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Exercise 2
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SS 08

Exercises on Elementary Particle Physics II

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1. *Cartan matrix of SU(n)*

   Let us work investigate the Lie algebra of SU(n) in detail. There, we will encounter many important concepts necessary for the understanding of any semisimple Lie algebra.[1] First, define the canonical basis $(e_{ab})_{ij} = \delta_{ai}\,\delta_{bj}$ of the $n \times n$-matrices.

   (a) Determine the Lie algebra $su(n)$, its dimension and find a basis, that generalizes the Pauli matrices. Perform a complex basis change analogous to Ex. 1.2,(b) for all non-diagonal matrices. *Hint: You should find e_{ab} as a basis after all.*

   (b) Define the set of commuting matrices $H := \{\sum_i \lambda_i e_{ii} \mid \sum_i \lambda_i = 0\}$ and check that H is a subalgebra of $su(n)$, i.e. $[g,h] = 0$ for $g,\, h \in H$.
   Check that
   $$ad(h)(e_{ab}) = [h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab} \qquad h \in H. \tag{1}$$
   Thus, the e_{ab} form an eigenbasis for the representation matrices $ad(h)$ with eigenvalue $(\lambda_a - \lambda_b)$, respectively.

Abstractly, H can be defined as the maximal set of commuting elements in $\mathfrak{g}$ with $ad(h)$ diagonalizable. It is called the **Cartan subalgebra** of $\mathfrak{g}$ and its dimension the **rank** r **of** $\mathfrak{g}$. The rank of $su(n)$ is $n-1$ and it is also called the algebra A_{n-1}.

Formalizing further, we can interpret (1) as a function $\alpha_{e_{ab}}$ on H defined by

$$\alpha_{e_{ab}}(h) := (\lambda_a - \lambda_b) \in \mathbb{R} \quad \text{for} \quad su(n)\,.$$

More abstract, $\alpha_\bullet$ is a mapping

$$\alpha_\bullet \,:\, \mathfrak{g}/H \to H^*\,, \quad t \mapsto (\alpha_t : h \mapsto \alpha_t(h))\;, \tag{2}$$

where H^* denotes the dual space to H. H^* is called the **root space** in this context. The mapping $\alpha_t \in H^*$ is called a **root**[2] and the r-dimensional vector $\left(\alpha_{e_{ab}}(e_{ii}\text{-}e_{i+1i+1})\right)_i$, $i = 1, \cdots, r$, the **root vector** of e_{ab}.

[1] A Lie algebra is called **semisimple** if it is a sum of abelian and simple Lie algebras. An **abelian Lie algebra** obeys $[g,h] = 0\ \forall g, h \in \mathfrak{g}$ and a **simple** one contains no proper ideal, i.e. invariant subspace.

[2] Not every $\tilde{\alpha} \in H^*$ is a root of $\mathfrak{g}$, i.e. there may be no $x \in \mathfrak{g}$ with $[h,x] = \alpha_x(h)x$ such that $\tilde{\alpha} = \alpha_x$. But every root of $\mathfrak{g}$ is in H^*.

Expand a root $\beta \in H^*$ into a basis $\alpha_1, \ldots, \alpha_r$ of roots as $\beta = \sum_i c^i \alpha_i$, β is called

- **positive (negative)**, $\beta > 0$, if the first non-zero $c^i > 0$ ($c^i < 0$). Accordingly, we write $\beta > \gamma$ for $\beta - \gamma > 0$.
- **simple root** if it is positive and cannot be written as $\beta = \sum_i d^i \tilde{\alpha}_i$ with $i > 1$, $d^i > 0$ $\forall i$ and $\tilde{\alpha}_i$ all positive.

Operators t associated to a root α by $\alpha = \alpha_t$ are also denoted by E_α. For $\alpha > 0$ ($\alpha < 0$) they are called **raising operators** (**lowering operators**).

(c) Show that the roots

$$\alpha_i(h) = \lambda_i - \lambda_{i+1} \quad i = 1, 2, ..., n-1.$$

are a basis of the root space for $su(n)$ and that they are positive with $\alpha_1 > \alpha_2 \cdots > \alpha_{n-1}$. Are these roots also simple roots?

