Exercises on Theoretical Astroparticle Physics

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Lie groups and Lie algebras are of highest importance for the understanding of the standard model (SM), grand unified theories (GUT's) and particle physics in general. The symmetries given by them imply that the particles are organized into multiplets which are built out of the representations of the groups. This way many particles have been predicted before their experimental detection. Therefore the first exercise sheets will deal with simple lie algebras, their representations and their classification. If you want to learn more about group theory, we recommend the books given in the bibliography at the end.

1. A **Lie group** G is roughly a group which at the same time is a differentiable manifold. The group action $\cdot : G \times G \to G$ and the inversion $()^{-1} : G \to G$ are also required to be differentiable maps. One can first divide them into Abelian and non-Abelian groups, but the Abelian ones are very easy to handle so our focus lies on the non-Abelian Lie groups. The simplest of them is SU(2), the group of **S**pecial **U**nitary 2×2 matrices.

$$\mathrm{SU}(2) := \left\{ U \in \mathbb{C}^{2\times 2} \;\middle|\; U^\dagger = U^{-1} \;,\; \det U = 1 \right\}$$

   (a) Show that SU(2) as a manifold is equivalent to a 3-sphere ($\mathrm{SU}(2) \cong S^3$) by parametrizing $U \in \mathbb{C}^{2\times 2}$ with real parameters and imposing constraints on them.

   (b) Since $U \in \mathrm{SU}(2)$ is invertible, it can be written as $U = \mathrm{e}^{iG}$. G is sometimes called the **generator** of U. Show that G is traceless and hermitian. *Hint: det exp = exp tr.*

   (c) The space of traceless hermitian matrices is a real vector space. (Is it also a complex space?) Its basis can be choosen as the set of the well-known Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Hence we can write $U = e^{i\vec{\omega}\cdot\vec{\sigma}}$, $\vec{\omega} \in \mathbb{R}^3$. Show that:

$$U = \cos(\omega)\cdot \mathbb{1} + \mathrm{i}\sin(\omega)\cdot(\vec{\omega}_0 \cdot \vec{\sigma}) \qquad \text{where } \vec{\omega} = \omega \cdot \vec{\omega}_0$$

   Hint: Expand the exponential and use the relation $\sigma_i\sigma_j = \delta_{ij}\,\mathbb{1} + \mathrm{i}\epsilon_{ijk}\sigma_k$ *. (Compute this relation if you don't know it.)*
   What is the parameter space of $\vec{\omega} = \omega \cdot \vec{\omega}_0$? What can you say about its boundary? After the proper identification again one can see that it is topologically equivalent to S^3.

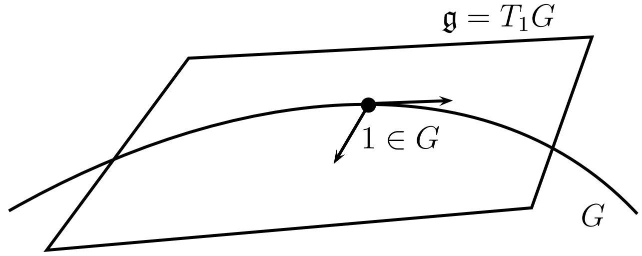

Figure 1: Lie group G and its Lie algebra $\mathfrak{g}$

(d) The **Lie algebra** $\mathfrak{su}(2)$ can be defined as the tangent space at the unit element, $\mathfrak{su}(2) := T_{\mathbb{1}}(SU(2))$, see also Figure 1. This means, given a local parametrization (e.g. $\vec{\omega}$), the elements of $\mathfrak{su}(2)$ are the first derivatives w.r.t. the parameters. Compute $\frac{\partial U(\vec{\omega})}{\partial \omega_i}\Big|_{\mathbb{1}} \in \mathfrak{su}(2)$. There is a natural group action on itself, the conjugation, defined by

$$\begin{aligned} f_h : \mathrm{SU}(2) &\to \mathrm{SU}(2) \\ f_h(g) &= hgh^{-1} . \end{aligned}$$

Now consider g and h to be infinitesimally small. Compute the pullback f^* of f on the Lie algebra $\mathfrak{su}(2)$. You should find that this is the commutator. *Hint: expand $g = e^{iG}$ and $f_h(g)$ to first order in G and H, where $h = e^{iH}$, and compare.* Show the following relations:

i. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ *Jacobi identity*

ii. $[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k$

iii. $[A, \sigma_i] = 0\ ,\ \forall i \ \Rightarrow\ A \propto \mathbb{1}$ *Schur's lemma*

The commutator is a natural bilinear skew-symmetric operation on every Lie-algebra $\mathfrak{g}$:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} .$$

Given a real basis T_i of $\mathfrak{g}$, one can expand every basic commutator in this basis:

$$[T_i, T_j] = \mathrm{i} f_{ijk} T_k . \tag{1}$$

The numbers f_{ijk} are called **structure constants** of $\mathfrak{g}$. They determine completely the properties of $\mathfrak{g}$. Normalizing the basis of $\mathfrak{su}(2)$ to $T_i = \sigma_i/2$, what are its structre constants?

