Physikalisches Institut | Exercise 6
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Exercises on Elementary Particle Physics

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1. *γ gymnastics and spinor identities*

   Using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\not{a} = a_\mu \gamma^\mu$, $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ and the normalisation $\bar{u}_r u_s = -\bar{v}_r v_s = \delta_{r,s}$

   (a) $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$,
   (b) $\not{a}_1 \not{a}_2 = 2a_1 \cdot a_2 - \not{a}_2 \not{a}_1 = a_1 \cdot a_2 + 2S^{\mu\nu} a_{1\mu} a_{2\nu}$,
   (c) $\mathrm{tr}(\gamma_5) = 0$,
   (d) $\mathrm{tr}(\not{a}_1 \not{a}_2 \ldots \not{a}_r) = 0$ if r is odd,
   (e) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$,
   (f) $\mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2) = 0$,
   (g) $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$,
   (h) $\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu = 4a_1.a_2$,
   (i) $\gamma_\mu \not{a}_1 \not{a}_2 \not{a}_3 \gamma^\mu = -2\not{a}_3 \not{a}_2 \not{a}_1$,
   (j) $\mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4) = 4i\epsilon_{\lambda\mu\nu\rho} a_1^\lambda a_2^\mu a_3^\nu a_4^\rho$,
   (k) $\sum_s u_s(p)\bar{u}_s(p) = \frac{1}{2m}(\gamma.p + m)$ (this is a projector!),
   (l) $\sum_s v_s(p)\bar{v}_s(p) = \frac{1}{2m}(\gamma.p - m)$.

   (*Hint: Use the symmetry of the trace, insert $(\gamma_5)^2 = 1$ at the right places, and use the equation of motions $(\gamma.p - m)u_s(p) = 0$ and $(\gamma.p + m)v_s(p) = 0$*)

2. *Kinematics*

   (a) Matrix elements of four particle processes can be described by the so-called Mandelstam variables

   $$s = (p_1 + p_2)^2 \qquad t = (p_1 - k_1)^2 \qquad u = (p_1 - k_2)^2 \,.$$

   Show that

   $$s + t + u = m_1^2 + m_2^2 + \mu_1^2 + \mu_2^2 \,,$$

   where m_1, m_2, μ_1, μ_2 are the masses of the particles with momentum p_1, p_2, k_1, k_2 respectively.

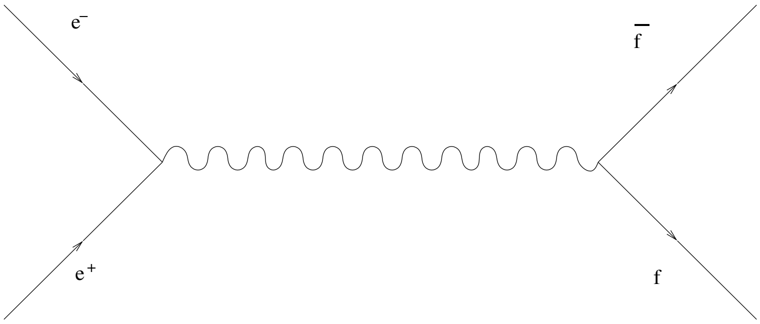

Figure 1: e^+e^- annihilation

(b) The differential cross-section can be split into two factors: first a part depending on the invariant amplitude $\mathcal{M}$ of the process (a Lorentz scalar!), second the kinematic and phasespace information. The general expression for spinless particles and photons is given by (read in a standard quantum field theory book how to derive that!)

$$\begin{aligned} \mathrm{d}\sigma &= \frac{1}{|\vec{v}_1 - \vec{v}_2|} \left(\frac{1}{2\omega_{p_1}}\right) \left(\frac{1}{2\omega_{p_2}}\right) |\mathcal{M}|^2 \frac{\mathrm{d}^3 k_1}{2\omega_1 (2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{2\omega_n (2\pi)^3} \\ &\quad \times (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_i k_i\right) S \end{aligned}$$

where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$ and $\vec{v}_i$, p_i are the velocities and momenta of the two incoming particles. The integration runs over the momenta k_i of all outgoing particles. S is an additional statistical factor that gives a factor of $1/m!$ for m identical particles, in general

$$S = \prod_i \frac{1}{m_i!} \; .$$

For fermions the factors $1/2\omega_{p_i}$ have to be replaced by m/E_{p_i}. Using this show that in a CMS (center of mass) system one gets the following equation for two incoming fermions with momenta p_1, p_2 and masses m_1, m_2, producing two outgoing fermions with momenta k_1, k_2 and masses μ_1, μ_2

$$\frac{d\sigma}{d\Omega} = \frac{m_1 m_2 \mu_1 \mu_2}{4\pi^2} \frac{1}{s} \frac{k_f}{p_i} |\mathcal{M}|^2$$

with $s = (p_1 + p_2)^2$ as above. p_i and k_f are the initial and final momentum in the CMS system and $d\Omega$ the spherical element.

3. e^+e^- *annihilation*

(a) Use the QED Feynman rules to calculate the matrix element $\mathcal{M}$ of the tree level process $e^+e^- \to f\bar{f}$ (s. figure 1). The fermion $f \neq e^\pm$ has a charge $q_f e$.

(b) Calculate the square of the amplitude $|\mathcal{M}|^2$ and split this expression into two factors $L_e^{\mu\nu}$, $L^f_{\mu\nu}$ where $L_e^{\mu\nu}$ only depends on the electron part, $L^f_{\mu\nu}$ only on the fermion part.

(c) Now assume the incoming beam to be unpolarised. Hence, we can average over the initial spins (i.e. sum over them with a suitable factor). Use the identities of (1k) and (1l) to transform both factors L_e and L_f – separately – into traces. (How and why can you reshuffle some of the Dirac spinors…?)

(d) We want to calculate the ultrarelativistic limit. Neglect the masses m_e, m_f of electron and fermion compared to their momenta and calculate $|\mathcal{M}|^2$ in that limit. Express the result in terms of Mandelstam variables.

(e) Finally, calculate the total cross section σ in that limit (using the formula of 2b) and the ratio

$$R = \frac{\sum_Q \sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \, .$$

The sum runs over all quarks that – given a certain total energy – can possibly contribute to this process.