Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 6
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General Relativity

Prof. Dr. H.-P. Nilles

1. *Schwarzschild solution*

One way of writing a spherically symmetric and static line element is

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{1}$$

Traditionally, this has been called the *standard form*. We will now try to determine $A(r)$ and $B(r)$ from the Einstein equations.

(a) What are the non-zero components of the metric tensor?

(b) Calculate all non-zero Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\,. \tag{2}$$

(You should find 8 distinct terms.)

(c) Calculate the terms of the Ricci tensor

$$R_{\mu\kappa} \equiv \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}\,. \tag{3}$$

(d) What are the Einstein field equations for empty space?
Use them to solve for $A(r)$ and $B(r)$.
*(Hints: Calculate $R_{rr}/A + R_{tt}/B$ to get a relation between A and B.
What form does $g_{\mu\nu}$ take for $r \to \infty$?)*

(e) Fix the remaining integration constant by using the Newtonian limit for the metric $g_{tt} = -1 + 2GM/r$. Why is only g_{tt} affected by Newtonian considerations?

(f) Write the Schwarzschild line element ds^2 in the standard form.
Use the substitution $r = \rho\left(1 + \frac{GM}{2\rho}\right)^2$ to write ds^2 in its *isotropic* form.