2. A **representation** ρ is a homomorphism from the Lie group into the general linear group of a vector space V

$$\rho : \ G \to \mathrm{GL}(V)$$

such that $\rho(U) \cdot \rho(U') = \rho(UU')$ for $U, U' \in G$. This of course corresponds to an homomorphism from the Lie algebra to the group of Endomorphisms of V, which we will also denote ρ.

$$\rho : \ \mathfrak{g} \to \mathrm{End}(V) ,$$

which satisfies $\rho([u,u']) = [\rho(u), \rho(u')]$ for $u, u' \in \mathfrak{g}$. One can also say, if one finds a set of matrices, which satisfy the commutation relation (1), one has an representation of $\mathfrak{g}$. If there is a nontrivial invariant subspace ($W \subsetneqq V$ with $W \neq \{0\}$ and $\rho(W) \subset W$), then ρ is called **reducible**. Otherwise, if the only invariant subspaces of V are $\{0\}$ and V, ρ is called **irreducible**. Now since all (finite dimensional) representations can be decomposed into irreducible ones, it will be enough to classify them. We will denote the **dimension** of a representation ρ by the dimension of V, i.e. $dim(\rho) := dim(V)$. Mostly the representations are just named by their dimensions. In the next exercise we will construct all irreducible representations (irreps) of SU(2). From now on we will write J_i for $\rho(J_i)$ if we refer just to one representation ρ.

(a) We make a complex basis change:

$$J_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) \qquad J_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) \qquad J_3 = \frac{1}{2}\sigma_3\,.$$

Verify the resulting commutators:

$$[J_3, J_+] = +J_+ \qquad [J_3, J_-] = -J_- \qquad [J_+, J_-] = 2J_3\,.$$

(b) Now consider the **Casimir operator** defined as $J^2 = \sum_i J_i^2$. Show that

$$[J^2, J_i] = 0 \quad \text{for} \quad i = 1,2,3\,.$$

Hint: [AB,C] = A[B,C] + [A,C],B. Now Schur's lemma tells us, that $\rho(J^2) = C(\rho) \cdot \mathbb{1}$, where $C(\rho)$ is a number which only depends on the representation ρ. Verify the relations

$$J^2 = J_3^2 - J_3 + J_+J_- = J_3^2 + J_3 + J_-J_+\,.$$

(c) In any irrep ρ, we can choose J_3 $(= \rho(J_3))$ to be diagonal. Therefore the space V decomposes into the eigenspaces of J_3, $V = \bigoplus_\alpha V_\alpha$, with $J_3 v = \alpha v$ for $v \in V_\alpha$. Show that $J_\pm v \in V_{\alpha\pm1}$. $J_\pm$ are therefore also called raising and lowering operators. Also show that for an irrep, $V = \bigoplus_{k\in\mathbb{Z}} V_{\alpha+k}$ for some α and that each V_α is one dimensional.

(d) Now since we consider finite dimensional representations, there must be a l such that $V_l \neq \{0\}$ and $V_{l+1} = \{0\}$. Determine $C(\rho)$ by the requirement that $J_+v_+ = 0$ for $v_+ \in V_l$. *Hint: Multiply with J_- and use the relations from ex.2b).* v_+ is then called **highest weight state**. Therefore we can write $V = \bigoplus_{m\in\mathbb{N}_0} V_{l-m}$ with $(J_-)^m v_+ \in V_{l-m}$. But there must also be a lowest weight state $v_- \in V_{l'}$ with $J_-v_- = 0$. Show that $l' = -l$ and determine the dimension of ρ. Which values can l have?

This result should remind you of the angular momentum in quantum mechanics. From there we know that a spin l particle appeares in $2l+1$ states, which corresponts to the dimension of the representation. It comes from the fact, that the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same. (But SU(2) is the double cover of SO(3), in other words, $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathbb{RP}_3$).

3. Now that we know all finite dimensional representations of $\mathfrak{su}(2)$, let us look at some representations in general. At first, for $\mathfrak{su}(N)$ there is of course the **fundamental** N-dimensional representation, where the group acts just by matrix multiplication.

   (a) Show that for a given representation $\rho(T_i)$, there exists the **conjugate representation** defined by

   $$\bar{\rho}(T_i) = -\rho(T_i)^*$$

   If ρ is an D dimensional representation, the conjugate of in is called $\bar{D}$, if it is not equivalent to D (In this case the representation is called **real**). The conjugate of the fundamental is also called antifundamental.

   (b) Show that the **adjoint**, defined by

   $$\mathrm{Ad}(T_i)_{jk} := \mathrm{i} f_{ijk}$$

   is a representation. *Hint: Jacobi identity.* Its dimension is the real dimension of the Lie algebra. Compute dim(Ad) for $\mathfrak{su}(N)$.

References

[1] J. Fuchs and C. Schweigert. *Symmetries, Lie Algebras and Representations.* Cambridge University Press, 2003.

[2] H. Georgi. *Lie Algebras in Particle Physics: From Isospin to Unified Theories.* Westview Press, 1999.