Universität Bonn
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Theoretische Physik

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Example sheet 8
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General Relativity

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1. *Precession of perihelia*

From the last sheet we know that the shape of Schwarzschild trajectories is given by

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2\,B(r)} = -\frac{E}{J^2}\,, \tag{1}$$

where E and J^2 are constants of the motion. Also, we find that

$$\phi = \pm \int \frac{\sqrt{A(r)}\,dr}{r^2\sqrt{\frac{1}{J^2\,B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}} \tag{2}$$

is a solution of (1). In weak fields, we can expand

$$A(r) = 1 + \frac{2\,MG}{r}\,, \quad B(r) = 1 - \frac{2\,MG}{r}\,. \tag{3}$$

(a) Determine E and J^2 by looking at the aphelion $r = r_+$ and perihelion $r = r_-$ of a planet in bound orbit around the sun, for general $B(r)$. *(At $r_\pm$, $dr/d\phi$ vanishes.)*

(b) Show that the amount of orbital precession per revolution is

$$\Delta\phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi\,, \tag{4}$$

where $\phi(r_+) - \phi(r_-)$

$$= \int_{r_-}^{r_+} \left[\frac{r_-^2\,(B^{-1}(r) - B^{-1}(r_-)) - r_+^2\,(B^{-1}(r) - B^{-1}(r_+))}{r_+^2 r_-^2\,(B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2}\right]^{-1/2} \times \frac{\sqrt{A(r)}\,dr}{r^2}\,. \tag{5}$$

(c) Show that for weak fields, we can use

$$B^{-1}(r) = 1 + \frac{2\,MG}{r} + \frac{4\,M^2G^2}{r^2} \tag{6}$$

which makes the first term in (5) quadratic in $1/r$, and that we can then write

$$\phi(r_+) - \phi(r_-) = \int\limits_{r_-}^{r_+} \left[C \left(\frac{1}{r_-} - \frac{1}{r} \right) \left(\frac{1}{r} - \frac{1}{r_+} \right) \right]^{-1/2} \times \frac{\sqrt{A(r)}\, dr}{r^2} \,. \tag{7}$$

(d) Determine C in the limit $r \to \infty$. You should get

$$C = 1 - \frac{4\,MG}{L} + \ldots \,, \tag{8}$$

where $\frac{1}{L} = \frac{1}{2} \left(\frac{1}{r_+} + \frac{1}{r_-} \right)$. Show that (5) now reduces to

$$\phi(r_+) - \phi(r_-) = \left(1 + \frac{2\,MG}{L} \right) \times \int\limits_{r_-}^{r_+} \frac{\left(1 + \frac{MG}{r}\right) dr}{r^2 \sqrt{\left(\frac{1}{r_-} - \frac{1}{r} \right) \left(\frac{1}{r} - \frac{1}{r_+} \right)}} \tag{9}$$

(e) Calculate $\Delta\phi$. *(You can approximate the result of the integral with* $\left(1 + \frac{MG}{L}\right) \pi$*.)*

(f) Determine the total precession $\Delta\phi$ for Mercury over the time of a century. (415 revolutions per century; $L = 55.3 \times 10^9$ m; $MG = 1475$ m).
The observed value is (43.11 ± 0.45) arcseconds.