Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 7
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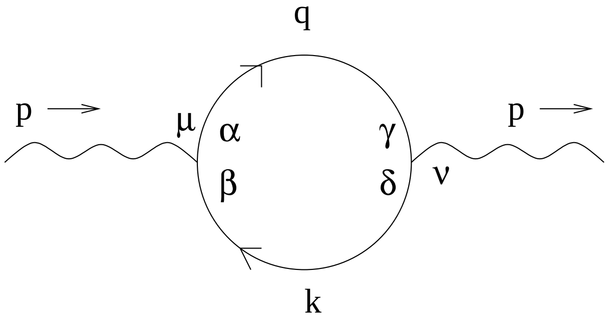
WS 01/02

Exercises on Elementary Particle Physics

Prof. H.-P. Nilles

1. *One-loop correction to the electric charge in QED*

   The one-loop correction $\Pi_{\mu\nu}(p)$ (legs amputated, momentum p), is given by the *Feynman* diagram



   We want to calculate this diagram. Proceed as follows:

   (a) Use the *Feynman* rules to show that the one-loop contribution can be written as
   $$\Pi_{\mu\nu}(p) = -e^2 \mathrm{tr} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \left(\frac{\not{q} + m}{q^2 - m^2 + i\epsilon} \right) \gamma_\nu \left(\frac{\not{q} - \not{p} + m}{(q-p)^2 - m^2 + i\epsilon} \right) . \tag{1}$$
   How divergent is this term for large $|q|$?

   (b) We now want to reexpress the denominator using the so called *Feynman* trick. It is clear that
   $$\frac{1}{ab} = \frac{1}{b-a} \int_a^b \frac{dx}{x^2} . \tag{2}$$
   Choose $x = az + b(1-z)$ and introduce $q' = q - p(1-z)$ to show that
   $$\frac{1}{q^2 - m^2 + i\epsilon} \, \frac{1}{(q-p)^2 - m^2 + i\epsilon} = \int_0^1 dz \left[q'^2 + z(1-z)p^2 - m^2 + i\epsilon \right]^{-2} . \tag{3}$$

   (c) To calculate the numerator use the trace identities of exercise 6.1. Relabel q' by q again (just omit the primes). What happens to the terms linear in q when integrated?

(d) Use your results and $q_\mu q_\nu = \eta_{\mu\nu} q^2/4$ (why?) to show that the one-loop propagator can be written as

$$\Pi_{\mu\nu}(p) = -4e^2 \int \frac{d^4q}{(2\pi)^4} \int dz \left\{ \frac{\eta_{\mu\nu} q^2}{2[q^2 + z(1-z)p^2 - m^2 + i\epsilon]^2} - \right. \tag{4}$$

$$\left. - \frac{\eta_{\mu\nu}}{[q^2 + z(1-z)p^2 - m^2 + i\epsilon]} - \frac{2z(1-z)(p_\mu p_\nu - \eta_{\mu\nu} p^2)}{[q^2 + z(1-z)p^2 - m^2 + i\epsilon]^2} \right\} . \tag{5}$$

How divergent are the different terms?

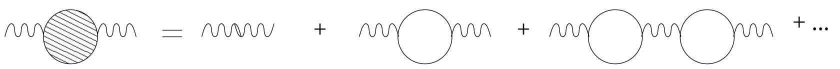
(e) To perform the q integration apply a *Wick* rotation $q_0 = iq_4$ and use polar variables, i.e. $|q| = r$ (*Hint:* Remember $\int d\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$). Of course the different terms are still divergent for $r^2 \to \infty$. To regularize we introduce a cut-off mass Λ, i.e. $\int_0^\infty d(r^2) \to \int_0^\Lambda d(r^2)$. After integration you should get

$$\Pi_{\mu\nu}(p) = -\frac{ie^2}{4\pi^2} \int dz \left\{ \frac{1}{2} \eta_{\mu\nu} \Lambda^2 + \eta_{\mu\nu} c^2 \ln \frac{\Lambda^2}{\Lambda^2 + c^2} - \right. \tag{6}$$

$$\left. - \frac{1}{2} \eta_{\mu\nu} c^2 + 2z(1-z) \eta_{\mu\nu} p^2 \left(\ln \frac{\Lambda^2}{c^2} - \Lambda \right) \right\} , \tag{7}$$

where $c^2 = m^2 - z(1-z)p^2$. Why can we discard the $p_\mu p_\nu$ term?

(f) Now just consider the terms $\sim \eta_{\mu\nu} Q$, where Q is independent of p^2. We can write the diagrams



as a sum $i\tilde{D} = iD + iD\Pi iD + iD\Pi iD\Pi iD + \ldots$. Compute $\tilde{D}$, interpret the result and argue why this term has to be omitted.

(g) Approximate the remaining terms of Π using $p^2 << \Lambda^2, m^2 << \Lambda^2$ and show that then

$$\Pi_{\mu\nu}(p) = -\frac{ie^2}{4\pi^2} \int_0^1 dz \left\{ -2\eta_{\mu\nu} z(1-z) p^2 \ln \frac{m^2}{\Lambda^2} \right\} , \tag{8}$$

where a finite term (which only depends on the regularization scheme) has been omitted.

(h) Integrate with respect to z and give the result. Again sum up the loop contributions as in f) and show that one can interpret the result as a renormalized electric charge with

$$e_R = \sqrt{Z_3}\, e \quad \text{with} \quad Z_3 = \left[1 - \frac{e^2}{12\pi^2 \ln \frac{m^2}{\Lambda^2}} \right]^{-1} .$$

(i) What can you derive for the coupling constant α? Now compare $1/\alpha$ for two different scales (approximate Z_3 to linear order). You get a so called renormalization group equation. Does it depend on the cut-off?