ii) The 'hook' law: Choose a box counting it once and count all boxes right to the box you choose. Add this number to the number of boxes under the box you choose. Write this number in the box you choose. Repeat this for every box of the tableaux. Again multiply all numbers and denote the product by, say, t.

Example:

$$\begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 3 & 1 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \Rightarrow t = 3 \cdot 1 \cdot 1 = 3\ .$$

Then the dimension d of the given irreducible representation is given by

$$d = \frac{s}{t}\ ,$$

in our example $d = 24/3 = 8$. Great, isn't it?

(a) Find the irreps of the product representations of exercise 2a), 2b) and 2d), but now using *Young* tableauxs. Have fun!