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Exercise 2
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SS 06

Exercises on Elementary Particle Physics II

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Exercise sheets in the Internet: www.th.physik.uni-bonn.de/nilles/

1. *Grassmann variables and Weyl spinors*

   Let $\theta_\alpha, \alpha = 1, 2$ be anti-commuting (Grassmann-) variables, i.e.

   $$\theta_\alpha \theta_\beta = -\theta_\beta \theta_\alpha.$$

   They transform in the $(1/2, 0)$ representation of the Lorentz group (left-chiral Weyl spinor) $\theta_\alpha \mapsto {(D_L)_\alpha}^\beta \theta_\beta$.

   (a) Verify that $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$, with $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ being the totally antisymmetric tensor, defined through $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{12} = 1$.

   (b) In Ex.4.1(b) of the last term, we have seen that the Pauli matrix σ_2 can be used as a spinor metric:

   $$\sigma_2 = (D_L)^T \sigma_2 D_L.$$

   Show that $\epsilon = i\sigma_2$ is an equivalent choice for the metric. Therefore, we can use the epsilon tensor to raise and lower spinor indices:

   $$\theta^\alpha := -\epsilon^{\alpha\beta}\theta_\beta$$

   What is the inverse of this relation?

   (c) The conjugate variable is defined by $\bar{\theta}^{\dot{\alpha}} := (\theta^\alpha)^\dagger$. Show that it transforms in the $(0, 1/2)$ representation of the Lorentz group (right-chiral Weyl spinor).
   Hint: Start with $\sigma_2 D_L \sigma_2 = D_R^*$, prove $-\epsilon D_L^T \epsilon = D_R^\dagger$ and finally compute the transformation of $\epsilon^{\alpha\beta}\theta_\beta$.

   (d) The conventions for index contraction are:

   $$\xi\psi := \xi^\alpha \psi_\alpha \quad \text{and} \quad \bar{\xi}\bar{\psi} := \bar{\xi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$$

   Verify the following identities:

   $$\xi\psi = \psi\xi \quad \text{and} \quad \bar{\xi}\bar{\psi} = \bar{\psi}\bar{\xi}$$

(e) Show that the following holds:

$$\xi^\alpha \psi_\alpha = -\xi_\alpha \psi^\alpha \quad \text{and} \quad \bar{\xi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}$$

(f) Prove the relations

$$\theta^\alpha \theta^\beta = \tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \qquad \theta_\alpha \theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta$$

$$\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = -\tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

2. *Pauli matrices and Weyl spinors*

(a) Use the explicit Lorentz transformation of left- and right-chiral Weyl spinors

$$\theta \mapsto D_L\theta = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\theta,$$

$$\bar{\theta} \mapsto D_R\bar{\theta} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\bar{\theta}.$$

from Ex.4.1(a) to transfer the above index convention to the spin generators $\sigma^{\mu\nu}$, $\bar{\sigma}^{\mu\nu}$ and to the Pauli matrices σ^μ and $\bar{\sigma}^\mu$.

(b) Check

$$\begin{aligned} (\bar{\sigma}^\mu)^T &= (-i\sigma_2)\sigma^\mu(i\sigma_2) \\ (\sigma^\mu)^{\alpha\dot{\beta}} &= (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \end{aligned}$$

(c) Verify the following identities:

$$\begin{aligned} (\bar{\xi}\bar{\sigma}^\mu\psi) &= -(\psi\sigma^\mu\bar{\xi}) \\ \psi_\alpha\bar{\xi}_{\dot{\beta}} &= \frac{1}{2}(\sigma^\mu)_{\alpha\dot{\beta}}(\psi\sigma_\mu\bar{\xi}) \\ (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta}) \end{aligned}$$

3. *Grassmann differentiation*

Define Grassmann differentiation

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad \text{and} \quad \bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}$$

by the relations $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$ and $\bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}}$. Note that the product rule must include a minus sign:

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma$$

(a) Show that $\partial^\alpha = \epsilon^{\alpha\beta}\partial_\beta$.

(b) Check that

$$\partial^\alpha\partial_\alpha(\theta\theta) = \bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = 4.$$