Furthermore, we can define an inner product on $\mathfrak{g}$. Let x, $y \in \mathfrak{g}$ be arbitrary, then define the **Killing form** κ by

$$\kappa(x, y) := tr\left(ad(x) \cdot ad(y)\right) .$$

Obviously it is bilinear and symmetric but not positive definite in general.

(d) Prove that the Killing form on the basis t_i of $\mathfrak{g}$ is given by

$$\kappa_{ij} = \kappa(ad(t_i) \cdot ad(t_j)) = -f_{iml} f_{jlm}$$

and the invariance $\kappa([x, y], z) = \kappa(x, [y, z])$. *Hint: Use the Jacobi identity.* Why is κ equivalently definied by firstly determining the basis expansion of

$$[x, [y, t_i]] = \sum_j \mathcal{K}_{ij} t_j$$

and secondly setting $\kappa(x, y) := \mathrm{Tr}(\mathcal{K})$. Use your favorite method to calculate $\kappa(h, h')$, where $h = \sum_i \lambda_i \, e_{ii}$, $h' = \sum_j \lambda'_j \, e_{jj}$ of $su(n)$.

The Killing form κ of any finite dimensional semisimple $\mathfrak{g}$ is non-degenerate on $\mathfrak{g}$ (proven by Cartan) as well as on H. Hence, there is a natural isomorphism

$$h_\bullet : H^* \to H, \quad \alpha \mapsto h_\alpha \qquad \text{defined by} \qquad \alpha(h) \stackrel{!}{=} \kappa(h_\alpha, h), \; \forall h \in H \, . \tag{3}$$

(e) Calculate $\kappa(h_{\alpha_i}, h)$ using (3) and find h_{α_i} by comparing to your result from (d).

Finally, we use the isomorphism (3) to define an inner product on H^*:

$$\langle \alpha, \beta \rangle := \kappa(h_\alpha, h_\beta), \qquad \alpha, \beta \in H^*.$$

(f) Calculate the **Cartan matrix** of $su(n)$, defined in general by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}\,,$$

where the α_i are simple roots. Note that $A_{ii} = 2$ and $A_{ij} \in \mathbb{Z}_{\leq 0}$, $i \neq j$, as a fact.

The information about the Lie algebra $\mathfrak{g}$ encoded in the Cartan matrix is equivalent to knowing all structure constants of $\mathfrak{g}$. It is convenient to represent the Cartan matrix in a pictorial way by drawing a **Dynkin diagram**: To every simple root α_i, associate a small circle and join the circles i and j with $A_{ij}A_{ji}$ (no $\sum_{ij}$, $i \neq j$) lines.

(g) Draw the Dynkin diagram for A_n (i.e. $SU(n+1)$).

(h) Consider the non-trivial example of $su(3)$. We use the standard basis for the hermitian 3×3 matrices, the so called **Gell-Mann matrices** defined by

$$\begin{aligned}
&\lambda_1 = e_{12} + e_{21}, \quad \lambda_2 = i(e_{21} - e_{12}), \qquad \lambda_3 = e_{11} - e_{22},\\
&\lambda_4 = e_{13} + e_{31}, \quad \lambda_5 = i(e_{31} - e_{13}),\\
&\lambda_6 = e_{23} + e_{32}, \quad \lambda_7 = i(e_{32} - e_{23}), \quad \lambda_8 = \tfrac{1}{\sqrt{3}}(e_{11} + e_{22} - 2e_{33}).
\end{aligned} \tag{4}$$

This is in complete accordance with the basis you should have found in Ex. 2.1, (a), except for the form of λ_8.

