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Exercise 5
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Exercises on Elementary Particle Physics

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1. *Spontaneous Symmetry Breakdown I (taken from exercise 4)*

   This is supposed to be a basic introductory example to show what happens in a theory where we have a classical ground state which exhibits less symmetry than the Lagrangian.

   (a) Let $\phi = (\phi_1, \ldots \phi_n)$ be an n component scalar field with Lagrangian density

   $$\mathscr{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) \quad , \quad V(\phi) = \frac{1}{8}g(\phi^2 - v^2)^2 \quad , g > 0\,.$$

   Why is the symmetry group O(n)? Expand the potential and identify the standard ingredients to a Lagrangian density (kinetic term, mass term, interactions..) – what is different? Draw the potential in a suitable projection!

   (b) Calculate the equation which determines the classical ground state(s) (states with lowest energy, hence potential). This equation defines an $n-1$ dimensional sphere (S^{n-1}), which breaks the higher symmetry of the Lagrangian.

   (c) Pick one ground state $\phi_0 = (0, \ldots, 0, v)$ and calculate the effective Lagrangian with low excitations around that ground state by expanding around it

   $$\phi = (\tilde{\phi}, v + f) \quad , \quad \tilde{\phi} = (\phi_1, \ldots, \phi_{n-1})\,.$$

   Give an interpretation for the different modes you find in the effectiv Lagrangian. What is the link to the Goldstone theorem.

   (d) What happens if one has the opposite sign in $V(\phi)$? (i.e. $V' = \frac{1}{8}g(\phi^2 + v^2)^2$)

2. *The Standard Model Higgs-effect*

   Consider a nonabelian gauge theory with gauge group $SU(2)_L \times U(1)_Y$. The $SU(2)_L$ corresponds to the so called leptonic isospin, the $U(1)_Y$ to the leptonic hypercharge. Let

   $$L_e = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \;;\; L_\mu = \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix} \;;\; e_R \;;\; \mu_R$$

   be isospin doublets and isospin singlets respectively. We assign analogs of hypercharges so that the *Gell-Mann-Nishijima* formaula $Q = T^3 + Y/2$ is still valid.

(a) You know the charges. Use that to calculate the hypercharges of the doublets and the singlets.

We introduce a complex *Higgs* (isospin) doublet $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, where the upper indices denote the charges.

(b) What's it's hypercharge?

We examine a part of the GWS (*Glashow-Weinberg-Salam*) Lagrangian

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \ ,$$

with

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \ , \ \mu^2, \lambda > 0$$

and

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi - \tfrac{i}{2} g_2 \tau^a W^a_\mu \Phi - \tfrac{i}{2} g B_\mu \Phi \ , \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \ , \\ G^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_2 \epsilon^{abc} W^b_\mu w^c_\nu \ , \end{aligned}$$

τ are the *Pauli* matrices.

(c) Comment on all appearing terms and equations?

(d) Find the minimum $\langle \Phi^\dagger \Phi \rangle \equiv \frac{v^2}{2}$ of $V(\Phi)$.

To leave the $U(1)_{em}$ unbroken we choose the vacuum expectation value (VEV)

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

(e) Reparametrize $\Phi(x)$ using

$$\Phi(x) = U^{-1}(\xi) \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \quad \text{with} \quad U(\xi) = \exp(i \xi(x)^a \tau^a / v) \ .$$

What can you say about the VEVs of $\xi(x)$ and $\eta(x)$?

Now we apply a gauge transformation according to $\Phi' = U(\xi)\Phi = \frac{v+\eta(x)}{\sqrt{2}} \chi$ with $\chi = (0 \ 1)^T$. From the local gauge invariance principle we know the transformation properties (if you don't, make it clear to yourself!)

$$\begin{aligned} \tfrac{\tau^a W'^a_\mu}{2} &= U(\xi) \left(\tfrac{\tau^a W^a_\mu}{2} \right) U^{-1}(\xi) - \tfrac{i}{g_2} \left[\partial_\mu U(\xi) \right] U^{-1}(\xi) \ , \\ B'_\mu = B_\mu \ , \ G'^a_{\mu\nu} G'^{\mu\nu a} &= G^a_{\mu\nu} G^{\mu\nu a} \ , \ F'_{\mu\nu} F'^{\mu\nu} = F_{\mu\nu} F^{\mu\nu} \ . \end{aligned}$$

(f) We now want to compute the transformed part of the Lagrangian

$$\tilde{L} = (D_\mu \Phi')^\dagger (D^\mu \Phi') - V(\Phi') \ .$$

Proceed as follows: First show that $V(\Phi) \to \mu^2\eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4$. Because we want to get information about the masses of the vector bosons, it's enough to restrict on the terms bilinear in Φ (why?). Define $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu'^1 \mp i W_\mu'^2)$ and show that the bilienar terms are given by

$$\frac{v^2}{8}\left[2g_2^2 W_\mu^- W^{+\mu} + (g_2 W_\mu'^3 - g B_\mu')\right] \ .$$

What's the $W^\pm$ mass?. Rewrite the second term as a bilinear in (W'^3, B'), i.e. define

$$\frac{1}{2}M_Z^2 Z_\mu Z^\mu = \frac{v^2}{8}(W_\mu'^3, B_\mu') M' (W^{\mu\prime 3}, B^{\mu\prime})^T \ .$$

Diagonalize M', call the eigenvectors Z and A and compute the masses of Z and A. Comment about the degrees of freedom before and after the symmetry breaking. Give an interpretation.

(g) The ρ parameter is defined by

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} \text{ with } \tan\theta_W = \frac{g}{g_2} \ .$$

Experiment says $\rho = 1$. Compute ρ.

*3. *Spontaneous Symmetry Breakdown II (taken from exercise 4)*

A field theory is described in terms of the elements of a complex $N \times N$ matrix M by a Lagrangian

$$\mathscr{L} = \mathrm{tr}(\partial^\mu M^\dagger \partial_\mu M) - \frac{1}{2}\lambda\ \mathrm{tr}(M^\dagger M M^\dagger M) - k\ \mathrm{tr}(M^\dagger M) \ ,$$

where tr denotes the matrix trace and $\lambda > 0$.

(a) Show that this theory is invariant under the symmetry group $U(N)\times U(N)/U(1)$ for transformations given by $M \to AMB^{-1}$, $A, B \in U(N)$, and where the $U(1)$ corresponds to $A = B = e^{i\theta}$.

(b) Show that for $k < 0$ spontaneous symmetry breakdown (as in problem 4) occursand that in the ground state $M_0^\dagger M_0 = v^2 \mathbb{1}$ for some v^2. What is the unbroken symmetry group and how many Goldstone bosons are there?

(c) If $\mathscr{L} \to \mathscr{L} + \mathscr{L}'$ with

$$\mathscr{L}' = h(\det(M) + \det(M^\dagger)) \ ,$$

what is the symmetry group and how many Goldstone bosons are there now after spontaneous symmetry breakdown? (Assume that the ground state still satisfies $M_0^\dagger M_0 = v^2 \mathbb{1}$).

(*Hint:* $U(N) = SU(N) \times U(1)/Z_N$, *where* Z_N *is the discrete symmetry group corresponding to the complex numbers* $e^{2\pi i k/N}, k = 0, \ldots, N-1$.)