Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 9
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WS 01/02

Exercises on Elementary Particle Physics

Prof. H.-P. Nilles

1. *Review* Review sheets 1 to 8 over the Christmas break and prepare questions ...

2. *Grassmann-Algebra* Let's also start to get familiar with one of the most important tools of supersymmetry – spinors.

   Let θ_α, $\alpha = 1, 2$ be anticommuting (Grassmann-)variables, i.e. $\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha$. They transform as a $(1/2, 0)$ representation of the Lorentz group, i.e. $\theta'_\alpha = M_\alpha^\beta \theta_\beta$.

   (a) Show that $\varepsilon_{\alpha\beta}$ is invariant under a Lorentz transformation, where $\varepsilon_{12} = +1$, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. Hence, we can use $\varepsilon_{\alpha\beta}$ for raising and lowering indices ($\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}$) and define a spinor with upper index $\theta_\alpha = \varepsilon_{\alpha\beta}\theta^\beta$. What is the inverse of this relation? How does θ^β transform under a Lorentz transformation? (*Hint: Use* $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = -1$ *and show* $M^{-T} = -\varepsilon M \varepsilon$ *remembering Exercise 2 Problem 2.*)

   (b) The hermitian conjugate variable is defined as $\bar{\theta}^{\dot{\alpha}} := (\theta^\alpha)^\dagger$. Show that the transformation of $\bar{\theta}^{\dot{\alpha}}$ under a Lorentz transformation is given by

   $$\begin{aligned} \bar{\theta}^{\dot{\alpha}\prime} &= (M^{-\dagger})^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\theta}^{\dot{\beta}} \,. \\ &= \bar{\theta}^{\dot{\beta}}((M^*)^{-1})_{\dot{\beta}}{}^{\dot{\alpha}} \end{aligned}$$

   Hence $\bar{\theta}^{\dot{\alpha}}$ transforms in a $(0, 1/2)$ representation of the Lorentz group. (Introducing such a strange notation $\dot{\alpha}$ for the indices is no mystery but just a way of clarifying the transformation properties of such a spinor on first glance!)

   (c) Use the above results to determine the tranformation properties of

   $$\bar{\theta}_{\dot{\alpha}}, \qquad \xi\theta \equiv \xi^\alpha\theta_\alpha, \qquad \bar{\theta}\bar{\xi} \equiv \bar{\theta}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}?$$

   (d) Use the exlicit Lorentz transformations of spinors $D_{R/L}$ calculated in Exercise 2 Problem 2 to transfer the above index convention to the spin generators $\sigma_{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$ and $\bar{\sigma}_{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ and the Pauli matrices $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$. You get e.g. $(\sigma_\mu)^{\alpha\dot{\beta}} = (\mathbb{1}, \vec{\sigma})_{\alpha\beta}$. Now show

   $$\sigma^\mu_{\gamma\dot{\delta}}\sigma_\mu^{\alpha\dot{\beta}} = 2\delta^\alpha_\gamma\delta^{\dot{\beta}}_{\dot{\delta}} \,.$$

(e) Now check the identities ($\theta^2 := \theta^\alpha \theta_\alpha$)

$$\xi\psi = \psi\xi, \qquad \bar{\theta}\bar{\xi} = \bar{\xi}\bar{\theta}, \qquad \theta_\alpha \theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta^2 \ .$$

(*Hint: Use* $\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$.)

(f) Use that to show

$$(\theta\sigma_\mu\bar{\theta})(\theta\sigma_\nu\bar{\theta}) = \frac{1}{2}\eta_{\mu\nu}\theta^2\bar{\theta}^2 \ .$$

(g) The (left) derivative w.r.t. a Grassmann variable θ^α is given by

$$\frac{\partial}{\partial\theta^\alpha}(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma \ .$$

Show that ($\partial_\alpha \equiv \partial/\partial\theta^\alpha$)

$$\varepsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha \equiv -\frac{\partial}{\partial\theta_\alpha}, \qquad \varepsilon^{\alpha\beta}\partial_\alpha\partial_\beta\theta^2 = -4$$

and calculate $\partial_\alpha f(\theta)$.