Proceed with the recipe described above. Perform the complex basis change to find the raising/lowering operators, show that $H_1 = \frac{1}{2}\lambda_3$, $H_2 = \frac{1}{2}\lambda_8$ are a basis of H and express them in terms of the e_{ab}. Determine the $\alpha_{e_{ab}}$ and draw them as 2-dimensional vectors with x-axis (y-axis) corresponding to H_1 (H_2). Define the simple basis $\alpha_1 := \alpha_{e_{12}}$ and $\alpha_2 := \alpha_{e_{23}}$ and determine the $H_{\alpha_i} \in H$. Finally, calculate the Cartan matrix and draw the Dynkin diagram of $su(3)$.

2. *Representations of $SU(N)$*

The whole analysis of Ex. 2.1 is nothing more than investigating the adjoint representation $(ad, \mathfrak{g})$ of $\mathfrak{g}$. For a general representation ρ we also use the maximal set of commuting matrices $\rho(h)$, $h \in H$. Thus, all $\rho(h)$ can be diagonalized simultaneously if every individual $\rho(h)$ can be diagonalized, what we assume.[3] Hence, there is a basis of eigenvectors v^i such that

$$\rho(h)v^i = M^i(h)v^i \qquad \forall h \in H,\, v^i \in V\,. \tag{5}$$

Again, the M^i are functions on H, thus, $M^i \in H^*$. They are the **weights** of the representation (ρ, V). Comparison with (1) shows that the roots α are just the weights of the adjoint representation $\rho = ad$. As the simple roots α_i span H^* we can expand

$$M^i = \sum_j c^{ij}\alpha_j\,.$$

[3]As in quantum mechanics we expect the eigenvalues w.r.t. to these operators to specify an element in any representation ρ of $\mathfrak{g}$ uniquely. However, it may happen that the eigenspace of 0 is degenerated (as for the adjoint itself) and that additional quantum numbers like Casimir eigenvalues are needed.

Hence, positivity of weights can be defined as for roots, compare Ex. 2.1 above. A weight is called **highest weight**, denoted by Λ, if $\Lambda > M^i$ for all $M^i \neq \Lambda$.

(a) Let v^i be a vector with weight M^i. Show that $\rho(E_\alpha)v^i$ has weight $M^i + \alpha^i$ if $\rho(E_\alpha)v^i \neq 0$. Make contact to the definition of raising/lowering operators.

(b) What are the weights of the complex conjugate representation $\bar{\rho}$ of ρ?
Hint: The Cartan generators are hermitan, hence, have real eigenvalues.

Consider the **defining representation 3** of $su(3)$ in terms of the Gell-Mann matrices, cf. Ex. 2.1, (h). Choose the canonical basis $(v^i)_j = \delta^i_j$, $i, j = 1, 2, 3$, of the 3-dimensional representation space.

(c) Find the weights M^i by applying H_1, H_2 to the v^i and expand them in terms of the roots α_1, α_2, cf. Ex. 2.1, (h). Which weight is Λ? Draw the M^i in a 2-dimensional picture. Note that the difference between two weights is a root!

(d) Consider the action of the raising/lowering operators on the v^i and compare to (a). Indicate the action of these operators on the weights by arrows in your 2-dimensional picture in (c).

Without any prove: A highest weight Λ of a representation ρ can be specified by a set of non-negative integers, called the **Dynkin coefficients**:

$$\Lambda_i = 2\frac{\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

Using the Λ_i it is possible to determine the corresponding highest weight and, furthermore, even all other weights of the representation ρ by the following recipe:

- To start, we need the Dynkin coefficients and the Cartan matrix.
- For each non-negative Dynkin coefficient Λ_i of the weight substract Λ_i-times the i-th row of the Cartan matrix, successively. For each step you will get the Dynkin coefficients of another weight.
- Repeat the last step for all weights, until all Dynkin coefficients are non-positive.

At the end, the number of Dynkin coefficients gives the number of different weights and therefore the number of linear independent weight vectors. Hence, this gives the dimension of the representation.[4]

SU(3) example

(e) Compute the Dynkin coefficients of the three weights and check that the **highest weight construction** is correct.

(f) Perform the highest weight construction for the Dynkin coefficients $(1, 1)$ of SU(3).

[4] This is not always true, consider e.g. (0...0) for the adjoint representation that is prescisely H.