Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 4
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Exercises on Elementary Particle Physics

Prof. H.-P. Nilles

1. *The free real scalar field*

   (a) The Lagrangian density of a free real scalar field with mass m is given as

   $$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)\,.$$

   Show that the Euler Lagrange equation of this Lagrangian density leads to the Klein Gordon equation. Argue that the $f_{\vec{k}}, f^*_{\vec{k}}$ defined as

   $$f_{\vec{k}} = \exp(i\vec{k}\vec{x} - i\omega_{\vec{k}} t)$$

   with $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$ give a complete system of solutions for this equation.

   (b) Calculate the canonical conjugate momentum density $\pi = \partial\mathscr{L}/\partial\dot{\phi}$ and the momentum

   $$P_\mu = \int d^3x (\pi \partial_\mu \phi - \eta_{0\mu} \mathscr{L})$$

   (c) The quantisation condition for these fields is given by

   $$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0, \qquad [\pi(\vec{x}, t), \phi(\vec{x}', t)] = -i\delta^{(3)}(\vec{x} - \vec{x}')$$

   Expand the field in the following form

   $$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[a_{\vec{k}} f_{\vec{k}}(x) + a^\dagger_{\vec{k}} f^*_{\vec{k}}(x)\right]$$

   and find the commutation relations for $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. (*Hint: First show* $a_{\vec{k}} = \int d^3x e^{ikx}(\omega_{\vec{k}}\phi(x) + i\pi(x))$...) Express P_μ in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

   (d) Calculate the commutators $[P_\mu, a_{\vec{k}}]$ and $[P_\mu, a^\dagger_{\vec{k}}]$ and – using that – describe the action of $a_{\vec{k}}$ und $a^\dagger_{\vec{k}}$ on eigenstates $|p\rangle$ of P_μ (i.e. $P_\mu|p\rangle = p_\mu|p\rangle$). What is the meaning of the operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ and how does one define the vacuum $|0\rangle$ of this theory?

2. *The free Dirac field*

(a) The Dirac Lagrangian density is given by (with ψ a Dirac spinor)

$$\mathscr{L}_D = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \,.$$

Show that the corresponding action $S = \int \mathrm{d}^4x\, \mathscr{L}$ is Lorentz invariant.

(b) Show that the Euler Lagrange equation leads to the Dirac equation, this time, and calculate the canonical conjugate momentum density $\xi_\alpha = \partial\mathscr{L}/\partial\dot{\psi}_\alpha$.

(c) The mode expansion of the Dirac field is given by

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left(a^s_{\underline{p}} u_s(p) e^{ip.x} + b^{s\dagger}_{\underline{p}} v_s(p) e^{-ip.x}\right)$$

where s is a spinor index and (in the Dirac representation of the γ matrices)

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{(\underline{\sigma}.\underline{p})\chi_s}{E_p+m} \end{pmatrix} \quad , \quad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{(\underline{\sigma}.\underline{p})\chi_s}{E_p+m} \\ \chi_s \end{pmatrix}$$

with $p = (E_p, \underline{p})$ and $\chi_{\frac{1}{2}} = \binom{1}{0}$, $\chi_{-\frac{1}{2}} = \binom{0}{1}$. Why? Show that

$$\begin{aligned} \bar{u}_s(p)u_{s'}(p) &= 2m\delta_{ss'} = -\bar{v}_s(p)v_{s'}(p) \qquad \bar{u}_s(p)v_{s'}(p) = 0 \\ u^\dagger_s(p)u_{s'}(p) &= 2E_p\delta_{ss'} = v^\dagger_s(p)v_{s'}(p) \qquad u^\dagger_s(-p)v_{s'}(p) = v^\dagger_s(-p)u_{s'}(p) = 0\,. \end{aligned}$$

(d) Show that imposing the anticommutation relations

$$\{a^s_{\underline{p}}, a^{s'}_{\underline{p}'}\} = \{b^s_{\underline{p}}, b^{s'}_{\underline{p}'}\} = (2\pi)^3\,\delta^{(3)}(\underline{p} - \underline{p}')\,\delta^{ss'}$$

(and all other combinations vanishing) one can reproduce the standard canonical field quantisation prescription

$$\begin{aligned} \{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{x}',t)\} &= \{\xi_\alpha(\vec{x},t), \xi_\beta(\vec{x}',t)\} = 0\,, \\ \{\xi_\alpha(\vec{x},t), \psi_\beta(\vec{x}',t)\} &= -i\delta_{\alpha\beta}\,\delta^{(3)}(\vec{x} - \vec{x}')\,. \end{aligned}$$

(e) Calculate the mode expansion of the number operator

$$N = \int \mathrm{d}^3x\; \psi(x)^\dagger \psi(x)\,.$$

(*Hint: Use the identities of point (c) before..*).

3. *Spontaneous Symmetry Breakdown I.* This is supposed to be a basic introductory example to show what happens in a theory where we have a classical ground state which exhibits less symmetry than the Lagrangian.

   (a) Let $\phi = (\phi_1, \ldots \phi_n)$ be an n component scalar field with Lagrangian density
   $$\mathscr{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi) \quad , \quad V(\phi) = \frac{1}{8} g(\phi^2 - v^2)^2 \quad , g > 0 \, .$$
   Why is the symmetry group O(n)? Expand the potential and identify the standard ingredients to a Lagrangian density (kinetic term, mass term, interactions..) – what is different? Draw the potential in a suitable projection!

   (b) Calculate the equation which determines the classical ground state(s) (states with lowest energy, hence potential). This equation defines an $n-1$ dimensional sphere (S^{n-1}), which breaks the higher symmetry of the Lagrangian.

   (c) Pick one ground state $\phi_0 = (0, \ldots, 0, v)$ and calculate the effective Lagrangian with low excitations around that ground state by expanding around it
   $$\phi = (\tilde{\phi}, v + f) \quad , \quad \tilde{\phi} = (\phi_1, \ldots, \phi_{n-1}) \, .$$
   Give an interpretation for the different modes you find in the effectiv Lagrangian. What is the link to the Goldstone theorem.

   (d) What happens if one has the opposite sign in $V(\phi)$? (i.e. $V' = \frac{1}{8} g(\phi^2 + v^2)^2$)

4. *Spontaneous Symmetry Breakdown II.* A field theory is described in terms of the elements of a complex $N \times N$ matrix M by a Lagrangian
   $$\mathscr{L} = \mathrm{tr}(\partial^\mu M^\dagger \partial_\mu M) - \frac{1}{2}\lambda \, \mathrm{tr}(M^\dagger M M^\dagger M) - k \, \mathrm{tr}(M^\dagger M) \, ,$$
   where tr denotes the matrix trace and $\lambda > 0$.

   (a) Show that this theory is invariant under the symmetry group $U(N) \times U(N)/U(1)$ for transformations given by $M \to AMB^{-1}$, $A, B \in U(N)$, and where the $U(1)$ corresponds to $A = B = e^{i\theta}$.

   (b) Show that for $k < 0$ spontaneous symmetry breakdown (as in problem 4) occursand that in the ground state $M_0^\dagger M_0 = v^2 \mathbb{1}$ for some v^2. What is the unbroken symmetry group and how many Goldstone bosons are there?

   (c) If $\mathscr{L} \to \mathscr{L} + \mathscr{L}'$ with
   $$\mathscr{L}' = h(\det(M) + \det(M^\dagger)) \, ,$$
   what is the symmetry group and how many Goldstone bosons are there now after spontaneous symmetry breakdown? (Assume that the ground state still satisfies $M_0^\dagger M_0 = v^2 \mathbb{1}$).
   (*Hint: $U(N) = SU(N) \times U(1)/Z_N$, where Z_N is the discrete symmetry group corresponding to the complex numbers $e^{2\pi i k/N}, k = 0, \ldots, N-1$.*)